Chapter Four. Philosophy of Stochastic Optimal Control Analysis (SOC)

Abstract:

I discuss the philosophy of the stochastic optimal control techniques used in later chapters. The net worth of the real estate sector in chapter five, and of AIG in chapter six, evolve dynamically. In the first case, debt is incurred in period t to purchase assets whose return is uncertain, and must be repaid in period $t+1$ at an uncertain interest rate. In the second case, insurance is sold in period t and the claims in period $t+1$ are uncertain. What is the optimal debt in the first case and what are the optimal insurance liabilities in the second case? I discuss the strengths and limitations of alternative criterion functions, what should the firm or industry maximize? How should risk aversion be taken into account? Then I discuss the modeling of reasonable stochastic processes of the uncertain variables. Given the criterion function, each stochastic process implies a different quantitative, but similar qualitative, optimum debt/net worth or insurance liabilities/net worth. Using SOC I derive quantitative measures of an optimal and an excessive leverage, or what is an excessive risk that increases the probability of a crisis? The optimal capital requirement or leverage balances risk against expected growth and return.

4.1. Why use Stochastic Optimal Control?

Neither the markets, the IMF nor the Central Banks anticipated the US financial crises until it was too late. The collapse of the housing market led to solvency problems – where debt exceeded assets - and liquidity problems, where key firms in the shadow banking system could not obtain sufficient liquid resources to service their debts. In many cases the failures led to bailouts, discussed in subsequent chapters. The basic question is: What is an “excessive” debt ratio for two key sectors in the crisis: the real estate sector and a large insurer such as AIG, that is likely to lead to a crisis? I explain why SOC is a valuable tool to derive theoretically based, not empirical ad hoc, measures of an excess debt that will be an Early Warning Signals (EWS) of a debt crisis.

Net worth is assets less debt. The crucial variable is leverage or the debt/net worth ratio. Leverage = assets/net worth = $1 + \text{debt/net worth}$. The growth of net worth is affected by leverage. An increase in debt at time t to finance the purchase of assets increases the growth of net worth by the return on investment. The return on investment has two components. The first is the productivity of assets and the second is the capital gain on the assets. The resulting higher debt decreases the subsequent growth of net worth by the associated interest payments and servicing of the debt. As a result an
increase in leverage will increase expected growth of net worth if the return on investment exceeds the interest rate on the higher debt. The productivity of assets is observed, but the future capital gain and the interest rates are unknown when the investment decision is made.

The basic equation for the growth of net worth is crucial to understanding optimal risk management and evaluation of desirable policy. Two cases are discussed in detail in the chapters below. In chapter five, the focus is upon the real estate sector and the US financial crisis. In chapter six, the focus is upon AIG which sold Credit Default Swaps (CDS) – a form of insurance – against the financial derivatives that were based upon debt from the real estate market. In both cases, I use the SOC approach.

The first case concerns the real estate market. The mortgagors/households borrow from banks, which then package the mortgages and sell these packages to investors. Focus upon the net worth of the mortgagors/real estate industry whose interest payments become income to the buyers of mortgages and their derivatives. The purchase of mortgages assets, which are investments in housing, implies a rise in the ratio of debt/net worth of the real estate sector. The income of the financial structure and the shadow banking system ultimately was derived from the ability of the mortgagors to service their debts. The increase in debt would raise the growth of net worth of the real estate sector if the capital gain on the assets – the appreciation of house prices - plus the rate of return on the assets exceed the interest rate, equation (4.1).

\[
\text{Change in net worth} = \left[\text{assets (capital gain + productivity of assets)}\right] - \text{interest rate (debt)} - (\text{consumption or dividends}) \tag{4.1}
\]

The capital gains and the interest rates are described in chapter three. The huge capital gains in housing and low interest rates, during the period 2002-2006 led to a rise in the ratio debt/net worth of the real estate sector. The first term in brackets was large so the private housing sector increased its debt directly to banks and indirectly to foreign investors. The investment in housing seemed to be profitable because the debt could be refinanced/repaid from the recent capital gains – not from the productivity of assets/the “marginal product of capital”. Those capital gains were not sustainable. The reason is that
insofar as the capital gain exceeds the appropriate interest rate, the present value of the asset diverges to infinity.

There is a dynamic process. Borrowing to purchase assets in period t has a return. The value of assets changes, resulting from the capital gain (or loss). At time (t + dt) the unit has a greater debt that must be serviced. The higher debt raises the term [(interest rate) (debt)] in period (t + dt), which tends to lower the growth of net worth. Actions taken at time t have consequences at time (t+dt). The risk is that with the higher debt ratios, there would be a period when the capital gains fell below the interest rate – such as occurred in 2007-08. In fact the capital gain term and interest rate term are negatively correlated. When interest rates declined, house prices rose – the boom/bubble. When interest rates rose, capital gains declined. When the growth of housing prices declined, then the large negative term – the return less the interest rate - is multiplied by a large debt ratio and the net worth of the housing industry vanishes. The housing sector defaults on their loans. Bank failures followed the collapse of the housing market. The government then intervened to avert the collapse of the financial system. The question is what is an optimal debt/net worth ratio at any time? How should one judge if the debt and risk were too large? The techniques of SOC are used in chapter five to answer this question.

The second case concerning AIG is discussed in chapter six. The AIG Financial Products AIGFP subsidiary of AIG sold credit default swaps (CDS’s) involving three parties. Party 1 the obligor sells the “reference securities” to party 2. The latter are security houses, hedge funds that purchase protection from party 3, who are banks and insurance companies, which issue the protection. In many cases municipalities are required to carry insurance in order to market their bonds, or to obtain a high credit rating. The CDS are privately negotiated contracts that function in a similar manner to insurance contacts, but their payoff structure is closer to a put option. The CDS liability requires that the insurer pay, or put up more collateral, if the market value of the securities insured falls below the notional amount insured. Surplus is the difference between assets and liabilities less initial contributed capital. The change in surplus is the sum of several components, equation (4.2).

\[ \text{Change in surplus} = (\text{premium rate}) \text{ liabilities} + (\text{return} + \text{capital gain}) \text{ assets} \]
(claims on the company related to liabilities)  

The insurer incurs a liability at time $t$, and receives a premium for the amount insured. This increases assets and surplus at time $t$. The capital gain (or loss) and claims at time $t+dt$ are unknown at time $t$ when the insurance is sold. The surplus at time $t+dt$ rises as a result of the return, equal to the productivity plus the capital gain on assets. Surplus declines by the amount of claims at time $t+dt$ against liabilities incurred earlier.

The stochastic variables are the capital gains on assets and claims against the insurer. They are unknown at time $t$ when the liabilities are incurred. The control variable of the insurance company is its liabilities, the insurance policies sold such as CDS. The question is: what is the optimum amount of insurance liabilities? When should the firm and/or regulators decide that liabilities are too large?

In eqn. (4.1) the control variable is the debt of the real estate sector and in equation (4.2) it is the liabilities of the insurer. The capital gains, interest rate or claims in the cases above are subject to imperfectly known disturbances that may be taken as random. Stochastic optimal control theory attempts to deal with models in which random disturbances are very important. I deal with the case where the controller knows the state of the system at each time $t$. In deterministic systems, if one knows the initial conditions, the model and controls, then the paths of the variables can be predicted perfectly. For stochastic systems, it is completely different. There are many paths that the system may follow from the controls and initial data. In the stochastic case, the best system performance depends on the information available to the controller at each time $t$. The method of dynamic programming is generally used. The basic references for techniques used in this approach are: Fleming-Rishel, Ch. V, VI, Fleming-Soner, Fleming-Stein and Stein (2004)(2005).
4.2. Research Strategy

My research strategy parallels the views of Fischer Black and Emanuel Derman (2004, ch. 16). I paraphrase/quote their views. Certain economic quantities are so hard to estimate that Black called them ‘unobservables’. One unobservable is expected return. So much of finance deals with this quantity unquestioningly. Yet our estimates of expected return are so poor, that Black said that they are almost laughable.

The Value at Risk VaR uses statistical distributions to assign probabilities of the firm’s value into the future, and statistics are inevitably based upon the past. But VaR is too simplistic. The statistical distributions are not necessarily stationary. The past does not repeat itself. People learn from experience, which subsequently changes the distributions. We are ignorant of the true probabilities – the extreme tails of price distributions.

Black described the approach of the finance profession as follows: “In the end a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that the theory is correct and relevant”. This “group think” behavior led to serious consequences as described in chapters two and three.

The question is what is the “right model” to analyze problems posed in part (4.1) above? The Quants are physicists, mathematicians and computer scientists who do “financial engineering”. They apply models from physics to the world of finance. Derman has moved between the world of Wall St. Quants and academia. He wrote that models in finance differ from those in physics. “Models are only models, not the thing in itself. We cannot therefore expect them to be truly right. Models are better regarded as a collection of parallel thought universes you can explore. Each universe should be consistent, but the actual financial and human world, unlike the world of matter, is going to be infinitely more complex than any model we make of it. We are always trying to shoehorn the real world into one of the models to see how useful approximation it is”.

His strategy, which I follow, can be summarized. One must ask: Does the model give you a set of plausible variables to describe the world, and a set of relationships
between them that permits its analysis and investigation. The real question is how useful is the theory? “The right way to engage with a model is, like a fiction reader or a really great pretender, to temporarily suspend disbelief, and then push it as far as possible”.

For these reasons, I consider a set of plausible models and derive the optimal controls, debt ratio or liability ratio, using SOC. Call the optimal ratio $f^*(t)$, which varies quantitatively among models. I then consider an upper bound on the optimal control for both models and call it $f^{**}(t)$. I compare the actual debt ratio or liability ratio, called $f(t)$, with this upper bound. Thereby, I derive a measure of “excess debt” $\Psi(t) = [f(t) - f^{**}(t)]$, the actual less upper bound optimal. I explain that the probability of a debt crisis is directly related to $\Psi(t)$ the excess debt. On the basis of this measure of excess debt, I derive early warning signals of the US financial crisis of 2008 and the debt crises of the 1980s.

### 4.3. Modeling the uncertainty, the stochastic variables

In general, focus upon the change in net worth, equation (4.1). It can be written as equation (4.3).

$$dX(t) = X(t)[(1 + f(t)) \left(\frac{dP(t)}{P(t)} + \beta(t) \, dt\right) - i(t)f(t) - c \, dt] \quad (4.3)$$

$X(t) = $ Net worth, $f(t) = $ debt/net worth = $L(t)/X(t)$, $dP(t)/P(t) = $ capital gain or loss, $i(t) = $ interest rate, $(1+f(t)) = $ assets/net worth, $\beta(t) = $ productivity of capital. $C(t)/X(t) = c(t) = $ consumption/net worth = $c$ is taken as given.

The sources of uncertainty are $dP(t)/P(t)$ the capital gain or loss, and $i(t)$ the interest rate. How should the stochastic variables be modeled? I consider in the chapters below two models that differ in their assumptions about the stochastic processes.

In model I, assume that the price grows at a trend rate and there is a deviation from the trend. The deviation from trend is a stochastic variable that has two parts. The first part converges to zero and the second is random with a mean of zero and a positive finite variance. This deviation is an Ornstein Uhlenbeck ergodic mean reverting process. This implies a stochastic differential equation for the capital gains term. The interest rate has two parts, a constant mean plus a diffusion or random term, with a zero mean and
finite variance. The capital gains term and interest rate are negatively correlated. Declines in interest rates lead to capital gains, and rises in interest rates lead to capital losses.

Since the growth of net worth equation (4.3) will differ according to the model assumed, the derived optimal debt ratio will differ in the two models. I examine the empirical data to decide which assumption is to be preferred. However, the tests do not allow one to reject either hypothesis.

Model II considers the return capital, the capital gain plus the productivity of capital, \( b(t) \) to be a stochastic variable. There are two parts to the return \( b(t) \). One is the mean \( b \, dt \) and the second is a diffusion \( \sigma_b \, dw_b \). The interest rate has a mean \( i \, dt \) and a diffusion \( \sigma_i \, dw_i \).

\[
b(t) = dP(t)/P(t) + \beta(t) \, dt = b \, dt + \alpha_b \, dw_b
\]

\[
i(t) = i \, dt + \sigma_i \, dw_i
\]

Many reasonable models have the form of stochastic differential equation (4.5) for the change in net worth. The model based upon equations (4.4) is a special case.

\[
dx(t) = A(t, X(t)) \, dt + v(t)dt,
\]

where \( X(t), v(t) \) are scalar valued. The mean is the first term and the term \( v(t)dt \) represents the disturbances. The finance and mathematics literature assume that the disturbance has the following properties. (i) \( v(t) \) is stationary with independent increments, (ii) The increment \( v(t) - v(r), t > r \), is Gaussian with mean of zero and a variance of \( \sigma^2(t-r) \), (iii) \( v(t) \) is Gaussian with a zero mean. Condition (ii) follows from the others. In that case, the stochastic differential equation is (4.6)

\[
dx(t) = A(t, X(t)) \, dt + \sigma \, dw(t)
\]

where the term \( w(t) \) is a Brownian motion and \( dw(t) \) is formally white noise. The expectation is zero \( E(dw) = 0 \), and the variance is \( \sigma^2 dt \).

In fact, assumptions (i) and (ii) do not accurately describe the empirical data. The disturbance term \( v(t)dt \) is generally serially correlated and is not normal. However, one may consider \( v(t)dt \) as wideband noise (Fleming-Rishel, p. 126). That is let \( v(t) \) be some
stationary process with a zero mean and known variance-covariance $R(r) = E[v(t), v(t+r)]$. If autocorrelation $R(r)$ is nearly 0 except in a small interval near $r = 0$, then $v$ is called wide band noise. White noise corresponds to the case where $R$ is a constant a times a Dirac delta function. Then $v(t)dt$ is replaced by $\sigma dw$, where $\sigma$ is a constant matrix. Then the corresponding diffusion leads to an approximation of solution to (4.5).

There are other realistic possibilities for modeling $v(t)$, but they involve mathematical complexities and do not lead to equation (4.6) that characterizes the “Merton type” models. My strategy is to work with models I and II that lead to a stochastic differential equation (4.6) and allow for model uncertainty in deriving the optimal debt ratio.

The next challenge is how to model and estimate the mean term or drift in (4.6), $E[dX(t)] = A(t, X(t)) dt$. The Quants assumed that the distribution of the capital gains was constant, so they could obtain estimates of the drift. They had to use historical data, say 1980 – 2007. Ex-post, the basic distribution of capital gains changed drastically from 2000 to 2010 as shown in chapters below. How then should one know if the drift or distribution has changed? The issues of a time varying drift and detection of its change were discussed in chapter three.

An important constraint must be introduced. The Quants and market assumed a drift of the capital gain based upon recent data. This was a period of rapidly rising house prices where the mean was considerably higher than the mean rate of interest. Mortgagors refinanced their homes and incurred a debt equal to the current high house prices. The latter exceeded the value of the initial mortgage. They consumed the monetary difference between the value of the new loan and the value of the old loan. In the next period, insofar as the house price appreciated at a greater rate than the interest rate, they were able to repeat the process: refinance and consume the difference between the value of the home and the new debt. This was a free lunch. The mortgagor may have been a NINJA (no income, no job, no assets) and the loan originator did not worry for two reasons. First, the equity in the home was rising due to the capital gain. Second, the originator sold the mortgages that were then packaged and sold to investors. Free lunches cannot continue.
therefore constrain the optimization: the drift of the capital gain should not exceed the mean rate of interest.

4.4. Criterion function

Chapters 2 and 3 explained that the finance industry has been poor at risk management and that the regulators have been ineffective. I assume that there is a hypothetical or ideal optimizer. In the financial market case it is the real estate industry. In the insurance case it is AIG. In each case the hypothetical optimizer selects a debt ratio that maximizes the expectation of a concave function of net worth Max EU(X(T)) at a terminal date T, subject to dynamic stochastic processes. This is a benchmark for optimality.

The criterion function should satisfy several requirements. (i) Losses are penalized more than equal gains are rewarded. This requirement seems to have been ignored or reversed in the finance industry/Wall St. prior to the crisis. What the market seemed to optimize differed from the requirements of an “optimal” solution. (ii) Very low values of X(T) should be very severely penalized. This too was ignored due to the moral hazard where major firms anticipated that a friendly Treasury/Federal Reserve would bail out “too big to fail” firms. In modeling, the distribution of the stochastic variable in equation (4.6) is assumed to be normal, but in reality it is not. There may be a fat tail, where there is a low probability of very large losses. Large losses may not be likely, but they should get very heavy weights in the criterion/utility function.

Utility $U(X-Y)$ is equation (4.7), using a second order approximation around $U(X)$ for small values of stochastic variable $Y$. It has a zero mean and a variance of $\sigma^2$

$$U(X-Y) = U(X) - U'(X)Y + (1/2)U''(X)Y^2$$  \hspace{1cm} (4.7)

$$E(Y) = 0.$$

$$E(Y^2) = \sigma^2.$$

The expected utility is (4.10). Divide by $U'(X) \neq 0$ and assume that $E(Y) = 0$. Therefore Equation (4.11) follows.
\[ E[U(X - Y)] = U(X) - U'(X)E(Y) + (1/2)(U''(X))E(Y^2) \]  \hfill (4.10)

\[ \frac{E[U(X-Y)] - U(X)}{U'(X)} = (1/2)\left[\frac{U''(X)}{U'(X)}\right]E(Y^2) \]  \hfill (4.11)

The loss of utility due to the probability of losses depends upon risk aversion \(U''(x)/U'(x))\), and risk \(\sigma^2\). When \(U''(X)/U'(X)\) is significantly negative, even if \(E(Y)\) is very small the variance imposes a large penalty.

Risk Aversion = \(-U''(X)/U'(X)\) = the price one would pay for certainty. \hfill (4.12)

There are many criterion functions that satisfy these requirements. Only a few can be solved analytically in the optimization process. The others must be calculated numerically. In table 4.1 below I consider three widely used functions. The first is often called HARA. The second is a logarithmic function, which is a special case of HARA when \(\gamma = 0\). The third is an exponential function.

These functions are different. In the HARA and exponential cases, values of \(X\) approaching zero are not severely penalized \(U(0) = 0\). In the logarithmic case, low values of \(X(T)\) are very severely penalized. In the HARA and logarithmic cases, the risk aversion coefficient in column three depends negatively upon net worth. In the exponential case, it is independent of net worth.
Table 4.1. Alternative Criteria/utility functions

<table>
<thead>
<tr>
<th>Criterion function</th>
<th>( \lim X(T) ) as ( X \to 0 )</th>
<th>Risk aversion coefficient ( \frac{U''}{U'}(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{\gamma}X(T) ), ( 1 &gt; \gamma \neq 0 )</td>
<td>0</td>
<td>(-\frac{1-\gamma}{X})</td>
</tr>
<tr>
<td>( \ln X(T) )</td>
<td>(-\infty)</td>
<td>(-\frac{1}{X})</td>
</tr>
<tr>
<td>( 1 - \exp(-\alpha X(T)) )</td>
<td>0</td>
<td>(-\alpha)</td>
</tr>
</tbody>
</table>
There are several great advantages to using the logarithmic criterion function. First, in all cases one can derive the stochastic optimal control (SOC) – debt or liability ratio – by using dynamic programming. However, in the logarithmic case, the SOC can be derived directly by integrating the Ito equation. Second, a Mean-Variance interpretation can be given to the determination of the optimal control, as I show later in this chapter. Third, the risk coefficients either $\gamma$ or $\alpha$ are arbitrary. Specifically, suppose that the debtor has a low risk aversion, $(1-\gamma)$ is close to zero in the HARA case. His optimum debt ratio would be very high, close to risk neutrality. In the LTCM case, the market view was that LTCM “ran in front of bulldozers to pick up nickels”. The lender is unlikely to be risk neutral. He may have a very high risk aversion, and be reluctant to accommodate the borrower’s desired debt. So the choice of $\gamma$ or $\alpha$ is arbitrary. There is a way to cope with this problem (Stein, 2007) in a general equilibrium framework, where the rate of interest equilibrates the demand and supply of debt.

There is another criterion. It involves selecting a control that maximizes the expected minimum utility of net worth or consumption: $\max \ E[\min U(X)]$. This approach leads into a differential game against Nature (Fleming, 2005). This model is deterministic. It has a lot of merits, but the key parameters are arbitrary.

4.5. Methods of solution of stochastic optimal control problem

The mathematics of solving for the optimal controls, which maximize the criterion function subject to the stochastic differential equations and constraints, are derived and discussed in Fleming-Rishel, Ch. V, VI, and Fleming and Stein (2004). In the appendix, I explain in detail the method of solution of the logarithmic case where dynamic programming is not needed. The Ito equation is sufficient. In this section, I relate the derivation, in model II, to a Mean-Variance interpretation.

I use Model II as an example of SOC. Using (4.3) and (4.4), the stochastic differential equation for the change in net worth is eqn. (4.13).

$$dX(t) = X(t)[A(f(t)) \ dt + B(f(t))] dt$$

$$A(t) = (1+f(t))b - if(t) - c$$
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\[ B(f(t)) = (1 + f(t)) \sigma_b^2 dw_b - f(t) \sigma_i^2 dw_i \]

The first term \( A(f(t)) \) is deterministic, call it the Mean. The second term \( B(f(t)) \) contains the two stochastic variables where the \( dw_i \) and \( dw_b \) terms are Brownian Motion (BM). The two BM terms are correlated. In the realistic case, the correlation is negative \( E(dw_b dw_i) = \rho \) dt, where \( 0 > \rho > -1 \). A decline in the interest rate increases the demand for houses and raises the price. Similarly, a rise in the interest rate decreases the demand for houses and lowers the price.

The solution of equation (4.13) involves the stochastic calculus. The criterion is to maximize the expectation of the logarithm of net worth at a terminal date. The Ito equation for the change in logarithm of net worth \( d(ln X(t)) \) implies equation (4.14), the expectation of the change in net worth. Fleming and Rishel, and Øksendal are excellent sources that explain the derivation.

\[ E[d(ln X(t))] = \{ M[f(t)] - R[f(t)] \} \ dt \]

**Mean** \( M[f(t)] = [(1 + f(t))b - if(t) - c] \) \hspace{1cm} (4.14)

**Risk** \( R(f(t)) = (1/2)[(1 + f(t))^2 \sigma_b^2 + f(t)^2 \sigma_i^2 - 2f(t)(1 + f(t)) \sigma_b \sigma_i \rho] \)

The optimum debt/net worth \( f(t) \) maximizes the difference between the mean term \( M(f(t)) \) and the risk term \( R(f(t)) \). Figure 4.1 graphs the two terms against the debt ratio. The mean term is the straight line \( M(f) \) and the risk term \( R(f) \) is the quadratic term.

The mean line is the growth of net worth *if there were no risks*. The slope of this line is \( b - i \), the difference between mean return on assets and the mean interest rate. The intercept is \( b - c \), the mean return on assets less the ratio of consumption/net worth. Risk is a quadratic in the debt ratio. The slope depends upon \( \sigma_b \), \( \sigma_i \), \( \rho \), the variances of the two stochastic variables and \( \rho \) the correlation between the two stochastic variables.

The optimum debt ratio is \( f^*(t) \), where the difference between line \( M(f(t)) \) and quadratic \( R(f(t)) \) is maximal. It is

\[ f^* = \text{argmax}_f [M(f(t)) - R(f(t))] = [(b - i) - (\sigma_b^2 - \rho \sigma_b \sigma_i)] / \sigma_i^2 \] \hspace{1cm} (4.15)
\[ \sigma^2 = \sigma_i^2 + \sigma_b^2 - 2\rho \sigma_i \sigma_b \]

It is positively related to \((b - i)\) the mean return on capital less the mean interest rate and negatively related to the risk elements. The negative correlation between the two BM terms increases \(\sigma^2\) the total risk.

The stochastic case in this book contrasts with the deterministic case. In the deterministic case there is no \(R(f)\) term. The economics literature assumes that the return on assets \(b(f)\) is a concave function. So more debt should be incurred as long as the slope of the term \((b - i)\) is positive, that as long as \(M'(f)\) is positive. The value of the debt ratio where \((b(f) - i)f\) is maximal, where \(M'(f) = 0\), may be very large. In the deterministic case, there is no concern that there will be unexpected “bad” realizations of \((b, i)\) that will render the firm unable to service its debt – no matter how large.
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Figure 4.1 Expected growth of net worth, \( E[\ln X(t)] = M[f(t)] - R[f(t)] \).

Variable \( f(t) = \text{debt/net worth or liabilities/surplus} \).
4.6. Loss of expected growth from misspecification

No one knows what is the correct model, or even the exact values of the parameters such as b and i. Suppose that the true model is described by figure 4.1 and the implied optimum debt ratio is f*(t). What happens if values (b₁, i₁) or (b₂, i₂) were used in the optimization (4.15), whereas the true unknown values are (b, i). Figure 4.2 describes the change in expected growth, which is the difference between the Mean M(f) and Risk R(f) in figure 4.1.
Figure 4.2. The change in expected growth equal to the difference between the Mean \( M(f) \) and Risk \( R(f) \) in figure 4.1.
The expected growth of net worth $E[d \ln X(t)]$ is maximal at the optimal debt ratio $f^*(t)$. As the actual debt ratio rises above it, the expected growth declines. At debt ratio $f$-max, the expected growth is zero and above $f$-max it is negative. The actual growth is the expected value plus a random term.

One can never know what is the “true” model and hence what is the “true” optimal debt ratio $f^*(t)$. One chooses what seems to be the optimal debt ratio based upon what seems to be the correct model. There will always be a specification error at any time $t$ measured as the excess debt $\Psi(t) = (f(t) - f^*(t))$.

Suppose that the true unknown model is described by the curve in figure 4.2. When $f^*(t)$ is selected, the expected optimum growth is maximal at $W^*(t)$. Given limited information, the optimizing agents have faulty estimates of the key parameters $(b, i)$. Some select $f_1$ and others select $f_2$ as their optimum debt ratios. Then, for example, the expected growth would be $W_1 = W_2 < W^*$. The loss of expected growth from model misspecification is $W^* - W_1$ in the cases drawn. Eqn. (4.16), which is the integral of eq. (14),

$$E[\ln X^*(T) - \ln X(T)] = \int_T^T [W^*(t) - W(t)] dt = (1/2) \int_T^T \sigma^2 \Psi(t)^2 dt. \quad (4.16)$$

states precisely what is the loss of expected growth of net worth as a result of using a non-optimal control, debt ratio $f(t)$. It is proportional to the square of the excess debt $\Psi(t)^2 = [f(t) - f^*(t)]^2$. Consequently, one can evaluate how sensitive the results, the expected growth of net worth, are to model specifications.

The relation between the optimal debt ratio and the fundamentals, equation (4.15), is described in figure 4.3. The debt ratio is on the ordinate. The fundamental, the mean net return $(b - i)$, is on the abscissa. The slope of the line is $1/\sigma^2$.

If the net return were $N(1)$, the optimal debt ratio is $f^*(1)$, which corresponds to $f^*(t)$ in figure 4.2. As the actual debt ratio rises above $f^*(1)$, the expected growth declines, as seen in figure 4.2. Risk, stated in equation (4.14), rises. If the debt ratio were less than $f^*(1)$, both the expected growth and risk decline.
The empirical work in the chapters below estimates the excess debt ratio, equal to the actual ratio less optimal debt ratio. Since there is model uncertainty, I derive an upper bound for the optimal ratio called $f^{**}(t)$, and calculate an excess debt as $\Psi(t) = [f(t) - f^{**}(t)]$. Then, on the basis of figure 4.2, the probability of a debt crisis is directly related to $\Psi(t)$.

The upper bound $f^{**}(t)$, discussed in chapter five, assumes in Model I that the mean interest rate is equal to the trend rate of growth of the price and there is no deviation of the price from trend. In Model II the upper bound $f^{**}(t)$ assumes that $(b - i) = \beta(t)$ the productivity of capital.
Figure 4.3. Optimum debt ratio and loss of expected net worth from model mis-specification.
4.7. Insurance

4.7.1 Cramér-Lundberg

In the AIG case, the question is how to estimate the optimal amount of insurance to offer. I follow Volchan’s (2007) exposition and use his notation. He writes that the Cramér/Lundberg (C-L) model is still a landmark of non-life insurance mathematics. It is to actuarial science what the Black-Scholes-Merton model is to finance. The decision process concerns the amount of reserve capital that the insurance company should keep in order to absorb the impact of possible losses due to adverse changes in value of claims. The basic equation is (4.17), used in all variants.

\[
\text{Change in surplus} = \text{(premium rate) liabilities} + \text{(return + capital gain) assets} - \text{(claims on the company related to liabilities)} \\
\text{(4.17)}
\]

The C-L model and its variants are concerned with the ruin problem, to be discussed below. In equation (4.18), \( U(t) \) is the surplus, equal to initial capital \( u \), plus total premium income \( \Pi(t) \) less aggregate claims \( S(t) \). The deterministic total of premium income \( \Pi(t) \) in equation (4.19) is the product of premium rate \( c > 0 \) and time \( t \). The aggregate claims \( S(t) \) is the sum of \( X_k(t) \), amounts to be paid out in the period, equation (4.20). The number of claims arriving in the interval \( (0,t) \) is \( N(t) \).

\[
U(t) = u + \Pi(t) - S(t) \quad (4.18) \\
\Pi(t) = ct. \quad (4.19) \\
S(t) = \sum X_k(t), \quad k = 0,1, ... N(t). \quad (4.20)
\]

The crucial assumption of the C-L model is that the claim number process \( \{N(t)\} \) is a homogeneous Poisson process equation (4.21). This implies that the probability that \( k \) claims will arrive in the interval is Poisson distributed. Thus \( P(0, \lambda t) = e^{-\lambda t} \) is the probability of no claims arriving during the interval of length \( t \). The aggregate claims \( \{S(t)\} \) is a compound Poisson process described by:

\[
P\{N(t) = k\} = e^{-\lambda t} (\lambda t)^k / k! \quad (4.21)
\]

The focus is upon equation (4.22) the probability \( \Theta(t) \) of bankruptcy/insolvency, that the surplus at time \( t \) will be non-positive.
\[ \Theta(t) = P[U(t) - u < 0], \text{ for some } t > 0. \quad (4.22) \]

The solution has the following characteristics. Using the above equations the expected value of the surplus \( E[U(t) - u] \) is equation (4.23). The expected number of claims \( E[N(t)] = \lambda t \) and \( \mu \) is the average cost per claim. Total premiums are ct. The net profit condition is that \( (c - \lambda \mu) > 0 \), the premium must exceed the average total claim.

\[ E[U(t)] - u = ct - \mu E[N(t)] = t(c - \lambda \mu). \quad (4.23) \]

Even though the premium \( c > \lambda \mu \), the process of \( U(t) \) can attain negative values in some subperiods. So it is important to have at least an estimate of \( \Theta(t) = P[U(t) - u < 0] \) guaranteeing that it is sufficiently small.

The Cramér-Lundberg analysis is inadequate to evaluate the risk/return in the AIG case. First, the asset side of the equation for the change in surplus is ignored. The insurance company has assets against the liabilities that bring in income. This is the second term in equation (4.17). Second, the value of the claims against AIG are highly negatively correlated with the value of the assets. When the market value of the insured assets decline, AIG must either compensate the insurer for the difference or put up more collateral. Third, the assets that are insured by AIG are quite closely correlated with the assets in AIG’s portfolio.

4.7.2. Ruin Analysis

The C-L model is closely related to the Ruin problems in the probability literature. See Feller I, II. The ruin literature is concerned with the question, what is the probability that a firm that starts with a capital \( z \) will ultimately be ruined? Thereby one has a framework to determine “optimal” capital.

The structure of the Ruin model is simple. In each period, a firm with a capital of \( z \) can either gain \$k \) with probability \( 1 > p > 0 \), or lose \$k \) with probability \( q = 1 - p \). Say that the firm has borrowed \$a. If the capital falls to zero, the firm is bankrupt: the probability of ruin is then one. If the capital rises to \$a, the probability of ruin is zero. The firm repays its debt and the story can start over.

Let \( Q(z) \) be the probability that a firm with a initial capital of \( z \) will go bankrupt, before repaying its debt. The terminal conditions are: \( Q(0) = 1, Q(a) = 0 \). The solution of
the stochastic difference equation for $Q(z)$ is (4.24), where $r = q/p$ is key parameter.

$$Q(z) = \frac{r^{a/k} - r^{z/k}}{r^{a/k} - 1}, \quad r = \frac{q}{p} \neq 1 \quad (4.2;\;)$$

Figure 4.4 graphs this equation for two values $r > 1$, and $r < 1$. At $z = 0$, $Q(0) = 1$ and at $z = a$, $Q(a) = 0$. The probability of ultimate ruin is clearly greater for $r > 1$ than for $r < 1$. If the “target” probability of ultimate ruin is $y$, then “optimal” initial capital is $z = v$ when $r < 1$, and $z = w$ when $r > 1$. The greater the odds of a gain relative to a loss, the smaller is the required initial capital.
Figure 4.4. Probability of ultimate ruin

\[ q(z) = \text{prob. ultimate ruin} \]
4.7.3. The Stochastic Optimal Control Approach to Insurance

The SOC analysis explained above is applied to the AIG case. Unlike the ruin problem or the Cramér-Lundberg approach, the criterion does not focus solely upon the probability of ruin. The criterion function, the maximization of the expected logarithm of surplus $X(T)$ at a terminal date $T$, contains both the growth of net worth and risk aversion. The logarithm function implies that an infinite penalty is placed upon bankruptcy. In this manner the SOC analysis contains the best points of the other approaches.

The dynamics of the process to derive $E[\ln X(T)]$ is based upon equation (4.25). There are several possibilities for modeling the stochastic processes for the claims $C(t)$ and capital gains $dP(t)/P(t)$. Assets = $A(t)$. Insurance liabilities = $L(t)$.

$$dX(t) = [\pi L(t)dt + \beta A(t)dt] + [dP(t)/P(t))A(t)] - C(t).$$  \hspace{1cm} (4.25)

Assume, for simplicity, that the premium rate $\pi$ is given. At this premium rate there is an elastic demand for insurance and the insurer decides how much insurance $L(t)$ to offer at that rate. Income from premiums is $\pi L(t)dt$. The insurer has $A(t)$ of assets whose value is very closely related to the assets that it has insured, the CDO’s in particular. The return on these assets has two parts, a rate of return $\beta$ that is deterministic and a future capital gain or loss $(dP(t)/P(t))A(t)$ that is unknown. The most important stochastic variable is $C(t)$, the future claims against the insurer against the liabilities – insurance policies.

The modeling consists of specifying the stochastic processes on the capital gain and the claims. Let the claims $C(t)$ be described by stochastic differential equation (4.26). They are proportional to $L(t)$ the amount of insurance liabilities. Claims are the required payments to the insured holders of CDS, due to either defaults of the obligors or for collateral calls when the prices of the insured securities decline. The latter led to the downfall of AIG. The mean of claims $C(t)$ is $cL(t)dt$. The variance of the claims is $\sigma_c^2L^2(t) dt$. Brownian Motion term $dw_c$ has independent and stationary increments, with zero expectations, as assumed in the classical literature.
Chapter Four. Philosophy of Stochastic Optimal Control

\[ C(t) = [c \, dt + \sigma_c dw_c]L(t). \quad (4.26) \]
\[ E(dw_c) = 0, \quad E(dw_c)^2 = dt. \]

Stochastic differential equation (4.27) concerns the capital gain/loss term \( dP(t)/P(t) \). The time varying drift is \( a(t)dt \) and the diffusion is \( \sigma_p dw_p \). This is a general formulation. Models 1 and 2 of the asset price or capital gain, discussed above, imply different values of the drift \( a(t) \). The variance of the capital gain is \( \sigma_p^2 dt \). The \( dw_p \) term has independent and stationary increments.

\[ dP(t)/P(t) = a(t) \, dt + \sigma_p dw_p \quad (4.27) \]
\[ E(dw_p) = 0, \quad E(dw_p)^2 = dt. \]

The two Brownian Motion terms are expected to be negatively correlated, equation (4.28): correlation coefficient \(-1 \leq \rho < 0\). When there is a period with capital losses \( dP(t)/P(t) < 0 \), then claims against AIG including collateral calls are most likely to be high. This condition accurately described the period of the AIG crisis with CDS in chapter six.

\[ E(dw_p dw_c) = \rho \, dt. \quad -1 \leq \rho < 0 \quad (4.28) \]

Using equations (4.26) – (4.28) in (4.25), derive the stochastic differential equation (4.29) for the change in surplus \( dX(t) \). The first set of brackets contains the deterministic part and the second set contains the stochastic part.

\[ dX(t) = [\pi L(t) + \beta A(t) - cL(t) + A(t)a(t)] \, dt + [A(t)\sigma_p dw_p - L(t)\sigma_c dw_c]. \quad (4.29) \]

\( X = A - L = \) surplus, \( A = \) assets, \( L = \) insurance liabilities, \( \beta = \) return on assets, \( \pi = \) premium rate, \( C = \) claims, \( a(t) = \) drift capital gain, stochastic BM terms \( dw_p, dw_c \). Correlation between them is \( \rho < 0 \). Leverage \( = A/X = (1+f) \), debt ratio \( f = L/X \).

The SOC analysis follows very closely to what was done in parts 4.5 and 4.6 above. The optimal ratio of insurance liabilities/surplus \( f^*(t) \) is (4.30).

\[ f^*(t) = [(\pi - c) + (\beta + a(t))] - (\sigma_p^2 - \rho \sigma_p \sigma_c)]/(\sigma_p^2 + \sigma_c^2 - 2 \rho \sigma_p \sigma_c) \quad (4.30). \]

The first term in brackets in (4.30) for the optimum ratio is \( (\pi - c) \) the premium less the drift of the claims plus \( (\beta + a(t)) \) the sum of the rate of return on assets plus the time
varying drift of the capital gain. The second term and the denominator involve the risks of the capital gains and claims. The negative correlation between the capital gain and the claims increases total risk and reduces the optimal ratio of insurance liabilities/surplus.

Figures 4.1 – 4.3 describing the determination of the loss of growth from model misspecification and the relation between the net return and the optimal ratio apply equally here for the optimal insurance/surplus.

AIGFP relied on Gorton’s actuarial model that did not provide a tool for monitoring the CDO’s market value. Gorton’s model had determined with 99.85% confidence that the owners of super-senior tranches of the CDO’s insured by AIGFP would never suffer real economic losses. The company’s auditor PWC apparently was also not aware of the collateral requirements. PWC concluded that “...the risk of default on the [AIG] portfolio has been effectively removed as a result of a risk management perspective…”.

The SOC analysis leads to a very different way of viewing AIG risk than was used by Gorton and the company’s auditor PWC.

4.8. The Endogenous Changing Distributions

Underlying all dynamic optimizations is the characterization of the price distributions. The usual assumptions are that the system parameters are unchanging. Ren Cheng (Fidelity Investments) gave an insightful presentation of the limitations of the existing approaches used in financial modeling. I conclude this chapter with a discussion of Chang’s presentation.

The finance models assume that the means-variances-covariances are stable. Each entity in the optimization is considered in isolation. There is no consideration of a feedback among units, or the global effects of the collective action upon the distribution functions. The latter are assumed to be normal, an assumption justified by the Central Limit Theorem (CLT). We know a lot about the decision making process of an individual unit in portfolio optimization. But we are ignorant of how the units interact and thereby change the distributions. In chapters five and six, I explain how the interactions among
units in the shadow banking system magnified the collapse of housing prices upon the financial system. The whole was much greater than the sum of its parts.

The most critical requirement for the CLT, and for almost all of the distributions used in probability theory, is the independence. Normality of sample distributions follows from the CLT. With dependence and feedbacks, the tail gets longer and heavier. The nature of financial markets changes as a result of the collective actions.

I can formalize Cheng’s views by showing how the ruin theorems and Cramér-Lundberg model are changed by interactions. In the ruin model section 4.7.2, the object is to determine the probability $Q(z)$ of being ruined as a function of capital $z$. When $z = 0$ the probability $Q(0) = 1$. When $z = a > 0$, then $Q(a) = 0$. The firm repays the debt and the process restarts. At each time the probability of gaining $1$ is $1 > p > 0$ and $q = 1 – p$ is the probability of losing $1$. The basic equation for the change in probability is

$$Q(z) = Q(z-1)p + Q(z+1)q$$

(4.31).

Thereby an equation like (4.24) was derived in the Gambler’s Ruin problem.

Consider an alternative to either this model or to the Bernoulli process that underlies the Binomial distribution. In the Bernoulli process there is an idealized urn with $b$ black and $r$ red balls. Sampling is done with replacement. The probability of drawing a black ball at any time is constant at $p = b/(b + r)$. Insofar as the samples are independent, the Bernoulli case, the probability of drawing $k$ black balls in $n > k$ trials is $Pr(k;n,p) = C(n,k) p^k q^{n-k}$, where $C(n,k)$ is the factorial term.

An alternative is an Urn process. When a ball is drawn, then $c$ balls of the same color are added. Calculate the probability that at the $n$th trial there will be $k$ black balls. The contrast between the two approaches is easily seen in an example. What is the probability of drawing two black balls in two trials $Pr(BB)$? In the Bernoulli independence case, since $Pr(B) = p = b/(b+r)$. It follows that

$$Pr(BB) = Pr(B)Pr(B) = p^2.$$  

(4.32)

In the Urn process the probability of drawing two black balls is $Pr(BB) = Pr(B|B)Pr(B)$. The probability of drawing a black ball on the first trial is $Pr(B) = b/(b+r)$. 
The urn now contains \((b+c)\) black balls and \(r\) red balls. The conditional probability of drawing a black ball on the second trial is \(\Pr(B|B) = \frac{(b + c)}{(b + r + c)} > \frac{b}{b + r}\). Therefore the conditional probability is greater than the marginal probability. As a result
\[
\Pr(BB) = \left[\frac{(b + c)}{(b + r + c)}\right] \left[\frac{b}{b + r}\right].
\]
(4.33)

Think of a black ball as a success, and a red ball as a failure. Then the Urn model implies that: Success breeds another success and a failure breeds another failure. The conditional probability is greater than the marginal probability.

In the financial market, if prices rise then firms buy and stimulate further price rises. A black ball drawn leads to \(c\) black balls added. If a red ball is drawn this indicates a decline in price. When prices fall firms sell which stimulates further declines. One adds \(c\) red balls. Conditional probabilities change endogenously. The history of the financial crisis in chapters five and six is consistent with this alternative model. Actions by firms change the probability distributions.

I conclude by repeating Derman’s view. “Models are only models, not the thing in itself. We cannot therefore expect them to be truly right. Models are better regarded as a collection of parallel thought universes you can explore. Each universe should be consistent, but the actual financial and human world, unlike the world of matter, is going to be infinitely more complex than any model we make of it. We are always trying to shoehorn the real world into one of the models to see how useful approximation it is”.

Mathematical Appendix

I show how the optimal debt ratio is derived in the logarithmic case.

The stochastic differential equation is (A1).

\[ dX(t) = X(t)[(1 + f(t)) \left( \frac{dP(t)}{P(t)} + \beta(t) \, dt \right) - i(t)f(t) \, dt - c \, dt] \quad (A1) \]

\( X(t) \) = Net worth, \( f(t) = \) debt/net worth = \( L(t)/X(t) \), \( dP(t)/P(t) = \) capital gain or loss, \( i(t) = \) interest rate, \( 1+f(t) = \) assets/net worth, \( \beta(t) = \) productivity of capital. \( C(t)/X(t) = c \, (t) = \) consumption/net worth = \( c \) is taken as given.

Let the price evolve as

\[ dP(t) = P(t)(a(t) \, dt + \sigma_p \, dw_p) \quad (A2) \]

where drift \( a(t) \) will depend upon the model I or II. The interest rate evolves as

\[ i(t) = i \, dt + \sigma_i \, dw_i \quad (A3). \]

Substitute (A2) and (A3) in (A1) and derive (A4)

\[ dX(t) = X(t)\{[(1+f(t))(a(t)dt + \beta(t)dt) – (if(t)dt + c dt)] + [(1+f(t))\sigma_p dw_p – \sigma_i f(t)dw_i] \}

\[ dX(t) = X(t)M(f(t))dt + X(t)B(f(t)) \quad (A4) \]

\( M(f(t)) = [(1+f(t))(a(t)dt + \beta(t))dt – (if(t) + c)] \)

\( B(t) = [(1+f(t))\sigma_p dw_p – \sigma_i f(t)dw_i] \)

\[ B^2(f(t) = (1+f(t)^2 \sigma_p^2 dt + f(t)^2 \sigma_i^2 dt - 2f(t)(1+f(t))\sigma_i \sigma_p dw_p dw_i \]

\[ Risk = R(f(t)) = (1/2)[(1+f(t)^2 \sigma_b^2 + f(t)^2 \sigma_i^2 - 2f(t)(1+f(t))\sigma_b \sigma_i \rho)] \]

\( M(f(t)) \) contains the deterministic terms and \( B(f(t)) \) contains the stochastic terms. To solve for \( X(t) \) consider the change in ln \( X(t) \), equation (A5). This is based upon the Ito equation of the stochastic calculus. A great virtue of using the logarithmic criterion is that one does not need to use dynamic programming. The expectation of \( d \ln X(t) \) is (A6).

\[ d \ln X(t) = (1/X(t))dX(t) – (1/2X(t)^2)(dX(t))^2 \quad (A5) \]
\[ E[d(ln X(t))] = [M[f(t)] - R[f(t)] \, dt \]  

(A6)

The correlation \( \rho \, dt = E(\, dw_p \, dw_i) \) is negative, which increases risk. \((dt)^2 = 0, dw \, dt = 0.\)

The optimal debt ratio \( f^* \) maximizes the difference between the Mean and Risk curves in figure 4.1.

\[ f^* = \arg \max \left\{ M[f(t)] - R[f(t)] = \left[ a(t) + \beta(t) - i \right] - (\sigma_p^2 - \rho \sigma_i \sigma_p) / \sigma^2 \right\} \]

(A7)

\( \sigma^2 = \sigma_i^2 + \sigma_p^2 - 2 \rho \sigma_i \sigma_p \)

Model I.

Model I assumes that the price \( P(t) \) has a trend \( rt \) and a deviation \( y(t) \) from it, equation (A8). The deviation \( y(t) \) follows an Ornstein-Uhlenbeck ergodic mean reverting process equation (A9). Coefficient \( \alpha \) is positive and finite. The interest rate is the same as in Model II.

\[ P(t) = P(0) \, exp[rt + y(t)]. \]

(A8)

\[ dy(t) = -\alpha y(t) + \sigma_p \, dw_p \]

(A9)

Therefore using the stochastic calculus \( a(t) \) in model I is the first term in (A10),

\[ dP(t)/P(t) = (r - \alpha y(t) + (1/2) \sigma_p^2) \, dt + \sigma_p \, dw_p. \]

(A10)

Substitute (A10) in (A7) and derive (A11), the optimal debt ratio in model I.

\[ f^*(t) = \frac{(r - i) + \beta - \alpha y(t) - (1/2) \sigma_p^2 + \rho \sigma_i \sigma_p}{\sigma^2}. \]

(A11)

Consider \( \beta(t) \) as deterministic.
Model II.

In Model II, the price equation is (A12). The drift is \( a(t) \, dt = \pi \, dt \) and the diffusion is \( \sigma_p \, dw_p \).

\[
dP(t)/P(t) = \pi \, dt + \sigma_p \, dw_p. \quad (A12)
\]

The optimal debt ratio \( f^*(t) \) is (A13). Consider \( \beta(t) \) as deterministic.

\[
f^*(t) = \left[ (\pi + \beta(t) - i) - (\sigma_p^2 - \rho \sigma_i \sigma_p) \right] / \sigma^2. \quad (A13)
\]

\[
\sigma^2 = \sigma_i^2 + \sigma_p^2 - 2\rho \sigma_i \sigma_p
\]
References

Cheng, Ren (2011), Fat Tail Genesis, power point presentation, Fidelity Asset Allocation


Fleming, Wendell H. and R. Rishel (1975), Deterministic and Stochastic Optimal Control, Springer-Verlag


