AM 166 Problem Set 1 2/7/03

#B1 $H_0: \mu = 10$ $\sigma^2 = 25$

$H_a: \mu = 5$ $\sigma^2 = 25$ known $\mu_a < \mu_0$

RR: $\bar{X} < \mu_0 - z_\alpha \sigma/\sqrt{n}$ $\alpha = 0.025$

$z_\alpha = 1.96$

$\bar{X} < 10 - (1.96)(5)/\sqrt{n}$

For $\beta = 0.025$, require $P_{\mu=5}(RR^c) = 0.025$

$P_{\mu=5}(\bar{X} \geq 10 - (1.96)(5)/\sqrt{n}) = P_{\mu=5}(\bar{X} - 5 \geq 5 - (1.96)(5)/\sqrt{n})$

$= P_{\mu=5}\left(\frac{\bar{X}-5}{\sigma/\sqrt{n}} \geq \frac{5}{5/\sqrt{n}} - 1.96 = \sqrt{n} - 1.96\right)$

$= 0.025$

$\Rightarrow \sqrt{n} - 1.96 \geq 1.96$; $\sqrt{n} \geq 3.92$

$\Rightarrow n \geq 15.36 \Rightarrow \underline{\underline{n \geq 16}}$

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10.83a. Under H_0 the likelihood function is

$$L(\theta_0) = \frac{1}{(2\theta_0^3)^4}\left(\prod_{i=1}^{4} y_i^2\right) e^{-\sum y_i/\theta_0}.$$

Under H_a it is

$$L(\theta_a) = \frac{1}{(2\theta_a^3)^4}\left(\prod_{i=1}^{4} y_i^2\right) e^{-\sum y_i/\theta_a}.$$

Using Theorem 10.1, we obtain the most powerful critical region as

$$\frac{L(\theta_0)}{L(\theta_a)} = \frac{\theta_a^{12}}{\theta_0^{12}} \exp\left[-\sum y_i \left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\right] \le k$$

or

$$\exp\left[\sum y_i \left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\right] \le k\left(\frac{\theta_0}{\theta_a}\right)^{12}$$

or

$$-\sum y_i \left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right) \le \ln k\left(\frac{\theta_0}{\theta_a}\right)^{12}$$

or

$$-\sum y_i \le \frac{\ln k\left(\frac{\theta_0}{\theta_a}\right)^{12}}{\left(\frac{1}{\theta_0}\right)-\left(\frac{1}{\theta_a}\right)}$$

or

$$\sum y_i \ge -k'.$$

If H_0 is true, Y_i has a gamma distribution with $\alpha = 3$ and $\beta = \theta_0$, and $\frac{2Y_i}{\theta_0}$ has a χ^2 distribution with 6 degrees of freedom. Hence $2\left(\sum Y_i\right)\theta_0$ has a χ^2 distribution with 24 degrees of freedom. (Recall the method of moment-generating functions.) The critical region can be written as

$$\frac{2\sum Y_i}{\theta_0} \ge \frac{-2k'}{\theta_0} = k''$$

where k'' is chosen so that the test will have size α.

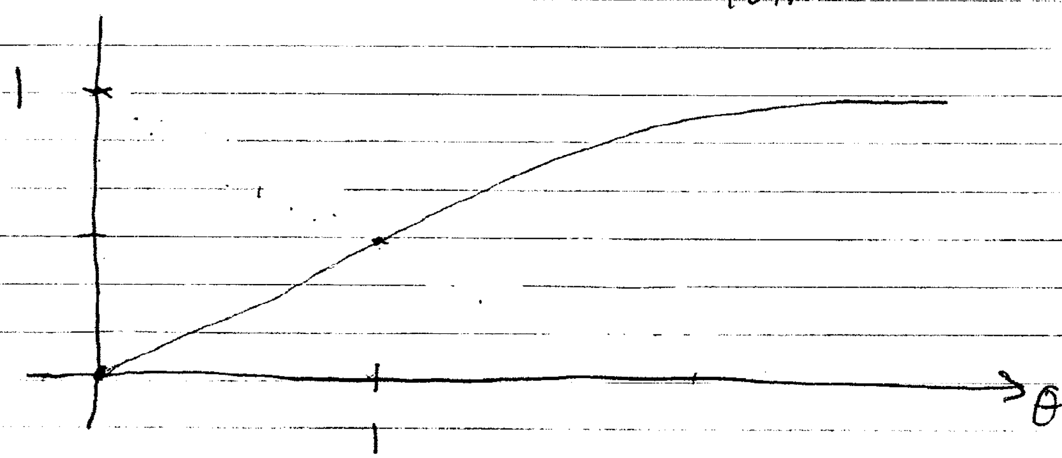
b. The choice of critical region did not depend on the particular value of θ_a but only upon the fact that $\theta_a > \theta_0$. Hence, for any $\theta > \theta_0$, the above critical region is most powerful and the test given in part **a** is uniformly most powerful for the alternative $\theta > \theta_0$.

$$f(y|\theta) = \begin{cases} \theta y^{\theta-1} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}, \quad \theta > 0$$

Y = Sample of size 1

a) Let $RR = \{Y > 0.5\}$

$$p(\theta) = P_\theta(Y > 0.5) = \int_{0.5}^{1} \theta y^{\theta-1} dy = y^\theta \Big|_{0.5}^{1}$$
$$= 1 - (0.5)^\theta$$



b) $H_0: \theta = 1$

$H_a: \theta > 1$ UMP test.

For now, fix $\theta = \theta_a > 1$ for an alternative hypothesis

$$L(\theta_0) = \theta_0 y^{\theta_0 - 1} \equiv 1 \;;\; L(\theta_a) = \theta_a y^{\theta_a - 1}$$

$$\lambda = \frac{L(\theta_0)}{L(\theta_a)} = \frac{1}{\theta_a} y^{-\theta_a + 1} \;;\; \lambda < k \iff y > c$$

$$\alpha = P_{\theta_0}(Y > c) \Rightarrow \int_c^1 1\, dy = \alpha \Rightarrow c = 1 - \alpha.$$

$RR: Y > 1-\alpha$ independent of $\theta_a > 1$. $\therefore$ UMP.

#85 $Y_1, \dots, Y_n$ iid, prob. fcn $p(y|\theta)$

$$p(y|\theta) = \begin{cases} \theta^2 & y=1 \\ 2\theta(1-\theta) & y=2 \\ (1-\theta)^2 & y=3 \end{cases} \qquad 0<\theta<1$$

N_i = # observations equal to i for $i=1,2,3$.

a) $$L(\theta) = \prod_{j=1}^{n} p(y_j|\theta) = [\theta^2]^{N_1}[2\theta(1-\theta)]^{N_2}[(1-\theta)^2]^{N_3}$$

$n = N_1+N_2+N_3$ $\qquad (N_3 = n-N_1-N_2)$

b) $H_0 : \theta=\theta_0$

$H_a : \theta=\theta_a$ $\qquad (\theta_a > \theta_0)$

$$\lambda = \frac{L(\theta_0)}{L(\theta_a)} = \left(\frac{\theta_0}{\theta_a}\right)^{2N_1}\left(\frac{\theta_0}{\theta_a}\right)^{N_2}\left(\frac{1-\theta_0}{1-\theta_a}\right)^{N_2}\left(\frac{1-\theta_0}{1-\theta_a}\right)^{2N_3}$$

substitute $N_3 = n-(N_1+N_2)$

$$\lambda = \left(\frac{\theta_0}{\theta_a}\right)^{2N_1+N_2}\left(\frac{1-\theta_0}{1-\theta_a}\right)^{N_2+2n-2(N_1+N_2)}$$

$$= \left(\frac{\theta_0}{\theta_a}\right)^{2N_1+N_2}\left(\frac{1-\theta_0}{1-\theta_a}\right)^{2n-(N_2+2N_1)} = f(2N_1+N_2)$$

∴ RR determined by $2N_1+N_2$.

$$\ln \lambda = (2N_1+N_2)\underbrace{\Big[\ln\Big(\underset{<1}{\frac{\theta_0}{\theta_a}}\Big) - \ln\Big(\underset{>1}{\frac{1-\theta_0}{1-\theta_a}}\Big)\Big]}_{\text{negative}} + \text{const.}$$

c)

$\therefore \lambda < k \iff (2N_1+N_2) > c$ some constant.

d) Test is UMP since RR is independent of value of θ_a.

10.87a. $L(\lambda) = \prod_{i=1}^{n} f\left(\frac{y_i}{\lambda}\right) = \frac{\lambda^{\Sigma y_i} e^{-n\lambda}}{\prod_{i=1}^{n} y_i!}$.

Then, by the Neymann–Pearson Lemma, the test that maximizes the power at θ_a has a rejection region determined by $\frac{L(\lambda_0)}{L(\lambda_1)} < k$ or

$$\frac{\left(\frac{\lambda_0^{\Sigma y_i} e^{-n\lambda_0}}{\prod_{i=1}^{n} y_i!}\right)}{\left(\frac{\lambda_a^{\Sigma y_i} e^{-n\lambda_a}}{\prod_{i=1}^{n} y_i!}\right)} < k$$

or

$$\left(\frac{\lambda_0}{\lambda_a}\right)^{\Sigma y_i} e^{n(\lambda_a - \lambda_0)} < k$$

or

$$\sum_{i=1}^{n} y_i \ln\left(\frac{\lambda_0}{\lambda_a}\right) + n(\lambda_a - \lambda_0) < \ln k$$

or

$$\sum_{i=1}^{n} y_i \ln\left(\frac{\lambda_0}{\lambda_a}\right) < \ln k - n(\lambda_a - \lambda_0)$$

or, since $\lambda_0 < \lambda_a$

$$\sum_{i=1}^{n} y_i > \frac{\ln k - n(\lambda_a - \lambda_0)}{\ln\left(\frac{\lambda_0}{\lambda_a}\right)}$$

or with $k' = \frac{\ln k - n(\lambda_a - \lambda_0)}{\ln\left(\frac{\lambda_0}{\lambda_a}\right)}$

we have $\sum_{i=1}^{n} Y_i > k'$.

b. $\sum_{i=1}^{n} Y_i \sim$ Poisson with mean $n\lambda$. Then, for a given α, the constant k' is the value such that $P\left(\sum_{i=1}^{n} Y_i > k' \text{ when } \lambda = \lambda_0\right) = \alpha$.

c. The form of the rejection region does not depend upon the particular value assigned to λ_a. Therefore, the test derived in part a is the uniformly most powerful for the composite hypothesis.

d. The form is similar to that in part **a.** We start with

$$\frac{L(\lambda_0)}{L(\lambda_a)} < k$$

or, since $\lambda_a < \lambda_0$,

$$\sum y_i \ln\left(\frac{\lambda_0}{\lambda_a}\right) < \ln k - n(\lambda_0 - \lambda_a)$$

$$\sum Y_i < \frac{\ln k - n(\lambda_0 - \lambda_a)}{\ln\left(\frac{\lambda_0}{\lambda_a}\right)}$$

or $\sum_{i=1}^{n} Y_i < k'$ with $k' = \frac{\ln k - n(\lambda_0 - \lambda_a)}{\ln\left(\frac{\lambda_0}{\lambda_a}\right)}$.

10.93 The null hypothesis specifies $\Omega_0 = \{\sigma^2\colon\ \sigma^2 = \sigma_0^2\}$, while $\Omega = \Omega_0 \cup \Omega_a = \{\sigma^2\colon\ \sigma^2 \geq \sigma_0^2\}$.

In the restricted space Ω_0, the likelihood function is

$$L(\Omega_0) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{1/2}\sigma_0} \exp\left[\frac{(y_i-\mu)^2}{2\sigma_0^2}\right]$$

The maximum likelihood estimate of μ is $\overline{Y}$, so that

$$L(\widehat{\Omega}_0) = \frac{1}{(2\pi)^{n/2}\sigma_0^n} \exp\left[\frac{\sum_{i=1}^{n}(y_i-\overline{y})^2}{2\sigma_0^2}\right].$$

Now consider the unrestricted space

$$L(\Omega) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left[\frac{\sum(y_i-\mu)^2}{2\sigma^2}\right].$$

The maximum likelihood estimate of μ is $\widehat{\mu} = \overline{Y}$, while

$$\widehat{\sigma}^2 = \max\left[\sigma_0^2, \widehat{\sigma}^2 = \frac{\sum(Y_i-\overline{Y})^2}{n}\right]$$

and

$$L(\widehat{\Omega}) = \frac{1}{(2\pi)^{n/2}\widehat{\sigma}^2} \exp\left[-\frac{\sum(y_i-\overline{y})^2}{2\widehat{\sigma}^2}\right].$$

The likelihood ratio statistic is

$$\lambda = \frac{L(\widehat{\Omega}_0)}{L(\widehat{\Omega})} = \left(\frac{\widehat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \exp\left[-\frac{\sum(y_i-\overline{y})^2}{2\sigma_0^2} + \frac{\sum(y_i-\overline{y})^2}{2\widehat{\sigma}^2}\right] = 1 \quad \text{if} \quad \widehat{\sigma} \leq \sigma_0$$

$$= \left[\frac{\sum(y_i-\overline{y})^2}{n\sigma_0^2}\right]^{n/2} \exp\left[-\frac{\sum(y_i-\overline{y})^2}{2\sigma_0^2}\right] e^{n/2} \quad \text{if} \quad \widehat{\sigma} > \sigma_0$$

Hence the rejection region $\lambda \leq k$ is equivalent to

$$g(\chi^2) = (\chi^2)^{n/2} e^{-\chi^2/2} n^{-n/2} e^{n/2} \leq k$$

where χ^2 is $\frac{(n-1)s^2}{\sigma_0^2}$, the χ^2 statistic given in Section 10.9.

Note that if $\widehat{\sigma} \leq \sigma_0$, $g(\chi^2) = 1$. Further, if $\widehat{\sigma} > \sigma_0$, $g(\chi^2)$ is a monotonically decreasing function of χ^2. Hence the region $\lambda \leq k$ is equivalent to $\chi^2 \geq c$, where c is determined so that the test has size α. A rough sketch of $g(\chi^2) = \lambda$ against χ^2 is shown in Figure 10.9.

Figure 10.9

10.94 The hypothesis of interest is $H_0\colon\ p_1 = p_2 = p_3 = p_4 = p$ against the alternative that at least one of these equalities is incorrect. In Ω, the likelihood function is

$$L(\Omega) = \prod_{i=1}^{4} \binom{200}{n_i} p_i^{n_i}(1-p_i)^{200-n_i}$$

and the maximum likelihood estimate of p_i is $\widehat{p}_i = \frac{n_i}{200}$.

In the restricted space Ω_0,

$$L(\Omega_0) = \prod_{i=1}^{4} \binom{200}{n_i} p_i^{n_i}(1-p_i)^{200-n_i} = Kp^{\sum n_i}(1-p)^{800-\sum n_i}$$

and

$$\ln L = \ln K + \sum n_i \ln p + (800 - \sum n_i)\ln(1-p).$$

One may easily verify that the maximum likelihood estimate of p is $\widehat{p} = \frac{\sum n_i}{800}$.

Then

$$\lambda = \frac{L(\widehat{\Omega}_0)}{L(\widehat{\Omega})} = \frac{\left(\frac{\sum n_i}{800}\right)^{\sum n_i}\left(\frac{800-\sum n_i}{800}\right)^{800-\sum n_i}}{\prod_{i=1}^{4}\left(\frac{n_i}{200}\right)^{n_i}\left(\frac{200-n_i}{200}\right)^{200-n_i}}$$

Since the n_i are large, Theorem 10.2 is applicable, and

$$-2\ln\lambda = -2\left\{\left(\sum n_i\right)\ln\left(\frac{\sum n_i}{800}\right) + \left(800-\sum n_i\right)\ln\left(1-\frac{\sum n_i}{800}\right) - \sum_{i=1}^{4}\left[n_i\ln\left(\frac{n_i}{200}\right) + (200-n_i)\ln\left(1-\frac{n_i}{200}\right)\right]\right\}$$

has an approximate χ^2 distribution with 3 degrees of freedom. For this exercise, $n_1 = 76$, $n_2 = 53$, $n_3 = 59$, $n_4 = 48$, and $\sum n_i = 236$, so that $-2\ln\lambda = -2(5.2676) = 10.54$. The rejection region, for $\alpha = .05$, will be $-2\ln\lambda > \chi_{.05,3} = 7.81$, and the null hypothesis is rejected. The fraction of voters favoring candidate A is not the same in all four wards.

10.114 From Section 10.3, we have

$$Z = \frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\sigma^2\left[\left(\frac{1}{n_1}\right)+\left(\frac{1}{n_2}\right)\right]}}.$$

As in Exercise 10.113, $\frac{\sum(X_i-\overline{X})^2+\sum(Y_i-\overline{Y})^2+\sum(W_i-\overline{W})^2}{\sigma^2}$ has a χ^2 distribution with $(n_1+n_2+n_3-3)$ degrees of freedom. The resulting t statistic is

$$T = \frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\times\frac{\sum(X_i-\overline{X})^2+\sum(Y_i-\overline{Y})^2+\sum(W_i-\overline{W})^2}{n_1+n_2+n_3-3}}}$$

$$= \frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\widehat{\sigma}^2}}$$

For the data given in this exercise, the test of hypothesis is as follows:

H_0: $\mu_1-\mu_2=0$ vs. H_a: $\mu_1-\mu_2\neq 0$

Calculate

$$\widehat{\sigma}^2 = \frac{36{,}950-\left[\frac{(600)^2}{10}\right]+25{,}850-\left[\frac{(500)^2}{10}\right]+49{,}900-\left[\frac{(700)^2}{10}\right]}{27} = 100$$

and

$$t = \frac{60-50}{\sqrt{\left(\frac{2}{10}\right)(100)}} = 2.326$$

The rejection region with $\alpha=.05$ and 27 degrees of freedom is $|t|>t_{.025,27}=2.052$. Hence the null hypothesis is rejected.