

APPLIED MATH 121: MidTerm Exam (open book)

Name:

ID:

1. (6 pts.) Turkeyco produces two types of turkey cutlets for sale to fast restaurants. Each type of cutlet consists of white meat and dark meat. Cutlet A sells for \$4/lb and must consist of at least 70% white meat. Cutlet B sells for \$3/lb and must consist of at least 60% white meat. At most 50 lb of cutlet A and 30 lb of cutlet B can be produced. The two types of turkey used to manufacture the cutlets are purchased from GobbleGobble Turkey Farm. Each type 1 turkey costs \$10 and yields 5 lb of white meat and 2 lb of dark meat. Each type 2 turkey costs \$8 and yields 3 lb of white meat and 3 lb of dark meat. Formulate an LP to maximize Turkeyco's profit.

Solution: Let

$$\begin{aligned}w_1 &= \text{amount of white meat in cutlet A} \\d_1 &= \text{amount of dark meat in cutlet A} \\w_2 &= \text{amount of white meat in cutlet B} \\d_2 &= \text{amount of dark meat in cutlet B} \\T_1 &= \text{number of Turkey 1 purchased} \\T_2 &= \text{number of Turkey 2 purchased.}\end{aligned}$$

The LP is

$$\text{Maximize } Z = 4(w_1 + d_1) + 3(w_2 + d_2) - (10T_1 + 8T_2)$$

under the constraints

$$\begin{aligned}w_1 + d_1 &\leq 50 \\w_2 + d_2 &\leq 30 \\w_1 + w_2 &\leq 5T_1 + 3T_2 \\d_1 + d_2 &\leq 2T_1 + 3T_2 \\w_1 &\geq 0.7(w_1 + d_1) \quad (\text{or } 0.3w_1 - 0.7d_1 \geq 0) \\w_2 &\geq 0.6(w_2 + d_2) \quad (\text{or } 0.4w_2 - 0.6d_2 \geq 0)\end{aligned}$$

and $w_1, w_2, d_1, d_2, T_1, T_2 \geq 0$.

2. (10 pts.) Consider the following LP:

$$\text{Maximize } Z = 4x_1 + x_2$$

under constraints

$$3x_1 + x_2 \leq 6$$

$$x_1 - 2x_2 \leq 0$$

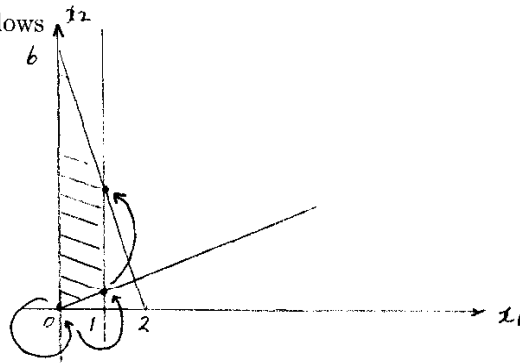
$$x_1 \leq 1$$

and $x_1, x_2 \geq 0$.

- (2 pts.) Draw the feasible region in (x_1, x_2) -space.
- (5 pts.) Use simplex algorithm in tabular form to find the optimal solution and the optimal value of the LP.
- (3 pts.) Describe graphically the sequence of vertices that the simplex algorithm has scanned.

Solution:

- (a) The feasible region is as follows



- (b) The optimal solution is $Z^* = 7$, $(x_1^*, x_2^*) = (1, 3)$. See the tableau next page.
 (c) The sequence of vertices scanned are

$$(0, 0) \rightarrow (0, 0) \rightarrow (1, 0.5) \rightarrow (1, 3)$$

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	-4	-1	0	0	0	0	
s_1	(1)	0	3	1	1	0	0	6	$6/3 = 2$
s_2	(2)	0	1*	-2	0	1	0	0	$0/1 = 0 \leftarrow \text{min}$
s_3	(3)	0	1	0	0	0	1	1	$1/1 = 1$

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	0	-9	0	4	0	0	
s_1	(1)	0	0	7	1	-3	0	6	$6/7$
x_1	(2)	0	1	-2	0	1	0	0	
s_3	(3)	0	0	2*	0	-1	1	1	$1/2 \leftarrow \text{min}$

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	0	0	0	-0.5	4.5	4.5	
s_1	(1)	0	0	0	1	0.5*	-3.5	2.5	$2.5/0.5 = 5 \leftarrow \text{min}$
x_1	(2)	0	1	0	0	0	1	1	
x_2	(3)	0	0	1	0	-0.5	0.5	0.5	

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	0	0	1	0	1	7	
s_2	(1)	0	0	0	2	1	-7	5	
x_1	(2)	0	1	0	0	0	1	1	
x_2	(3)	0	0	1	1	0	-3	3	

3. (14 pts.) Farmer Leary grows wheat and corn on his 45 acre farm. He can sell at most 140 bushels of wheat and 120 bushels of corn. Each planted acre yields either 5 bushels of wheat or 4 bushels of corn. Wheat sells for \$30 per bushel and corn sell for \$50 per bushel. 6 hours of labor is needed to harvest an acre of wheat, and 10 hours are required to harvest an acre of corn. Up to 350 hours of labor can be purchased at \$10 per hour. With

$$x_1 = \text{acres planted with wheat}$$

$$x_2 = \text{acres planted with corn}$$

$$x_3 = \text{hours of labor purchased,}$$

The LP Farmer Leary should solve to optimize profit is

$$\text{Maximize } Z = 5 \cdot 30 \cdot x_1 + 4 \cdot 50 \cdot x_2 - 10x_3$$

under constraints

$$x_1 + x_2 \leq 45 \quad (1)$$

$$6x_1 + 10x_2 - x_3 \leq 0 \quad (2)$$

$$x_3 \leq 350 \quad (3)$$

$$5x_1 \leq 140 \quad (4)$$

$$4x_2 \leq 120 \quad (5)$$

and $x_1, x_2, x_3 \geq 0$.

Adding slack variables, the simplex algorithm yields the following optimal tableau:

Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	RHS
Z	(0)	1	0	0	0	75	12.5	2.5	0	0	4250
x_2	(1)	0	0	1	0	-1.5	0.25	0.25	0	0	20
x_1	(2)	0	1	0	0	2.5	-0.25	-0.25	0	0	25
s_4	(3)	0	0	0	0	-12.5	1.25	1.25	1	0	15
x_3	(4)	0	0	0	1	0	0	1	0	0	350
s_5	(5)	0	0	0	0	6	-1	-1	0	1	40

- (a) (2 pts.) Which constraints in the LP are tight for the optimal solution?
- (b) (4 pts.) Write down the dual of this LP, and find the optimal solution and optimal value for this dual LP.
- (c) (2 pts.) What is the most that Leary should pay for an additional hour of labor?
- (d) (3 pts.) Suppose the DEM declare that 5 acres of Leary's land to be wetlands, and therefore unusable. What would be the new profit?
- (e) (3 pts.) Farmer Leary is considering growing barley. The market for barley is unlimited. An acre yields 4 bushels of barley and requires 3 hours of labor to harvest. If barley sells at \$30 per bushel, should Leary grow *any* barley?

Solution:

- (a) In the optimal solution $(s_1^*, s_2^*, s_3^*) = (0, 0, 0)$, $(s_4^*, s_5^*) = (15, 40)$. Therefore, the constraints (1), (2), (3) are tight, whereas (4), (5) are not tight.
- (b) The dual of this LP is

$$\begin{aligned} &\text{Minimize } W = 45y_1 + 350y_3 + 140y_4 + 120y_5 \\ &\text{under constraints} \\ &\begin{array}{rccccccc} y_1 & + & 6y_2 & & & + & 5y_4 & & & \geq & 150 \\ y_1 & + & 10y_2 & & & & & + & 4y_5 & \geq & 200 \\ & & - & y_2 & + & y_3 & & & & \geq & -10 \end{array} \end{aligned}$$

$$\text{and } y_1, y_2, y_3, y_4, y_5 \geq 0.$$

From the optimal tableau of the primal LP, the optimal solution for the dual is

$$(y_1^*, y_2^*, y_3^*, y_4^*, y_5^*) = (75, 12.5, 2.5, 0, 0), \quad W^* = Z^* = 4250.$$

- (c) It is just the shadow price of the labor, which is the $y_3^* = 2.5$.
- (d) In this case, the right-hand side of the constraints change, and

$$\Delta b = \bar{b} - b = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The original set of optimal basic variables is still optimal since

$$\begin{aligned} \mathbf{B}^{-1}\bar{b} &= \mathbf{B}^{-1}b + \mathbf{B}^{-1}\Delta b = \begin{bmatrix} 20 \\ 25 \\ 15 \\ 350 \\ 40 \end{bmatrix} + \begin{bmatrix} -1.5 & 0.25 & 0.25 & 0 & 0 \\ 2.5 & -0.25 & -0.25 & 0 & 0 \\ -12.5 & 1.25 & 1.25 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 6 & -1 & -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 20 \\ 25 \\ 15 \\ 350 \\ 40 \end{bmatrix} + \begin{bmatrix} 7.5 \\ -12.5 \\ 72.5 \\ 0 \\ -30 \end{bmatrix} = \begin{bmatrix} 27.5 \\ 12.5 \\ 77.5 \\ 350 \\ 10 \end{bmatrix} \geq 0. \end{aligned}$$

Therefore, the new profit is

$$\begin{aligned} \mathbf{c}_B^T \mathbf{B}^{-1}\bar{b} &= \mathbf{c}_B^T \mathbf{B}^{-1}b + \mathbf{c}_B^T \mathbf{B}^{-1}\Delta b = 4250 + [75 \ 12.5 \ 2.5 \ 0 \ 0] \cdot \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= 4250 - 375 = 3875. \end{aligned}$$

(e) We are going to add a new variable

$x_4 =$ acres planted with barley,

with

$$c_4 = 4 \cdot 30 = 120, \quad A_4 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We just need to add a column of x_4 in the simplex tableau. The original set of optimal basic variables will remain optimal as long as the coefficient of x_4 in Row (0) is non-negative. However, the coefficient of x_4 in Row (0) is indeed

$$\mathbf{c}_B^T \mathbf{B}^{-1} A_4 - c_4 = [75 \quad 12.5 \quad 2.5 \quad 0 \quad 0] \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 120 = 112.5 - 120 = -7.5 < 0.$$

Therefore, the original optimal solution is no longer optimal, and Leary should grow barley.