Variants of Simplex Method

All the examples we have used in the previous chapter to illustrate simple algorithm have the following common form of constraints; i.e.

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i, \quad i = 1, 2, \cdots, m$$

with all these b_i being *non-negative*. The significance of b_i being non-negative is that the *initialization* step of the Simplex algorithm becomes very simple — we just put in the slack variables $\{s_i\}$, and obtain a BFS by setting $x_j = 0$ for all original decision variables and $s_i = b_i$, for every slack variable.

Indeed, initialization is the serious problem caused by constraints of other forms. In this chapter, we will introduce the concept of *artificial variable* to find a starting BFS, and the Big-M method, as well as the Two-Phase Method, that solves the expanded LP problem.

1 The Big-M method

We will illustrate the main idea by solving the following simple example.

Example: Solve the LP problem:

$$\text{Minimize} \qquad Z = 2x_1 + 3x_2$$

under constraints

$$\begin{array}{rcrcl} 2x_1 & + & x_2 & \ge & 4 \\ -x_1 & + & x_2 & \le & 1 \end{array}$$

and $x_1, x_2 \ge 0$.

Solution: This problem can be transformed into canonical form by adding slack variables and change the minimization to maximization:

$$\text{Maximize} \qquad \bar{Z} = -2x_1 - 3x_2$$

such that

$$\begin{array}{rcrcrcrcl} 2x_1 & + & x_2 & - & s_1 & & & = & 4 \\ -x_1 & + & x_2 & & & + & s_2 & = & 1 \end{array}$$

and $x_j, s_i \ge 0$. If we did it the old way, setting $x_i = 0$, we have $s_1 = -4$ and $s_2 = 1$, which is *not* feasible since s_1 is now negative. To overcome this difficulty, we introduce an *artificial variable* $\bar{a}_1$ and write the constraints as

$$\begin{array}{rcrcrcrcrcl} 2x_1 & + & x_2 & - & s_1 & & & + & \bar{a}_1 & = & 4 \\ -x_1 & + & x_2 & & & + & s_2 & & & = & 1 \end{array}$$

with $x_j \geq 0, s_i \geq 0$ and $\bar{a}_1 \geq 0$.

However, the consequences of the introduction of the artificial variable are two-folds: (1) it is easy now to determine an initial BFS corresponding to the new constraints, namely,

$$NBV = (x_1, x_2, s_1) = 0, \qquad BV = (s_2, \bar{a}_1) = (1, 4).$$

(2) there is, however, no guarantee that the optimal solution to the new constraints will be the same as that to the original constraints; in other words, the artificial variable in the new optimal solution may take strictly positive value. Indeed, if we solve the LP with new constraints, we will have an optimal solution

$$(x_1^*, x_2^*) = (0, 0), \quad (s_1^*, s_2^*) = (0, 0), \quad \bar{a}_1 = 4,$$

and the optimal value $\bar{Z}^* = 0$, which is wrong because the graphical method will easily show that the optimal solution is

$$(x_1^*, x_2^*) = (2, 0), \quad \bar{Z}^* = -4, \quad Z^* = -\bar{Z}^* = 4.$$

Remark: Why the optimal solution could be different? Indeed, the introduction of artificial variable could change the feasible region – the expanded LP is now equivalent to

$$\text{Maximize} \qquad \bar{Z} = -2x_1 - 3x_2$$

such that

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & s_1 & \leq & 4 \\ -x_1 & + & x_2 & & & \leq & 1, \end{array}$$

which is further equivalent to (why?)

$$\text{Maximize} \qquad \bar{Z} = -2x_1 - 3x_2$$

such that

$$-x_1 + x_2 \leq 1.$$

The change of feasible region could very possible change the optimal solution.

Big-M method: One way to guarantee that the new optimal solution is optimal for the original LP, is to modify the objective function, so that the artificial variable will take value zero in the new optimal solution. In other words, a "very large" *penalization* is added to the objective function if the artificial variable takes positive value.

Consider the following LP:

$$\text{Maximize} \qquad \bar{Z} = -2x_1 - 3x_2 - M\bar{a}_1$$

such that

$$\begin{array}{rcrcrcrcrcl} 2x_1 & + & x_2 & - & s_1 & & & + & \bar{a}_1 & = & 4 \\ -x_1 & + & x_2 & & & + & s_2 & & & = & 1. \end{array}$$

Here M is a symbolic "big" positive number. It is so big that even if $\bar{a}_1$ is slightly big than 0, the penalization $-M\bar{a}_1$ will be very severe. In this case, it is reasonable that the optimal solution to this new LP will take value 0 for the artificial variable $\bar{a}_1$, and hence an optimal solution for the original LP.

Let us perform the simplex algorithm to find the optimal solution to this new LP. We write down the following table:

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	$\bar{a}_1$	RHS	Ratios
Z	(0)	1	2	3	0	0	M	0	
$\bar{a}_1$	(1)	0	2	1	-1	0	1	4	
s_2	(2)	0	-1	1	0	1	0	1	

You might want to jump to the conclusion that the current BFS is optimal, since all the coefficients in Row (0) is non-negative. **This is wrong!** The validity of this optimality test is based on the assumption that the objective function Z is written in terms of non-basic variable only. However, in the above table, Row (0) contains a basic variable $\bar{a}_1$ with non-zero coefficient. Therefore, we need to perform Gaussian elimination first to make the coefficients of all basic variables be 0. We have

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	$\bar{a}_1$	RHS	Ratios
Z	(0)	1	2-2M	3-M	M	0	0	-4M	
$\bar{a}_1$	(1)	0	2*	1	-1	0	1	4	$4/2 = 2 \leftarrow$min
s_2	(2)	0	-1	1	0	1	0	1	

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	$\bar{a}_1$	RHS	Ratios
Z	(0)	1	0	2	1	0	M-1	-4	
x_1	(1)	0	1	0.5	-0.5	0	0.5	2	
s_2	(2)	0	0	1.5	-0.5	1	0.5	3	

We have the optimal solution

$$(x_1^*, x_2^*) = (2, 0), \quad (s_1^*, s_2^*) = (0, 3), \quad \bar{a}_1^* = 0, \quad \bar{Z}^* = -4, \quad Z^* = -\bar{Z}^* = 4.$$

Remark: The symbol M represents a huge number, and it will only appear in Row (0). The symbolic quantity $aM + b$ (in case $a \neq 0$) almost equals aM since M is much larger compared with the constant b. Therefore, $aM + b$ is positive (resp. negative) as long as a is positive (resp. negative). We have, for example

$$0.1M - 1000 > 0, \quad -0.1M + 100 < 0, \quad 2M - 100 > M + 1000, \quad -2M + 100 < -M - 1000.$$

Remark: It is possible that in the new optimal solution the artificial variable takes a strictly positive value. If this is the case, the original problem has *no* feasible solution. Indeed, for example, say we are to

$$\text{Maximize} \qquad Z = c_1 x_1 + c_2 x_2 - Ma$$

where a is an artificial variable. Suppose in the optimal solution to this LP, the artificial variable $a^* > 0$, then we have

$$Z^* = -a^* M + b, \qquad \text{for some constant } b.$$

If the original LP has a feasible solution, say $(\tilde{x}_1, \tilde{x}_2)$, then $(\tilde{x}_1, \tilde{x}_2, a = 0)$ is a feasible solution to the expanded LP.

$$Z^* \geq c_1 \tilde{x}_1 + c_2 \tilde{x}_2 - M \cdot 0 = c_1 \tilde{x}_1 + c_2 \tilde{x}_2.$$

Combining these two inequalities, we have

$$-a^* M + b \geq c_1 \tilde{x}_1 + c_2 \tilde{x}_2,$$

which is impossible since M is a really big constant, and a^* is strictly positive. A contradiction. Hence the original LP has no feasible solution.

Equality constraints: Suppose one of the constraints is, say,

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i.$$

in a maximization LP. Without loss of generality, assume $b_i \geq 0$ (otherwise, just multiply -1 on both sides). One can write this equality constraint as two inequality constraints (i.e. "$\geq$" and "$\leq$") and then use the Big-M method. However, more conveniently, we can directly introduce the artificial variable $\bar{a}$ and rewrite the LP as

$$\text{Maximize} \qquad Z = \cdots - M\bar{a}$$

while change the constraint to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + \bar{a} = b_i$$

Example: Solve the following LP:

$$\text{Maximize} \qquad Z = -x_1 + x_2$$

under constraints

$$\begin{array}{rcrcl} x_1 & + & x_2 & \geq & 1 \\ 3x_1 & + & 2x_2 & = & 6. \end{array}$$

Solution: The new LP will be

$$\text{Maximize} \qquad Z = -x_1 + x_2 - M\bar{a}_1 - M\bar{a}_2$$

under constraints

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & - & s_1 & + & \bar{a}_1 & & & = & 1 \\ 3x_1 & + & 2x_2 & & & & & + & \bar{a}_2 & = & 6. \end{array}$$

We have the following table:

Basic Variable	Row	Z	x_1	x_2	s_1	$\bar{a}_1$	$\bar{a}_2$	RHS	Ratios
Z	(0)	1	1	-1	0	M	M	0	
$\bar{a}_1$	(1)	0	1	1	-1	1	0	1	
$\bar{a}_2$	(2)	0	3	2	0	0	1	6	

To start the simplex algorithm, we need to write the value function Z as a function of NBV variables.

Basic Variable	Row	Z	x_1	x_2	s_1	$\bar{a}_1$	$\bar{a}_2$	RHS	Ratios
Z	(0)	1	1-4M	-1-3M	M	0	0	-7M	
$\bar{a}_1$	(1)	0	1$*$	1	-1	1	0	1	$1/1 = 1 \leftarrow$min
$\bar{a}_2$	(2)	0	3	2	0	0	1	6	$6/3 = 2$

Basic Variable	Row	Z	x_1	x_2	s_1	$\bar{a}_1$	$\bar{a}_2$	RHS	Ratios
Z	(0)	1	0	M-2	-3M+1	4M-1	0	-3M-1	
x_1	(1)	0	1	1	-1	1	0	1	
$\bar{a}_2$	(2)	0	0	-1	3$*$	-3	1	3	$3/3 = 1 \leftarrow$min

Basic Variable	Row	Z	x_1	x_2	s_1	$\bar{a}_1$	$\bar{a}_2$	RHS	Ratios
Z	(0)	1	0	$-\frac{5}{3}$	0	M	M-$\frac{1}{3}$	-2	
x_1	(1)	0	1	$\frac{2}{3}*$	0	0	$\frac{1}{3}$	2	$2/2 \cdot 3 = 3 \leftarrow$min
s_1	(2)	0	0	$-\frac{1}{3}$	1	-1	$\frac{1}{3}$	1	

Basic Variable	Row	Z	x_1	x_2	s_1	$\bar{a}_1$	$\bar{a}_2$	RHS	Ratios
Z	(0)	1	2.5	0	0	M	$M + 0.5$	3	
x_2	(1)	0	1.5	1	0	0	0.5	3	
s_1	(2)	0	0.5	0	1	-1	0.5	2	

Therefore, the optimal solution is

$$(x_1^*, x_2^*) = (0, 3), \quad s_1^* = 2, \quad (\bar{a}_1^*, \bar{a}_2^*) = (0, 0), \quad Z^* = 3.$$

2 Two-Phase method

The two-phase method and big-M method are equivalent. In practice, however, most computer codes utilizes the two-phased method. The reasons are that the inclusion of the big number M may cause round-off error and other computational difficulties. The two-phase method, on the other hand, does not involve the big number M and hence all the problems are avoided.

The two-phase method, as it is called, divides the process into two phases. *Phase 1:* The goal is to find a BFS for the original LP. Indeed, we will ignore the original objective for a while, and instead try to *minimize* the sum of all artificial variable. At the end of phase 1, a BFS is obtained if the minimal value of this LP is zero (why?). *Phase 2:* Drop all the artificial variables, change the objective function back to the original one. Use just the regular simplex algorithm, with the starting BFS obtained in Phase 1.

We will illustrate the idea re-doing the last example in the preceding section.

Example *(revisited)***:** Solve the following LP:

$$\text{Maximize} \qquad Z = -x_1 + x_2$$

under constraints

$$\begin{array}{rcrcl} x_1 & + & x_2 & \geq & 1 \\ 3x_1 & + & 2x_2 & = & 6. \end{array}$$

Solution: Phase 1: The LP for this phase is

$$\text{Minimize} \qquad Z = \bar{a}_1 + \bar{a}_2 \quad \text{or} \quad \text{Maximize} \qquad Z = -\bar{a}_1 - \bar{a}_2$$

under constraints

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & - & s_1 & + & \bar{a}_1 & & & = & 1 \\ 3x_1 & + & 2x_2 & & & & & + & \bar{a}_2 & = & 6. \end{array}$$

We have the following table.

Basic Variable	Row	Z	x_1	x_2	s_1	$\bar{a}_1$	$\bar{a}_2$	RHS	Ratios
Z	(0)	1	0	0	0	1	1	0	
$\bar{a}_1$	(1)	0	1	1	-1	1	0	1	
$\bar{a}_2$	(2)	0	3	2	0	0	1	6	

Note in this first table, the objective function $Z = -\bar{a}_1 - \bar{a}_2$ is written in term of the basic variables, so first we need to express it in terms of the non-basic variable. This leads to the following table, from which the simplex algorithm starts.

Basic Variable	Row	Z	x_1	x_2	s_1	$\bar{a}_1$	$\bar{a}_2$	RHS	Ratios
Z	(0)	1	-4	-3	1	0	0	-7	
$\bar{a}_1$	(1)	0	1*	1	-1	1	0	1	$1/1 = 1 \leftarrow$min
$\bar{a}_2$	(2)	0	3	2	0	0	1	6	$6/3 = 2$

Basic Variable	Row	Z	x_1	x_2	s_1	$\bar{a}_1$	$\bar{a}_2$	RHS	Ratios
Z	(0)	1	0	1	-3	4	0	-3	
x_1	(1)	0	1	1	-1	1	0	1	
$\bar{a}_2$	(2)	0	0	-1	3*	-3	1	3	$3/3 = 1 \leftarrow$min

Basic Variable	Row	Z	x_1	x_2	s_1	$\bar{a}_1$	$\bar{a}_2$	RHS	Ratios
Z	(0)	1	0	0	0	1	1	0	
x_1	(1)	0	1	$\frac{2}{3}$	0	0	$\frac{1}{3}$	2	
s_1	(2)	0	0	-$\frac{1}{3}$	1	-1	$\frac{1}{3}$	1	

This is the end of phase 1, and we indeed obtain an BFS with

$$NBV = (x_2, \bar{a}_1, \bar{a}_2) = (0, 0, 0), \qquad BV = (x_1, s_1) = (2, 1).$$

Phase 2: We will just drop all the columns of artificial variables in the final tableau of Phase 1, and rewrite the objective function back to the original one.

Basic Variable	Row	Z	x_1	x_2	s_1	RHS	Ratios
Z	(0)	1	1	-1	0	0	
x_1	(1)	0	1	$\frac{2}{3}$	0	2	
s_1	(2)	0	0	-$\frac{1}{3}$	1	1	

However, again, the objective function Z is not expressed in terms of non-basic variables x_2, we will do Gaussian elimination to make it so.

Basic Variable	Row	Z	x_1	x_2	s_1	RHS	Ratios
Z	(0)	1	0	$-\frac{5}{3}$	0	-2	
x_1	(1)	0	1	$\frac{2}{3}$*	0	2	$2/2 \cdot 3 = 2 \leftarrow$min
s_1	(2)	0	0	-$\frac{1}{3}$	1	1	

Basic Variable	Row	Z	x_1	x_2	s_1	RHS	Ratios
Z	(0)	1	2.5	0	0	3	
x_2	(1)	0	1.5	3	0	3	
s_1	(2)	0	0.5	0	1	2	

The optimal solution is

$$(x_1^*, x_2^*) = (0, 3), \quad s_1^* = 2, \quad Z^* = 3.$$

Remark: If in Phase 1, the optimal value for $\bar{a}_1 + \bar{a}_2$ is strictly positive (or $-\bar{a}_1 - \bar{a}_2$ is strictly negative), then the original Lp has *no* feasible solution. Because otherwise, there must exist a feasible solution with $(\bar{a}_1, \bar{a}_2) = 0$, which is a contradiction.

Remark: In the example given, in the final tableau of the LP in Phase 1, the artificial variables are both non-basic variables. It is possible that in the final tableau of Phase 1, some artificial variables are basic variable, while the value of Z already reaches 0. In this case, the problem is degenerate, and we will not go into details about this. But keep in mind, even in this case, it is also very easy to move to Phase 2, and find the optimal solution for the original LP.

Remark: When a variable, say x_1, has no sign constraint, we just write

$$x_1 = u_1 - v_1, \qquad u_1 \geq 0, \ v_1 \geq 0,$$

and then perform the simplex algorithm. It is easy to see that, in all the tables, the column for u_1 is always the negative of that of v_1. Therefore, it will never be the case that u_1 and v_1 are basic variables simultaneously (why?). In other words, it will never be the case that u_1, v_1 are both taking strictly positive values.