

Duality Theory

Every LP is associated with another LP, called the **dual** (in this case, the original LP is called the **primal**). The relation between an LP and its dual is extremely important for understanding the linear programming (and non-linear programming, indeed). It also provides insights into the so called *sensitivity analysis*.

1 What is the dual of an LP in standard form?

Consider an LP in standard form:

$$\text{Maximize } Z = c^T x$$

such that

$$Ax \leq b, \quad x \geq 0.$$

Here

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Its **dual** is the following minimization LP:

$$\text{Minimize } W = b^T y$$

such that

$$A^T y \geq c, \quad y \geq 0.$$

Here

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Example: Suppose that the primal LP is

$$\text{Maximize } Z = 2x_1 + 3x_2$$

under constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ -x_1 + x_2 &\leq 1 \\ -3x_1 + x_2 &\leq -1 \end{aligned}$$

and $x_1, x_2 \geq 0$.

In this case,

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -3 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$$

Therefore, the dual is:

$$\text{Minimize} \quad W = b^T y = 4y_1 + y_2 - y_3$$

under constraints

$$\begin{aligned} 2y_1 - y_2 - 3y_3 &\geq 2 \\ y_1 + y_2 + y_3 &\geq 3 \end{aligned}$$

and $y_1, y_2, y_3 \geq 0$.

2 What is the meaning of the dual? How can we write down a dual for LPs of other forms?

The dual LP in the preceding section seems a bit out of the blue. Actually, the value of the dual LP defines an *upper bound* for the value of the primal LP. More precisely, we claim that:

Theorem: Suppose x is feasible for the primal, and y is feasible for the dual. Then

$$c^T x \leq b^T y.$$

Proof: By definition, $A^T y - c \geq 0$. Since x is non-negative component-wise, we have

$$(A^T y - c)^T \cdot x \geq 0 \quad \Rightarrow \quad y^T Ax - c^T x \geq 0.$$

However, since $Ax \leq b$ and y is non-negative component-wise, we have

$$y^T b - c^T x \geq y^T Ax - c^T x \geq 0.$$

This completes the proof. □

Remark: The optimal value of the primal (maximization LP) is less than or equal to that of the dual (minimization LP).

Remark: Suppose there exist x^* (feasible for the primal) and y^* (feasible for the dual) such that

$$c^T x^* = b^T y^*.$$

Then x^* is an optimal for the primal LP, and y^* is an optimal solution to the dual LP (why?).

The given proof is very simple, but does not shed much light to the construction of the dual of LP in other forms. We should instead specialize to the concrete example from the previous section, and give a detailed account of how the dual is obtained.

Example (revisited): Suppose that the primal LP is

$$\text{Maximize} \quad Z = 2x_1 + 3x_2$$

under constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ -x_1 + x_2 &\leq 1 \\ -3x_1 + x_2 &\leq -1 \end{aligned}$$

and $x_1, x_2 \geq 0$.

Please keep in mind that we are searching for an LP (the dual) serving as an upper-bound of the primal LP.

Construction of the dual: Let $x = (x_1, x_2)^T$ be a feasible solution to the primal LP, and $y = (y_1, y_2, y_3)^T \in \mathbb{R}^3$.

Consider the function

$$\begin{aligned} f &= y_1(2x_1 + x_2) + y_2(-x_1 + x_2) + y_3(-3x_1 + x_2) \\ &= x_1(2y_1 - y_2 - 3y_3) + x_2(y_1 + y_2 + y_3) \end{aligned}$$

1. Since all the constraints in the primal LP is “ \leq ”, we need to assign sign constraints $y_1 \geq 0$, $y_2 \geq 0$ and $y_3 \geq 0$, which leads to

$$f \leq 4y_1 + y_2 - y_3 = b^T y.$$

2. Since $x_1 \geq 0$ and $x_2 \geq 0$ in the primal, we need to assign constraints

$$\begin{aligned} 2y_1 - y_2 - 3y_3 &\geq 2 \\ y_1 + y_2 + y_3 &\geq 3 \end{aligned}$$

which leads to

$$f \geq 2x_1 + 3x_2 = c^T x.$$

This ends the construction. □

Construction dual for an LP in general form: The constraint of “ \geq ” type is not a big issue, since one can change it into “ \leq ” by multiplying -1 on both sides.

As for the constraint of “ $=$ ” type, say, in the above example, the constraint is indeed $-x_1 + x_2 = 1$, then we do not need to restrict y_2 to be positive, and still have the desired inequality.

If one decision variable, say x_1 has no sign constraint, then we have to restrict

$$2y_1 - y_2 - 3y_3 = 2$$

in order for the inequalities to work.

We have the following correspondence in building the dual of an LP in general form.

| | Primal | Dual |
|--------------------|--|--|
| Objective function | Max $Z = c^T x$ | Min $W = b^T y$. |
| Row (i) | $a_{i1}x_1 + \dots + a_{in}x_n = b_i$ | no sign constrain on y_i |
| Row (i) | $a_{i1}x_1 + \dots + a_{in}x_n \leq b_i$ | $y_i \geq 0$ |
| Variable (j) | $x_j \geq 0$ | $a_{1j}y_1 + a_{2j}y_2 + \dots + a_{mj}y_m \geq c_j$ |
| Variable (j) | x_j has no sign constraint | $a_{1j}y_1 + a_{2j}y_2 + \dots + a_{mj}y_m = c_j$ |

The dual of LP in canonical form: Suppose that the primal LP is in canonical form:

$$\text{Maximize } Z = c^T x, \quad \text{such that } Ax = b, \quad x \geq 0.$$

Its dual is

$$\text{Minimize } W = b^T y, \quad \text{such that } A^T y \geq c \quad (\text{no sign constraints on } y).$$

Example: Find the dual of the following LPs.

$$\text{Maximize } Z = 2x_1 + x_2$$

under constraints

$$\begin{aligned} x_1 + x_2 &\geq 4 \\ -x_1 + 2x_2 &\leq 1 \\ -3x_1 + x_2 &= -1 \end{aligned}$$

and $x_1 \geq 0, x_2 \in \mathbb{R}$.

Solution: The dual can be found as follows: Convert the first \geq constraints to

$$-x_1 - x_2 \leq -4$$

| | Primal | Dual |
|--------------------|------------------------------|-------------------------------|
| Objective function | Max $Z = 2x_1 + x_2$ | Min $W = -4y_1 + y_2 - y_3$. |
| Row (1) | $-x_1 - x_2 \leq -4$ | $y_1 \geq 0$ |
| Row (2) | $-x_1 + 2x_2 \leq 1$ | $y_2 \geq 0$ |
| Row (3) | $-3x_1 + x_2 = 1$ | no sign constraint on y_3 |
| Variable (1) | $x_1 \geq 0$ | $-y_1 - y_2 - 3y_3 \geq 2$ |
| Variable (2) | x_2 has no sign constraint | $-y_1 + 2y_2 + y_3 = 1$ |

Question: As we have discussed before, a general LP can always be expressed in the canonical form (or standard form), with (possible) introduction of slack variables. Both of the original LP and this (equivalent) expanded LP will have its corresponding dual LP. The question is: are the two duals from equivalent primal LPs equivalent?

The answer is **affirmative**, as the following example shows. Consider an LP in standard form

$$\text{Maximize } Z = 3x_1 + x_2$$

under constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 2 \\ -x_1 + x_2 &\leq 1 \end{aligned}$$

and $x_1, x_2 \geq 0$. The expanded LP in canonical form is

$$\text{Maximize } Z = 3x_1 + x_2$$

under constraints

$$\begin{aligned} 2x_1 + x_2 + s_1 &= 2 \\ -x_1 + x_2 + s_2 &= 1 \end{aligned}$$

The dual LP for the LP in the standard form is

| | Primal | Dual |
|--------------------|----------------------|------------------------|
| Objective function | Max $Z = 3x_1 + x_2$ | Min $W = 2y_1 + y_2$. |
| Row (1) | $2x_1 + x_2 \leq 2$ | $y_1 \geq 0$ |
| Row (2) | $-x_1 + 2x_2 \leq 1$ | $y_2 \geq 0$ |
| Variable (1) | $x_1 \geq 0$ | $2y_1 - y_2 \geq 3$ |
| Variable (2) | $x_2 \geq 0$ | $y_1 + y_2 \geq 1$ |

The dual for the expanded LP in canonical form is

| | Primal | Dual |
|--------------------|------------------------|------------------------------|
| Objective function | Max $Z = 3x_1 + x_2$ | Min $W = 2y_1 + y_2$. |
| Row (1) | $2x_1 + x_2 + s_1 = 2$ | y_1 has no sign constraint |
| Row (2) | $-x_1 + x_2 + s_2 = 1$ | y_2 has no sign constraint |
| Variable (1) | $x_1 \geq 0$ | $2y_1 - y_2 \geq 3$ |
| Variable (2) | $x_2 \geq 0$ | $y_1 + y_2 \geq 1$ |
| Slack Variable (1) | $s_1 \geq 0$ | $y_1 \geq 0$ |
| Slack Variable (2) | $s_2 \geq 0$ | $y_2 \geq 0$ |

The two dual LP are obviously equivalent. □

3 The dual of the dual is the primal

The following result establishes the *dual relation* between the primal and the dual.

Theorem: The dual of the dual is the primal.

Proof. Without loss of generality, we will restrict ourselves to primal LPs in standard form. Suppose the primal is

$$\text{Maximize } Z = c^T x$$

such that

$$Ax \leq b, \quad x \geq 0.$$

Its dual is

$$\text{Minimize } W = b^T y$$

such that

$$A^T y \geq c, \quad y \geq 0,$$

which can be written in standard form

$$\text{Maximize } -W = (-b)^T y$$

such that

$$(-A)^T y \leq -c, \quad y \geq 0.$$

The dual of the dual is therefore

$$\text{Minimize } \bar{Z} = (-c)^T x$$

such that

$$((-A)^T)^T x \geq -b, \quad x \geq 0,$$

But this equivalent to

$$\text{Maximize } Z = c^T x$$

such that

$$Ax \leq b, \quad x \geq 0.$$

We complete the proof. □

4 The dual principle

In this section we study the extremely important **dual theorem**. We should state the theorem and give a proof. The proof relies on some important formulae from simplex algorithm.

Dual Theorem: An LP has an optimal solution if and only if its dual has an optimal solution, and in this case their optimal values are equal.

An immediate consequence of the above result is the following:

Corollary: Exactly one of these three cases occurs:

1. Both the Primal and the dual have no feasible solution.
2. One is unbounded, and the other has no feasible solution.
3. (normal case) Both have optimal solutions, and the optimal values are identical.

Proof: Suppose that none of the primal and the dual has optimal solutions. This means that both LP are either unbounded or infeasible. We only need to argue that the two LPs cannot be unbounded at the same time. If so, the optimal value of the dual (minimization LP) is $-\infty$ and the optimal value of the primal (maximization LP) is $+\infty$. However, we know the value of the dual dominates the value of the primal, we have $-\infty \geq +\infty$, a contradiction. □

In fact, each of the three cases can occur.

Exercise: Here is an example of normal case (check!)

$$\text{Maximize } Z = 3x_1 + 4x_2, \text{ such that } x_1 + 2x_2 \leq 1, \quad x_1, x_2 \geq 0.$$

and its dual

$$\text{Minimize } Z = y_1, \text{ such that } y_1 \geq 3, \quad 2y_1 \geq 4, \quad y_1 \geq 0.$$

Both optimal values equal 3.

Exercise: Here is an example where both primal and dual have no feasible solution (check!).

$$\text{Maximize } Z = 2x_1 + 4x_2, \text{ such that } x_1 \leq -5, \quad -x_2 \leq -2, \quad x_1, x_2 \geq 0.$$

and its dual

$$\text{Minimize } Z = -5y_1 - 2y_2, \text{ such that } y_1 \geq 2, \quad -y_1 \geq 4, \quad y_1, y_2 \geq 0.$$

Exercise: Here is an example where the primal is unbounded, and the dual has no feasible solution.

$$\text{Maximize } Z = x_1, \text{ such that } -x_1 \leq 3, \quad -x_1 \leq 2, \quad x_1 \geq 0.$$

and its dual

$$\text{Minimize } Z = 3y_1 + 2y_2, \text{ such that } -y_1 - y_2 \geq 1, \quad y_1, y_2 \geq 0.$$

Can you construct an example where the primal has no feasible solution, and the dual is unbounded?

Before the proof of the dual theorem, we will prepare ourselves in the next section with some useful observations and formulae from simplex algorithm. The proof is given in the section after.

5 Some important formulae: getting the simplex tableau at any stage in terms of the original LP

In this section we discuss some important results regarding the simplex algorithm; i.e. how an LP's optimal tableau (indeed, any tableau) can be expressed in terms of the parameters from original LP? These results are not only useful for the proof of the dual theorem, but also for sensitivity analysis, and other advanced LP topics.

5.1 Getting the simplex tableau

Consider the following LP in canonical form:

$$\begin{aligned} &\text{Maximize } Z = c^T x \\ &\text{such that } Ax = b, \quad x \geq 0. \end{aligned}$$

Here A is an $m \times n$ matrix.

Question: Suppose we are informed that, in a simplex tableau the basic variables are

$$\mathbf{x}_B = [x_{i_1}, \dots, x_{i_m}].$$

Can we construct the tableau (without going through the simplex from them beginning)?

Answer: Well, let us start by defining

$$\mathbf{x}_{NB} = [x_{i_{m+1}}, \dots, x_{i_n}]$$

as the list of non-basic variables. Let A_i be the i -th column of A , and

$$\mathbf{B} = [A_{i_1}, \dots, A_{i_m}], \quad N = [A_{i_{m+1}}, \dots, A_{i_n}].$$

In other words, \mathbf{B} and N are the coefficient (sub)matrix for the basic variables and the non-basic variables, respectively. Similarly, define

$$\mathbf{c}_B = \begin{bmatrix} c_{i_1} \\ \vdots \\ c_{i_m} \end{bmatrix}, \quad \mathbf{c}_{NB} = \begin{bmatrix} c_{i_{m+1}} \\ \vdots \\ c_{i_n} \end{bmatrix},$$

which are the coefficients for BV and NBV in the objective function, respectively.

It follows that the original constraint $Ax = b$ can be rewritten as

$$A_1x_1 + A_2x_2 + \dots + A_nx_n = b,$$

which can be further expressed as

$$Ax = \mathbf{B}\mathbf{x}_B + N\mathbf{x}_{NB} = b.$$

By multiplying \mathbf{B}^{-1} on both sides, we have

$$(1) \quad \mathbf{B}^{-1}Ax = \mathbf{x}_B + \mathbf{B}^{-1}N\mathbf{x}_{NB} = \mathbf{B}^{-1}b.$$

If you read this equation in more details, it says

$$x_{i_k} + \text{a linear combination of non-basic variables} = \text{some constant}.$$

However, we know that in the simplex tableau with basic variables \mathbf{x}_B have the following property: the row corresponding to basic variable x_{i_k} can be interpreted as

$$x_{i_k} + \text{a linear combination of non-basic variables} = \text{RHS}.$$

The preceding two equations, therefore, have to be identical. In other words, the simplex tableau at this stage is indeed,

$$(2) \quad \mathbf{B}^{-1}Ax = \mathbf{B}^{-1}b.$$

But what about Row (0)? We know that Row (0) starts off with

$$0 = Z - c^T x = Z - \mathbf{c}_B^T \mathbf{x}_B - \mathbf{c}_{NB}^T \mathbf{x}_{NB}.$$

We want to express this equation in terms of Z and \mathbf{x}_{NB} only. However, it follows from (1) that

$$\mathbf{c}_{\mathbf{B}}^T \mathbf{x}_{\mathbf{B}} + \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} N \mathbf{x}_{\text{NB}} = \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} b,$$

which in turn implies

$$(3) \quad Z + (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} N - \mathbf{c}_{\text{NB}}^T) \mathbf{x}_{\text{NB}} = \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} b.$$

Note in equation (3) the coefficients of basic variables are zero, exactly as they should be. This gives the Row (0) for the simplex tableau. To write it more clearly, we can do the following:

$$\begin{aligned} & Z + (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} N - \mathbf{c}_{\text{NB}}^T) \mathbf{x}_{\text{NB}} \\ = & Z + (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{B} - \mathbf{c}_{\mathbf{B}}^T) \mathbf{x}_{\mathbf{B}} + (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} N - \mathbf{c}_{\text{NB}}^T) \mathbf{x}_{\text{NB}} \\ = & Z + \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} \mathbf{B} \mathbf{x}_{\mathbf{B}} + \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} N \mathbf{x}_{\text{NB}} - (\mathbf{c}_{\mathbf{B}}^T \mathbf{x}_{\mathbf{B}} + \mathbf{c}_{\text{NB}}^T \mathbf{x}_{\text{NB}}) \\ = & Z + \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} (\mathbf{B} \mathbf{x}_{\mathbf{B}} + N \mathbf{x}_{\text{NB}}) - (\mathbf{c}_{\mathbf{B}}^T \mathbf{x}_{\mathbf{B}} + \mathbf{c}_{\text{NB}}^T \mathbf{x}_{\text{NB}}) \\ = & Z + \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} A x - c^T x. \end{aligned}$$

In other words, Row (0) is

$$(4) \quad Z + (\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} A - c^T) x = \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} b.$$

In conclusion, the simplex tableau for an LP in canonical form is

| Basic Variable | Row | Z | x | RHS |
|---------------------------|----------|--------------|---|---|
| Z | (0) | 1 | $\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} A - c^T$ | $\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} b$ |
| $\mathbf{x}_{\mathbf{B}}$ | \vdots | $\mathbf{0}$ | $\mathbf{B}^{-1} A$ | $\mathbf{B}^{-1} b$ |

Remark: Suppose we are only interested in a specific column, say, the column of x_j . The above formulae imply that it is indeed

| Basic Variable | Row | Z | $\cdots x_j \cdots$ | RHS |
|---------------------------|----------|--------------|---|---|
| Z | (0) | 1 | $\cdots \mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} A_j - c_j \cdots$ | $\mathbf{c}_{\mathbf{B}}^T \mathbf{B}^{-1} b$ |
| $\mathbf{x}_{\mathbf{B}}$ | \vdots | $\mathbf{0}$ | $\cdots \mathbf{B}^{-1} A_j \cdots$ | $\mathbf{B}^{-1} b$ |

with A_j be the j -th column in matrix A .

Example: Consider the following LP:

$$\begin{aligned} \text{Maximize} \quad & Z = -x_1 + x_2 \\ \text{such that} \quad & 2x_1 + x_2 \leq 4 \\ & x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Suppose you are told that the basic variable in a simplex tableau is $BV = (s_1, x_2)$. Construct the simplex table. Is this the optimal table?

Solution: We should first add slack variables and turn the LP into canonical form.

$$\begin{aligned} \text{Maximize} \quad & Z = -x_1 + x_2 + 0s_1 + 0s_2 \\ \text{such that} \quad & 2x_1 + x_2 + s_1 + 0s_2 = 4 \\ & x_1 + x_2 + 0s_1 + s_2 = 2 \\ & x_1, x_2, s_1, s_2 \geq 0. \end{aligned}$$

We have

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{x}_B = [s_1, x_2], \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A bit algebra leads to

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

and

$$\begin{aligned} \mathbf{B}^{-1}A &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \\ \mathbf{B}^{-1}b &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{c}_B^T \mathbf{B}^{-1}b = [0 \ 1] \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \end{aligned}$$

and

$$\mathbf{c}_B \mathbf{B}^{-1}A - c^T = [0 \ 1] \cdot \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix} - [-1 \ 1 \ 0 \ 0] = [2 \ 0 \ 0 \ 1].$$

Therefore, the simplex tableau is

| Basic Variable | Row | Z | x_1 | x_2 | s_1 | s_2 | RHS |
|----------------|-----|---|-------|-------|-------|-------|-----|
| Z | (0) | 1 | 2 | 0 | 0 | 1 | 2 |
| s_1 | (1) | 0 | 1 | 0 | 1 | -1 | 2 |
| x_2 | (2) | 0 | 1 | 1 | 0 | 1 | 2 |

This table is optimal. □

Exercise: Use simplex-algorithm to the above example to double-check the tableau.

Example: Consider the LP in canonical form

$$\begin{aligned} \text{Maximize} \quad & Z = x_1 + x_2 + 2x_3 \\ \text{such that} \quad & 2x_1 + x_2 + 2x_3 = 4 \\ & 2x_2 + x_3 = 5 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Suppose someone tells you that in the optimal tableau, the basic variables are $BV = (x_1, x_2)$. Without solving the LP, check whether this information is valid.

Solution: In this case, we have

$$\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

One can compute

$$\mathbf{B}^{-1} = \begin{bmatrix} 0.5 & -0.25 \\ 0 & 0.5 \end{bmatrix}, \quad \mathbf{B}^{-1}A = \begin{bmatrix} 0.5 & -0.25 \\ 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.75 \\ 0 & 1 & 0.5 \end{bmatrix}$$

Therefore the coefficients for the decision variables are

$$\mathbf{c}_B \mathbf{B}^{-1}A - c^T = \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0.75 \\ 0 & 1 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -0.75 \end{bmatrix}.$$

Since there is a decision variable has negative coefficient in the table, it is not optimal. □

5.2 Getting the simplex tableau: specialization to slack variables

Consider the following LP of standard form:

$$\begin{aligned} \text{Maximize} \quad & Z = c^T x \\ \text{such that} \quad & Ax \leq b, \quad x \geq 0. \end{aligned}$$

If the vector $b \geq 0$, then we can add slack variables, and the LP will turn into canonical form

$$\begin{aligned} \text{Maximize} \quad & Z = [c \ \mathbf{0}]^T \cdot \begin{bmatrix} x \\ \mathbf{s} \end{bmatrix} \\ \text{such that} \quad & [A \ \mathbf{I}] \cdot \begin{bmatrix} x \\ \mathbf{s} \end{bmatrix} = b, \quad x, \mathbf{s} \geq 0. \end{aligned}$$

Now suppose \mathbf{x}_B (could contain slack variables) is basic variables in a tableau, a direct consequence of the general formula we have obtained in the preceding subsection yields the whole tableau as

| Basic Variable | Row | Z | x | s | RHS |
|----------------|----------|--------------|--|----------------------------------|------------------------------------|
| Z | (0) | 1 | $\mathbf{c}_B^T \mathbf{B}^{-1} A - c^T$ | $\mathbf{c}_B^T \mathbf{B}^{-1}$ | $\mathbf{c}_B^T \mathbf{B}^{-1} b$ |
| \mathbf{x}_B | \vdots | $\mathbf{0}$ | $\mathbf{B}^{-1} A$ | \mathbf{B}^{-1} | $\mathbf{B}^{-1} b$ |

The main observation is: *The matrix \mathbf{B}^{-1} is already present in the tableau.* This observation is very useful in sensitivity analysis.

Example: Let us revisit the following LP:

$$\begin{aligned} \text{Maximize} \quad & Z = -x_1 + x_2 \\ \text{such that} \quad & 2x_1 + x_2 \leq 4 \\ & x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Suppose $BV = (s_1, x_2)$. Then we have from before that

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Now perform the simplex algorithm, we have

| Basic Variable | Row | Z | x_1 | x_2 | s_1 | s_2 | RHS | Ratios |
|----------------|-----|---|-------|-------|-------|-------|-----|---------------------------|
| Z | (0) | 1 | 1 | -1 | 0 | 0 | 0 | |
| s_1 | (1) | 0 | 2 | 1 | 1 | 0 | 4 | $4/1 = 4$ |
| s_2 | (2) | 0 | 1 | 1* | 0 | 1 | 2 | $2/1 = 2 \leftarrow \min$ |

| Basic Variable | Row | Z | x_1 | x_2 | s_1 | s_2 | RHS | Ratios |
|----------------|-----|---|-------|-------|-------|-------|-----|--------|
| Z | (0) | 1 | 2 | 0 | 0 | 1 | 2 | |
| s_1 | (1) | 0 | 1 | 0 | 1 | -1 | 2 | |
| x_2 | (2) | 0 | 1 | 1 | 0 | 1 | 2 | |

Observe \mathbf{B}^{-1} is exactly the matrix by the columns of slack variables.

6 Proof of the dual theorem

Proof: We will assume that the primal LP is in canonical form

$$\text{Maximize} \quad Z = c^T x, \quad \text{such that } Ax = b, \quad x \geq 0.$$

Its dual is

$$\text{Minimize } W = b^T y, \quad \text{such that } A^T y \geq c \quad (\text{no sign constraints on } y).$$

Step 1: Suppose \mathbf{x}_B is the basic variables in the optimal BFS (say x^*) for the primal LP. It follows from the above discussion that Row (0) of the optimal tableau will be

| Basic Variable | Row | Z | x | RHS |
|----------------|-----|---|--|------------------------------------|
| Z | (0) | 1 | $\mathbf{c}_B^T \mathbf{B}^{-1} A - c^T$ | $\mathbf{c}_B^T \mathbf{B}^{-1} b$ |

It follows that, if we let

$$\hat{y} = (\mathbf{B}^{-1})^T \mathbf{c}_B,$$

we have

$$\hat{y}^T A - c^T \geq 0, \quad \text{or} \quad A^T \hat{y} \geq c.$$

In other words, y^* is dual feasible.

Step 2: The dual objective function takes value $b^T \hat{y}$ at feasible solution \hat{y} . However, it is not difficult to see that

$$b^T \hat{y} = \hat{y}^T b = \left((\mathbf{B}^{-1})^T \mathbf{c}_B \right)^T b = \mathbf{c}_B^T \mathbf{B}^{-1} b = c^T x^*,$$

which is the optimal value for the primal LP.

Step 3: We have found a dual feasible solution \hat{y} and a primal feasible solution x^* , for which $b^T \hat{y} = c^T x^*$. It follows that \hat{y} is optimal for the dual, and

$$\text{optimal value for the dual} = b^T \hat{y} = c^T x^* = \text{optimal value for the primal}.$$

We complete the proof. □

Corollary: It follows from Step 1 in the proof that a BFS with basic variables \mathbf{x}_B is optimal if and only if $(\mathbf{B}^{-1})^T \mathbf{c}_B$ is dual feasible. And in this case, $(\mathbf{B}^{-1})^T \mathbf{c}_B$ is indeed dual optimal.

Some comment on the slack variables: Consider the following LP of standard form:

$$\text{Maximize } Z = c^T x, \quad \text{such that } Ax \leq b, \quad x \geq 0.$$

If the vector $b \geq 0$, then we know Row (0) of the simplex tableau corresponding basic variables \mathbf{x}_B is of form

| Basic Variable | Row | Z | x | s | RHS |
|----------------|-----|---|--|----------------------------------|------------------------------------|
| Z | (0) | 1 | $\mathbf{c}_B^T \mathbf{B}^{-1} A - c^T$ | $\mathbf{c}_B^T \mathbf{B}^{-1}$ | $\mathbf{c}_B^T \mathbf{B}^{-1} b$ |

This implies that *the optimal value of the i -th dual variable is the coefficient of the slack variable s_i in Row (0) of the optimal tableau.*

Remark: The simplex algorithm tableau is not optimal if there are some negative coefficients in Row (0). This means that the corresponding $(\mathbf{B}^{-1})^T \mathbf{c}_\mathbf{B}$ is not dual feasible. Therefore the simplex algorithm can be regarded as keeping the primal feasibility while trying to reach dual feasibility.

Example: Consider the LP problem

$$\begin{aligned} \text{Maximize} \quad & Z = 4x_1 + x_2 \\ \text{such that} \quad & 3x_1 + 2x_2 \leq 6 \\ & 6x_1 + 3x_2 \leq 10 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Suppose in solving this LP, Row (0) of the optimal tableau is found to be

| Basic Variable | Row | Z | x_1 | x_2 | s_1 | s_2 | RHS |
|----------------|-----|---|-------|-------|-------|-------|------|
| Z | (0) | 1 | 0 | 2 | 0 | 1 | 20/3 |

Use the dual theorem to prove that the computations must be incorrect.

Solution: The dual LP is

$$\begin{aligned} \text{Minimize} \quad & W = 6y_1 + 10y_2 \\ \text{such that} \quad & 3y_1 + 6y_2 \geq 4 \\ & 2y_1 + 3y_2 \geq 1 \\ & y_1, y_2 \geq 0. \end{aligned}$$

Suppose that the computation is correct, we know

$$\hat{y} = (0, 1)$$

is dual feasible and optimal. It is indeed feasible, and we expect

$$W^* = b^T \hat{y} = 6 \cdot 0 + 10 \cdot 1 = 10.$$

But

$$W^* \neq Z^* = \frac{20}{3}$$

is in contradiction with the dual theorem. The computations must be wrong. □

7 Complete slackness theorem

The complete slackness theorem is a very important result that allows us to determine if a pair of vectors, respectively primal and dual feasible, are primal optimal and dual optimal. Rough speaking, the result says that *“If a dual variable is non-zero, the corresponding primal constraint must be tight. If a primal variable is non-zero, the corresponding dual constraint must be tight.”* The following theorem is stated for LP in standard form for the sake of simplicity, though the result is true for general LP.

Complete slackness theorem: Consider a primal LP in standard form:

$$\text{Maximize } Z = c^T x \quad \text{such that } Ax \leq b, \quad x \geq 0$$

and its dual

$$\text{Minimize } W = b^T y \quad \text{such that } A^T y \geq c, \quad y \geq 0.$$

Let \hat{x} be primal feasible and \hat{y} be dual feasible. Then \hat{x} is primal optimal and \hat{y} is dual optimal if and only if

$$(b - A\hat{x})^T \hat{y} = 0 \quad \text{and} \quad (A^T \hat{y} - c)^T \hat{x} = 0.$$

Proof: Define

$$V = (b - A\hat{x})^T \hat{y} + (A^T \hat{y} - c)^T \hat{x}.$$

Since $b - A\hat{x} \geq 0$, $\hat{y} \geq 0$, $A^T \hat{y} - c \geq 0$, $\hat{x} \geq 0$, it is clear that $V \geq 0$, and

$$V = 0 \quad \text{if and only if} \quad (b - A\hat{x})^T \hat{y} = 0 \quad \text{and} \quad (A^T \hat{y} - c)^T \hat{x} = 0.$$

Moreover, it is not difficult to compute that

$$V = b^T \hat{y} - \hat{x}^T A^T \hat{y} + \hat{y}^T A \hat{x} - c^T \hat{x} = b^T \hat{y} - c^T \hat{x},$$

since $\hat{x}^T A^T \hat{y} = (\hat{x}^T A^T \hat{y})^T = \hat{y}^T A \hat{x}$ (it is a real number).

“ \Rightarrow ”: suppose \hat{x} is primal optimal and \hat{y} is dual optimal, we have

$$V = b^T \hat{y} - c^T \hat{x} = 0,$$

thanks to the dual theorem. This implies that

$$(b - A\hat{x})^T \hat{y} = 0 \quad \text{and} \quad (A^T \hat{y} - c)^T \hat{x} = 0.$$

“ \Leftarrow ”: It follows that $V = 0$, which implies

$$b^T \hat{y} - c^T \hat{x} = 0.$$

This implies that \hat{x} is primal optimal, and \hat{y} is dual optimal. This completes the proof. \square

8 Economic interpretation of the dual LP

Consider a LP in standard form

$$\text{Maximize } Z = c^T x, \quad \text{such that } Ax \leq b, \quad x \geq 0,$$

and its dual

$$\text{Minimize } W = b^T y, \quad \text{such that } A^T y \geq c, \quad y \geq 0.$$

We also assume vector $b \geq 0$. Think of the primal LP as a production problem, x_j being the output of the j -th item, c_j being the profit from the sale of one unit of j -th item, b_i being the available amount of i -th resource, and a_{ij} as the amount of i -th resource required for the production of one unit of j -th item. The primal LP is then just the problem of maximizing profit, subject to the total supply of the resources.

The primal LP can be written in canonical form with slack variables. Suppose the optimal basic variables is \mathbf{x}_B , then the optimal tableau is

| Basic Variable | Row | Z | x | \mathbf{s} | RHS |
|----------------|----------|--------------|--|----------------------------------|------------------------------------|
| Z | (0) | 1 | $\mathbf{c}_B^T \mathbf{B}^{-1} A - c^T$ | $\mathbf{c}_B^T \mathbf{B}^{-1}$ | $\mathbf{c}_B^T \mathbf{B}^{-1} b$ |
| \mathbf{x}_B | \vdots | $\mathbf{0}$ | $\mathbf{B}^{-1} A$ | \mathbf{B}^{-1} | $\mathbf{B}^{-1} b$ |

and the optimal solution to the dual problem is $y^* = (\mathbf{c}_B^T \mathbf{B}^{-1})^T = (\mathbf{B}^{-1})^T \mathbf{c}_B$.

Suppose now the total available amount of resource slightly increases from b to $b + \Delta b$, with Δb being the small increment. The primal LP becomes

$$\text{Maximize } Z = c^T x, \quad \text{such that } Ax \leq b + \Delta b, \quad x \geq 0,$$

and its dual

$$\text{Minimize } W = (b + \Delta b)^T y, \quad \text{such that } A^T y \geq c, \quad y \geq 0.$$

If we replace b in the above (previously) optimal tableau, we have

| Basic Variable | Row | Z | x | \mathbf{s} | RHS |
|----------------|----------|--------------|--|----------------------------------|---|
| Z | (0) | 1 | $\mathbf{c}_B^T \mathbf{B}^{-1} A - c^T$ | $\mathbf{c}_B^T \mathbf{B}^{-1}$ | $\mathbf{c}_B^T \mathbf{B}^{-1} (b + \Delta b)$ |
| \mathbf{x}_B | \vdots | $\mathbf{0}$ | $\mathbf{B}^{-1} A$ | \mathbf{B}^{-1} | $\mathbf{B}^{-1} (b + \Delta b)$ |

Note that *the coefficients for the decision and slack variables in Row (0) remains unchanged* — they are all non-negative. Therefore, the basic variables \mathbf{x}_B is still optimal for this new primal LP as long as $\mathbf{B}^{-1}(b + \Delta b) \geq 0$ (i.e., the basic solution corresponding to the basic variables \mathbf{x}_B is still feasible in this new LP). One condition to guarantee this, for example, is that $\mathbf{B}^{-1}b > 0$ (or

equivalently, the optimal BFS for the original primal LP is non-degenerate) and the increment Δb is sufficiently small. Suppose this is the case, then $y^* = (\mathbf{B}^{-1})^T \mathbf{c}_B$ is still optimal for the new dual LP, and the increment of maximal profit is

$$\Delta Z^* = \mathbf{c}_B^T \mathbf{B}^{-1} \Delta b = y^* \Delta b,$$

which is also the increment of the optimal value for the dual LP.

For this reason, the dual variables y are often referred to as the **shadow price** (or *marginal value*) for the resources: if the supply of the i -th resource is increase by Δb_i , the profit will increase by $y_i^* \Delta b_i$.

9 Farkas theorem: an application of dual theory

Farkas theorem is a remarkably simple characterization of those linear systems that have solutions. Via the duality theorem, the proof is trivial.

Farkas theorem: The equation $Ax = b$, $x \geq 0$ has a solution if and only if there is no vector y such that $A^T y \geq 0$ and $b^T y < 0$.

Proof: Consider LP in canonical form

$$\text{Maximize } Z = 0^T x \quad \text{such that } Ax = b, \quad x \geq 0,$$

and its dual

$$\text{Minimize } W = b^T y \quad \text{such that } A^T y \geq 0.$$

“ \Rightarrow ”: Suppose $Ax = b$, $x \geq 0$ has a solution. Then the primal LP is feasible and the optimal value is $Z^* = 0$. It follows from the dual theorem that the dual LP is feasible with the same optimal value 0. Now suppose there exists a y such that $A^T y \geq 0$ and $b^T y < 0$. Then the optimal value of the dual is less than 0, a contradiction.

“ \Leftarrow ”: Since the dual problem clearly has a feasible solution $Y = 0$, the dual is either unbounded or has an optimal solution. However, by assumption, the optimal value for the dual LP is clearly bounded from below by 0. In other words, the dual LP has to have an optimal solution. Thanks to the dual theorem, so does the primal LP, which implies that $Ax = b$, $x \geq 0$ has a solution.

Remark: The Farkas theorem is equivalent to the dual theorem in the sense that we can prove the dual theorem from Farkas theorem. We have shown the reverse in the above proof.

9.1 Application of Farkas theorem to the study of Markov chains

Suppose that a particle can be in any one of the states numbered $1, 2, \dots, n$. If the particle is in state i , let p_{ij} denote the probability of a transition (a jump) to state j . We require $p_{ij} \geq 0$ and

$$p_{i1} + p_{i2} + \dots + p_{in} = 1.$$

At a certain instant, let x_i equal the probability that the particle is in state i . Then

$$x_1 + \dots + x_n = 1, \quad x_i \geq 0, \quad \forall i.$$

After a transition, the particle will be in state j with probability

$$y_j = p_{1j}x_1 + p_{2j}x_2 + \cdots + p_{nj}x_n.$$

Evidently, all y_j are non-negative and $\sum y_j = 1$ (why?).

If we write $y = (y_1, \dots, y_n)^T$ and $x = (x_1, \dots, x_n)^T$, we have

$$y = P^T x, \quad \text{with } P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}.$$

The matrix P is called the Markov matrix.

A steady state is a state vector x that goes to itself:

$$x = P^T x, \quad x \geq 0, \quad \sum x_i = 1.$$

We should use Farkas theorem to prove the following result.

Theorem: Every Markov matrix P has a steady state.

Proof: The Markov matrix P has a steady state if and only if

$$Ax = b \quad \text{has a solution } x \geq 0,$$

with

$$A = \begin{bmatrix} P^T - I \\ u^T \end{bmatrix}, \quad \text{with } u = (1, 1, \dots, 1)^n,$$

and

$$b = (0, 0, \dots, 0, 1)^T.$$

If the Markov matrix P has no steady state, Farkas theorem implies that there exists a vector y such that

$$A^T y \geq 0, \quad b^T y < 0.$$

Write $y = (y_1, \dots, y_n, -\lambda)^T = (z^T, -\lambda)$. Then we have

$$A^T y = \begin{bmatrix} P - I & u \end{bmatrix} \cdot \begin{bmatrix} z \\ -\lambda \end{bmatrix} = (P - I)z - \lambda u \geq 0,$$

$$b^T y = -\lambda < 0.$$

However, this implies that

$$\sum_{j=1}^n p_{ij}z_j - z_i \geq \lambda > 0, \quad \text{for all } i.$$

Let $z_m = \max z_i$. We have

$$\sum_{j=1}^n p_{mj}z_j - z_m \geq \lambda > 0.$$

However, the left hand side is

$$\sum_{j=1}^n p_{mj}z_j - z_m \leq z_m \sum_{j=1}^n p_{mj} - z_m \leq 0.$$

A contradiction. □