SAMPLE SOLUTIONS   MIDTERM II   AM 165

11/14/01

1. The annual rainfall in Cleveland, OH has roughly a normal distribution with mean 40.2 inches and standard deviation 8.4 inches. Suppose the annual rainfalls in different years are independent.

(a) What is the probability that next year's rainfall will exceed 44 inches?

(b) What is the probability that the yearly rainfalls in exactly three of the next seven years will exceed 44 inches?

(c) Over the next 10 years, in how many years would one expect the rainfall to be between 35 inches and 45 inches?

Let X denote the annual rainfall next year.

(a) $$P(X \geq 44) = P\left(\frac{X-40.2}{8.4} \geq \frac{44-40.2}{8.4}\right) = P(Z \geq .4524)$$

$$= .3264(-.0009) = \underline{\underline{0.3255}}$$

from table → (interpolating)

(b) Binomial, $n = 7$, $k = 3$, $p = 0.3255$

$$P(\text{exactly 3 years} \geq 44) = \binom{7}{3}(.3255)^3(.6745)^4$$

$$= \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}(.3255)^3(.6745)^4 = \underline{\underline{.2498}}$$

(c) Binomial, $n = 10$, $p = P(35 \leq X \leq 45) = P\left(\frac{35-40.2}{8.4} \leq \frac{X-40.2}{8.4} \leq \frac{45-40.2}{8.4}\right)$

$$p = P(-.619 \leq Z \leq .5714) = 1 - P(Z > .5714) - P(Z > .619)$$

$$= 1 - .2846 - .2673 = .4481.$$

(interpolated values) ∴ Expected number $= np = \underline{\underline{4.481}}$

2. If the waiting time for a taxi to arrive at a popular taxi stand is exponentially distributed with mean β minutes and there are n people in a queue waiting for taxis, then the wait for the last person in line is a random variable X having a gamma distribution with parameters $\alpha = n$ and β. X/n is proposed as an estimator of β.

(a) What is the expected value of X/n?

(b) How large would n have to be in order that the **relative** error of X/n be less than 0.2 with probability at least 0.75, i.e.,

$$P(|\frac{X}{n} - \beta| < 0.2\beta) \geq 0.75?$$

Be sure to justify your answer.

(a) Since X is gamma with $\alpha = n$, $EX = \alpha\beta = n\beta$.

$$E(\frac{X}{n}) = \frac{1}{n} EX = \frac{1}{n} n\beta = \underline{\underline{\beta}}$$

(b) $$Var(\frac{X}{n}) = \frac{1}{n^2} Var(X) = \frac{1}{n^2} \alpha\beta^2 = \frac{1}{n^2} n\beta^2 = \frac{\beta^2}{n}$$

Tchebysheff says:

$$P(|\frac{X}{n} - \beta| < 0.2\beta) \geq 1 - \frac{\beta^2/n}{(.2\beta)^2} = 1 - \frac{1}{.04n}$$

Choose n so $1 - \frac{1}{.04n} = .75$, $.25 = \frac{1}{.04n}$

$$n = \frac{1}{(.04)(.25)} = \underline{\underline{100}}$$

3. The apeed of a molecule in a uniform gas at equilibrium is a random variable X whose probability density function is given by $f(x) = ax^2 \exp(-bx^2)$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. (b is a physical constant, $b = m/2kT$, where m is the mass of the molecule, k is Boltzmann's constant, and T is temperature. The form of b doesn't concern us here, but what matters is that it is known.)

(a) Solve for a in terms of b. (*You should be able to do this without doing any nasty integration.*) Explain your work.

(b) Solve for EX in terms of a and b. (*The integration is quite tractable.*)

(a) We need to find a satisfying $1 = \int_0^\infty a x^2 e^{-bx^2} dx.$

The integral is the same form as the one for the variance of a zero mean Normal r.v.

$$\sigma^2 = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-x^2/(2\sigma^2)} dx$$

Substitute $b = \frac{1}{2\sigma^2}$ or $\sigma^2 = \frac{1}{2b}$

$$\frac{1}{2b} = \frac{1}{\sqrt{2\pi \frac{1}{2b}}} \int_{-\infty}^{\infty} x^2 e^{-bx^2} dx = \frac{2\sqrt{b}}{\sqrt{\pi}} \int_0^\infty x^2 e^{-bx^2} dx$$

$\leftarrow$ changing lower limit

$$\therefore \; 1 = \underline{\frac{4b^{3/2}}{\sqrt{\pi}} \int_0^\infty x^2 e^{-bx^2} dx} . \quad \therefore \; a = \underline{\underline{4b^{3/2}/\sqrt{\pi}}}$$

(b) $EX = \int_0^\infty x \cdot a x^2 e^{-bx^2} dx$ One can either integrate by parts or Substitute $u = x^2$, $du = 2x\,dx$

$$= \int_0^\infty \frac{1}{2} . a u e^{-bu} du = \frac{1}{2} \frac{a}{b} \int_0^\infty u \cdot b e^{-bu} du \quad (\text{Exponential mean}, \mu = \tfrac{1}{b})$$

$$= \frac{1}{2} \frac{a}{b} \cdot \frac{1}{b} = \underline{\underline{\frac{a}{2b^2}}}$$

4. Let X_1, X_2 and X_3 be iid observations from a distribution with mean value μ and variance σ^2. Suppose μ is unknown. The experimentalist who collected these observations speculates that the most recent data might be "more relevant" to the estimation of μ than the earlier observations and proposes

$$\hat{\mu} = \frac{1}{6}X_1 + \frac{2}{6}X_2 + \frac{3}{6}X_3$$

to estimate μ.

(a) Find the expected value and variance of $\hat{\mu}$.

(b) Compare $\hat{\mu}$ to the sample mean $\bar{X}$ and explain which of these two estimators is better.

(linearity)

(a) $E\hat{\mu} = \frac{1}{6}EX_1 + \frac{2}{6}EX_2 + \frac{3}{6}EX_3 = \frac{\mu}{6} + \frac{\mu}{3} + \frac{\mu}{2} = \underline{\underline{\mu}}$

$Var(\hat{\mu}) = Var\left(\frac{X_1}{6}\right) + Var\left(\frac{2X_2}{6}\right) + Var\left(\frac{3X_3}{6}\right)$ (independence)

$= \frac{\sigma^2}{36} + \frac{4\sigma^2}{36} + \frac{9\sigma^2}{36} = \frac{14}{36}\sigma^2 = \underline{\underline{\frac{7}{18}\sigma^2}}.$

(b) $E(\bar{X}) = E\left(\frac{1}{3}\sum_{i=1}^{3} X_i\right) = \frac{1}{3}\sum_{i=1}^{3} EX_i = \frac{1}{3}\cdot 3\mu = \underline{\underline{\mu}}$ Same mean as $\hat{\mu}$

$Var(\bar{X}) = Var\left(\frac{1}{3}\sum_{i=1}^{3} X_i\right) = \frac{1}{9}\sum_{i=1}^{3} Var(X_i) = \frac{1}{9}\cdot 3\sigma^2 = \underline{\underline{\frac{\sigma^2}{3}}}$

$Var(\bar{X}) = \frac{\sigma^2}{3} = \frac{6}{18}\sigma^2 < \frac{7}{18}\sigma^2 = Var(\hat{\mu})$

Smaller variance $\Rightarrow$ $\bar{X}$ is better than $\hat{\mu}$.

5. The randon variables X and Y have joint density function

$$f(x,y) = 12xy(1-x) \qquad 0 < x < 1, \quad 0 < y < 1$$

and equal to 0 otherwise.

(a) Are X and Y independent? Explain.

(b) Determine EX and $\mathrm{Var}(X)$.

(c) Determine $\mathrm{Cov}(X,Y)$.

(a) $f(x,y) = 12\,xy(1-x) = \big(12x(1-x)\big)\cdot(y) \qquad 0<x<1,\ 0<y<1$

$= g(x)\cdot h(y)$

Since f factors, <u>X and Y are independent</u> (See Theorem 5.5, text)

(b) Look at f again to figure out the density of X & Y separately.

$f_2(y) = 2y$ is a density on $[0,1]$ (It's integral is 1.)

So $f_1(x)$ must be $\underline{\underline{6x(1-x)}} = f_1(x)$ on $[0,1]$

Check: $\int_0^1 6x(1-x)\,dx = 3x^2 - 2x^3\Big|_0^1 = 1$ ✓

$\therefore\ EX = \int_0^1 x\cdot 6x(1-x)\,dx = \int_0^1 6x^2 - 6x^3\,dx = 2x^3 - \frac{3}{2}x^4\Big|_0^1 = \underline{\underline{\frac{1}{2}}}$

$EX^2 = \int_0^1 x^2\,6x(1-x)\,dx = \int_0^1 6x^3 - 6x^4\,dx = \frac{3}{2}x^4 - \frac{6}{5}x^5\Big|_0^1 = \underline{\underline{0.3}}$

$\mathrm{Var}(X) = E(X^2) - (EX)^2 = 0.3 - 0.25 = \underline{\underline{0.05}}$

(c) $\mathrm{Cov}(X,Y) = 0$ since they are independent.