**APMA 0410 MIDTERM EXAM**

**October 14, 2012**

**Three Poisson neurons, A, B and C, fire independently of each other. Their rates are, respectively, 3 spikes/s, 5 spikes/s, and 9 spikes/s.**

**Problem 1.**

**We denote the numbers of spikes of A, B and C during a given 5-second period by *X*A, *X*B and *X*C. Compute the expected values and variances of the following three random variables:**

***Y*1 = 2*X*A; *Y*2 = *X*A + *X*B + *X*C; *Y*3 = 2*X*A – 3*X*B + *X*C.**

*E*(*Y*1) = *E*(2*X*A) = 2 *E*(*X*A) = 2×3×5 = 30

*V*(*Y*1) = *V*(2*X*A) = 4 *V*(*X*A) = 4×3×5 = 60

*E*(*Y*2) = *E*(*X*A + *X*B + *X*C) = *E*(*X*A) + *E*(*X*B) + *E*(*X*C) = 3×5 + 5×5 + 9×5 = 85

*V*(*Y*2) = *V*(*X*A + *X*B + *X*C) = *V*(*X*A) + *V*(*X*B) + *V*(*X*C) = 3×5 + 5×5 + 9×5 = 85

*E*(*Y*3) = *E*(2*X*A – 3*X*B + *X*C) = 2*E*(*X*A) – 3*E*(*X*B) + *E*(*X*C) = 2×3×5 + – 3×5×5 + 9×5 = 0

*V*(*Y*3) = *V*(2*X*A – 3*X*B + *X*C) = 4*V*(*X*A) + 9*V*(*X*B) + *V*(*X*C) = 4×3×5 + 9×5×5 + 9×5 = 330

**Problem 2.**

**During the period of time 1 ms to 1000 ms, it is observed that the last firing of neuron B occurred at time *t* = 900 ms. Given that observation, what is the conditional probability that the next spike of B will occur after time *t* = 1200 ms?**

Since the firing of B is a Poisson process, the waiting time until the next spike is exponentially distributed. The exponential distribution is memoryless. Therefore, the conditional distribution of the waiting time given any past observation is the same as the unconditional distribution. In other words, the information about the last firing before *t* = 1000 ms is irrelevant to the question. Let *X* be the waiting time until the next spike, i.e., the next spike occurs at 1 + *X* (sec). We want to find *P*(*X* > 0.2 sec). Since the firing of B is a Poisson process with mean 5 spikes/s, *X* ~ Exponential*b* , with *b*, the expected value of *X*, equal to 1/5 sec. Thus, the pdf of *X* is *f*(*x*) = (1/*b*)exp(–*x*/*b*) = 5exp(–5*x*). We then have:

Therefore, *P*(*X* > 0.2 sec) = 1 – *P*(*X* < 0.2 sec) = exp(–5×0.2) = exp(–1) = 1/*e*.

**Problem 3.**

**When neuron C is not stimulated, it fires at 9 spikes/s as mentioned above. However, when stimulated, it fires at some higher rate. Find a number *r* such that if neuron C is observed to fire more than *r* spikes during a given period of 9 seconds you would conclude at a significance level of 0.025 that it was stimulated during that period. Explain your answer.**

The Null Hypothesis H0 is that C is not stimulated, and the alternative Ha is that it is. Let *X* be the number of spikes in a 9 sec period. Under H0, *X* ~ Poisson 81. Therefore, under H0, *E*(*X*) = *V*(*X*) = 81, and std(*X*) = 9. A Poisson distribution with a mean of 81 is very well approximated by a normal with a mean of 81 and a standard deviation of 9. Using the rule of thumb, 95% of the distribution is within two standard deviations of the mean, and 2.5% is above *r* = mean + 2 standard deviations. We use a one-sided test, since the alternative Ha tells us that the neuron fires at a higher rate than under H0. Choosing *r* = 99 spikes, we will reject H0 in favor of Ha if we observe more than *r* spikes in a period of 9 sec.

**Problem 4. Neurons D, E and F have refractory periods such that, during a given short enough period of time *T*, they can fire at most one spike each. D and E fire independently of each other. The probability that D fires during *T* is 0.3 and the probability that E fires during *T* is 0.5. Neuron F receives an excitatory synapse from neuron D and an inhibitory synapse from neuron E. As a result, the probability that neuron F will fire during *T* is:**

**0.3 if both D and E fired during *T*,**

**0.1 if E but not D fired during *T*,**

**0.8 if D but not E fired during *T*,**

**0.5 if neither D nor E fired during *T*.**

**Given that F was observed to fire during *T*, what is the probability that D fired during *T*?**

Let *D*, *E* and *F* denote the random variables that are 1 if the corresponding neuron fired, 0 if it didn’t. Our goal is to compute *P*(*D*|*F*). We are told the following:



We first compute *P*(*F*), noting that there are 4 possibilities altogether regarding the firing of neurons D and E. Thus:

= 0.3 × 0.15 + 0.8 × 0.15 + 0.1 × 0.35 + 0.5 × 0.35

= 0.045 + 0.12 + 0.035 + 0.175

= 0.375.

We note that , hence . We use Bayes rule to compute each of these two conditional probabilities:

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Adding these two, we get:

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**Problem 5.**

**We record simultaneously from 100 motor cortical neurons for a period *T* of length 10 ms, under one of two conditions: Rest (R) or Movement (M). In each of the two conditions, all neurons are Poisson and independent. Under condition R, all neurons fire at 5 spikes/s. Under condition M, all neurons fire at 100 spikes/s.**

**We denote by *Ui* the number of spikes of neuron *i* for any trial such that the condition during *T* was R, and by *Vi*  the number of spikes of neuron *i* for any trial such that the condition during *T* was M. We also denote:**

***U* = *U*1 + *U*2 +… + *U*100**

***V* = *V*1 + *V*2 +… + *V*100.**

**Is the distribution of *U* binomial? Is it approximately Poisson? Is it approximately normal? Specify the parameter(s) of the distribution for each positive answer.**

**Is the distribution of *V* binomial? Is it approximately Poisson? Is it approximately normal? Specify the parameter(s) of the distribution for each positive answer.**

The 100 neurons are recorded for 10 milliseconds, hence there are 100×10 = 1000 possible spike occurrences. Since all neurons are Poisson and independent, these 1000 events are all independent of each other. All occur with same probability *p*, where *p* = .005 under R and *p* = .1 under M. In other words, this is equivalent to a single experiment consisting of 1000 independent and identically distributed coin flips. Thus, the total number of spikes is *exactly* Binomial *N*, *p*, with *N* = 1000; *p* = .005 under R; *p* = .1 under M.

*U* is Binomial1000, 0.005.

*U* is, to a very good approximation, Poisson 5, since 1000 is a large number.

*U* is not approximately normally distributed, since 0.005 is too small, equivalently 5 is too small, for the normal approximation to be valid: the Poisson 5 distribution, peaking so close to 0, is skewed.

*V* is Binomial1000, 0.1

*U* is, to a very good approximation, Poisson 100, since 1000 is a large number.

*U* is also, to a very good approximation, Normal 100, 100, since 100 is a large number, hence Poisson100 approximates to Normal 100, 100.

**Bonus problem.**

**We denote by *Wi* the number of spikes of neuron *i* during *T* on trials where we don’t observe the condition but are told that R and M occur with probability 0.5 each. We also denote:**

***W* = *W*1 + *W*2 +… + *W*100**

**Are *W*1 and *W*2 independent?**

**Sketch the distribution of *W*. Is it binomial, approximately Poisson, approximately normal?**

We can think of this situation by imagining many trials, with roughly half of these trials in condition R and the rest in condition M. For the trials in the R condition, we have: *W*1 ~ Binomial10, 0.005 and *W* ~ Binomial1000, 0.005 ~ Poisson 5. For the trials in the M condition, we have: *W*1 ~ Binomial10, 0.1 and *W* ~ Binomial1000, 0.1 ~ Poisson 100. Since the trials distribute evenly between the two conditions, we get the following global distribution for *W*:



This distribution is a “mixture” of the two distributions, Poisson 5 and Poisson 100. It is a bimodal distribution, i.e., it has two distinct peaks. It clearly is neither a Binomial, nor a Poisson, nor a Normal. The first mode, the pdf of Poisson5, has standard deviation , whereas the second mode, the pdf of Poisson100, has standard deviation 10, i.e., it is about 5 times broader. Since *P*(M) = *P*(R) = .5, the areas under each of the two modes is 0.5, hence the narrower mode is higher.

Are the firings of neurons 1 and 2 are independent? Remember that, in the R condition, *W*1 ~ Binomial10, 0.005; *W*2 of course has the same distribution. In the M condition, *W*1 ~ Binomial10, 0.1, and *W*2 has the same distribution. So, for instance, P(*W*1 = 0 | R) = .99510 ≈ .95 whereas P(*W*1 = 0 | M) = .910 ≈ .35. Now say we observe *W*1 = 2. This makes it nearly impossible that the trial was in the R condition: the probability that it was M is very close to 1. Therefore, P(*W*2 = 0 | *W*1 = 2) ≈ .35. On the other hand, the unconditional probability is P(*W*2 = 0) ≈ 0.5×.35 + 0.5×.95 = .65. This proves that *W*1 and *W*2are dependent: observing the value of *W*1 sometimes gives away the condition, which affects the probability of *W*2 .

Note that it is only when the condition is random and *unknown* that *W*1 and *W*2 are dependent. If the condition is known to be either R or M, the firings of neurons 1 and 2 are independent. *W*1 and *W*2 are said to be “conditionally independent,” given the condition.