The Simplex Method — Canonical Form Summary Sheet

Standard/Original form LP ($A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c, x \in \mathbb{R}^n$)

$$\begin{array}{lrl} \text{Maximize} & Z &= c^\mathsf{T} x \\ \text{subject to} & Ax &\preceq b \\ & x &\succeq 0. \end{array}$$

Augmented form LP ($x_s \in \mathbb{R}^m$)

$$\begin{array}{lrl} \text{Maximize} & Z = c^\mathsf{T} x + 0x_s & \\ \text{subject to} & Ax + Ix_s &= b \\ & x, x_s &\succeq 0. \end{array}$$

Canonical form (Upon choosing $x_N \in \mathbb{R}^n, x_B \in \mathbb{R}^m$—see below)

$$\begin{array}{lrl} \text{(Increase)} & Z + \bar{c}^\mathsf{T} x_N + 0x_B &= \bar{Z} \\ \text{subject to} & \bar{A}x_N + Ix_B &= \bar{b} \\ & x_N, x_B &\succeq 0. \end{array}$$

$$\text{i.e.} \qquad \underbrace{\begin{bmatrix} 1 & \bar{c}^\mathsf{T} & 0 \\ 0 & \bar{A} & I \end{bmatrix}}_{M} \begin{pmatrix} Z \\ x_N \\ x_B \end{pmatrix} = \begin{pmatrix} \bar{Z} \\ \bar{b} \end{pmatrix}$$

Matrix representation of the canonical form

$$\begin{array}{lrl} \text{(Increase)} & Z + (-c^\mathsf{T} + c_B^\mathsf{T} A_B^{-1} A)x + (c_B^\mathsf{T} A_B^{-1})x_s &= c_B^\mathsf{T} A_B^{-1} b \\ \text{subject to} & (A_B^{-1}A)x + A_B^{-1}x_s &= A_B^{-1}b \\ & x, x_s &\succeq 0. \end{array}$$

$$\text{i.e.} \qquad \underbrace{\begin{bmatrix} 1 & c_B^\mathsf{T} A_B^{-1} \\ 0 & A_B^{-1} \end{bmatrix}}_{Q_B} \underbrace{\begin{bmatrix} 1 & -c^\mathsf{T} & 0 \\ 0 & A & I \end{bmatrix}}_{M_0} \begin{pmatrix} Z \\ x \\ x_s \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & c_B^\mathsf{T} A_B^{-1} \\ 0 & A_B^{-1} \end{bmatrix}}_{Q_B} \begin{pmatrix} 0 \\ b \end{pmatrix}$$

That is, $M = Q_B M_0 P^{-1}$. (see below)

Tableau representation of the canonical form

BV	Z	$x_1 \dots x_n$	$x_{s,1} \dots x_{s,m}$	
	1	$-c^\mathsf{T} + c_B^\mathsf{T} A_B^{-1} A$	$c_B^\mathsf{T} A_B^{-1}$	$c_B A_B^{-1} b$
$x_{B,1}$	0			
$\vdots$	$\vdots$	$A_B^{-1}A$	A_B^{-1}	$A_B^{-1}b$
$x_{B,m}$	0			

Note: The entries in the tableau representation are exactly the same entries in the matrix $Q_b M_0$ and $Q_B \binom{0}{b}$ of the matrix representation

Basic and nonbasic variables for the canonical form. Upon choosing x_N, x_B, we may elucidate a permutation matrix[1] P such that $\boxed{\binom{x_N}{x_B} = P\binom{x}{x_s}}$, from which we obtain

$$[A_N \quad A_B] = [A \quad I]P^{-1}, \quad \text{and} \quad [c_N^\mathsf{T} \quad c_B^\mathsf{T}] = [c^\mathsf{T} \quad 0]P^{-1}$$

Specifically, if $x_N = (x_{N,1}, \ldots, x_{N,n}) = (x_{j_1}, \ldots, x_{j_n})$ and $x_B = (x_{B,1}, \ldots, x_{B,m}) = (x_{k_1}, \ldots, x_{k_m})$, then

$A_N = [C_{j_1} \ \ldots \ C_{j_n}] \in \mathbb{R}^{m\times n}$	The jth column of A_N is the column of $[A\ I]$ corresponding to $x_{N,j}$
$A_B = [C_{k_1} \ \ldots \ C_{k_m}] \in \mathbb{R}^{m\times m}$	The jth column of A_B is the column of $[A\ I]$ corresponding to $x_{B,j}$
$c_N = (c_{j_1} \ \ldots \ c_{j_n})^\mathsf{T} \in \mathbb{R}^{n}$	The jth entry of c_N is the entry of $[c^\mathsf{T}\ 0]$ corresponding to $x_{N,j}$
$c_B = (c_{k_1} \ \ldots \ c_{k_n})^\mathsf{T} \in \mathbb{R}^{m}$	The jth entry of c_B is the column of $[c^\mathsf{T}\ 0]$ corresponding to $x_{B,j}$

where C_j is the jth column of $[A\ I]$, and c_j is the jth entry of $[c^\mathsf{T}\ 0]$.
Consequently,

$$\boxed{\begin{array}{ll} \bar{A} = A_B^{-1}A_N & \bar{b} = A_B^{-1}b \\ \bar{c}^\mathsf{T} = c_B^\mathsf{T}\bar{A} - c_N^\mathsf{T} & \bar{Z} = c_B^\mathsf{T}\bar{b} \end{array}}$$

Connection between canonical form and matrix/tableau representations. Suppose you already know the matrix/tableau representation, and want to obtain the canonical form. Since $M = Q_B M_0 P^{-1}$,

$\bar{A} =$	$\mathbb{R}^{m\times n}$ matrix whose jth column is the column of $[A_B^{-1}A \quad A_B^{-1}]$ corresponding to $x_{N,j}$
$I =$	$\mathbb{R}^{m\times m}$ matrix whose jth column is the column of $[A_B^{-1}A \quad A_B^{-1}]$ corresponding to $x_{B,j}$
$\bar{c} =$	$\mathbb{R}^{n}$ vector whose jth entry is the entry of $[-c^\mathsf{T}+c_B^\mathsf{T}A_B^{-1}A \quad c_B^\mathsf{T}A_B^{-1}]$ corresponding to $x_{N,j}$
$0 =$	$\mathbb{R}^{m}$ vector whose jth entry is the entry of $[-c^\mathsf{T}+c_B^\mathsf{T}A_B^{-1}A \quad c_B^\mathsf{T}A_B^{-1}]$ corresponding to $x_{B,j}$

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[1] Recall that a permutation matrix P has exactly one 1 in each row and each column, with all other entries being 0, and that $P^{-1} = P^\mathsf{T}$. For any matrix X, Y, one can check that left-multiplying PX will permute the *rows* of X, whereas right-multiplying YP^T will permute the *columns* of Y in exactly the same manner that the rows of X were permuted by PX.