5.53. Independent, since $f(y_1, y_2)$ can be factored.

Problem Set 8

**5.59** **a.** Because of the independence of $Y_1$ and $Y_2$,

$$f(y_1, y_2) = f(y_1)f(y_2) = \frac{1}{9}e^{-(y_1+y_2)/3}$$

for $y_1 > 0, y_2 > 0$.

**b.** The probability of interest is the shaded area in Figure 5.11. Hence

$$P(Y_1 + Y_2 \le\ ) = \int_0^1 \int_0^{1-y_2} f(y_1, y_2)\, dy_1\, dy_2$$

$$= \int_0^1 \left[1 - e^{-(1-y_2)/3}\right] \frac{1}{3} e^{-y_2/3}\, dy_2$$

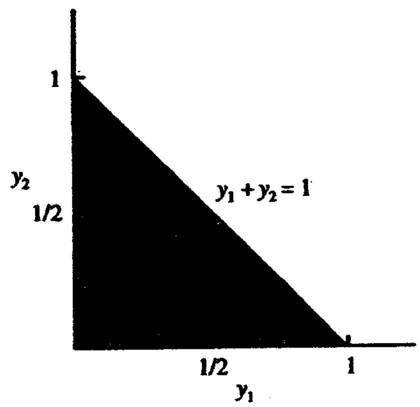


Figure 5.11

$$= \int_0^1 \left(\frac{1}{3}e^{-y_2/3} - \frac{1}{3}e^{-1/3}\right) dy_2 - e^{-y_2/3}\Big]_0^1 - \frac{1}{3}e^{-1/3} = 1 - \frac{4}{3}e^{-1/3}$$

**5.61** Let $Y_1$ = calling time to the switchboard of the first call, then

$$f(y_1) = 1; \qquad 0 \le y_1 \le 1$$

$Y_2$ = calling time to the switchboard of the second call, then

$$f(y_2) = 1; \qquad 0 \le y_2 \le 1$$

Then we have $f(y_1, y_2) = 1$.

**a.** $P\left(Y_1 \le \frac{1}{2}, Y_2 \le \frac{1}{2}\right) = \left(\int_0^{1/2} 1\, dy_1\right)\left(\int_0^{1/2} 1\, dy_2\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

(since $Y_1$ and $Y_2$ are independent).

**b.** Note that 5 minutes $= \frac{1}{12}$ of 1 hour.

$$P\left(|Y_1 = Y_2| < \frac{1}{12}\right) = \int_0^{1/12} \int_0^{y_1+(1/12)} dy^2\, dy_1 + \int_{1/12}^{11/12} \int_{y_1-(1/12)}^{y_1+(1/12)} dy_2\, dy_1$$

$$+ \int_{11/12}^{1} \left[\left(\frac{13}{12}\right) - y_1\right] dy_1$$

$$= \left(\frac{y_1^2}{2}\right) + \frac{y_1}{12}\Big]_0^{1/12} + \frac{2y_1}{12}\Big]_{1/12}^{11/12} + \left(\frac{13y_1}{12}\right) - \frac{y_1^2}{2}\Big]_{11/12}^{1} = \frac{46}{288} = \frac{23}{144}.$$

**5.64** Refer to Exercises [redacted] 5.22. Recall $f_1(y_1) = 2y_1$ for $0 \le y_1 \le 1$.

**a.** $E(Y_1) = \int_0^1 2y_1 y_1 \, dy_1 = \int_0^1 2y_1^2 \, dy_1 = \frac{2}{3}$

**b.** $E(Y_1^2) = \int_0^1 2y_1^3 \, dy_1 = \frac{1}{2}$ so that $V(Y_1) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$.

**c.** Since $E(Y_2) = \int_0^1 2y_2^2 \, dy_2 = \frac{2}{3}$, $E(Y_1 - Y_2) = 0$.

In all the above, we use the following computation (which is also required for 5.22):

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 = \int_0^1 4y_1 y_2 \, dy_2$$

$$= (4y_1) \left. \frac{y^2}{2} \right|_0^1 = 2y_1 \qquad \text{for } 0 \le y_1 \le 1.$$

which provides the marginal density function for $Y_1$.

**5.69** Since $Y_1$ and $Y_2$ are independent, with $f_1(y_1) = \frac{1}{4} y_1 e^{-y_1/2}$ and $f_2(y_2) = \frac{1}{2} e^{-y_2/2}$,

$$E\left(\frac{Y_2}{Y_1}\right) = E\left(\frac{1}{Y_1}\right) E(Y_2) = \frac{1}{8} \int_0^\infty e^{-y_1/2}\, dy_1 \int_0^\infty y_2 e^{-y_2/2}\, dy_2$$

$$= \frac{1}{8}\left[-2e^{-y_1/2}\right]_0^\infty (4) = \frac{1}{4}(4) = 1$$

since the second integral is the variable factor of a gamma distribution with $\alpha = 2$, $\beta = 2$ and integrates to $\Gamma(2)2^2 = 4$.

**5.70** The marginal distribution of $Y_1$ is $f_1(y_1) = 1$ for $0 \le y_1 \le 1$, so that $E(Y_1) = \int_0^1 y_1 \, dy_1 = \frac{1}{2}$. Using the joint distribution of $Y_1$ and $Y_2$, we obtain

$$E(Y_2) = \int_0^1 \int_0^{y_1} \frac{y_2}{y_1} \, dy_2 \, dy_1 = \int_0^1 \frac{y_1^2}{2y_1} \, dy_1 = \left.\frac{y_1^2}{4}\right]_0^1 = \frac{1}{4}$$

Thus, $E(Y_1 - Y_2) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

**5.77** From Exercise 5.64, $E(Y_1) = E(Y_2) = \frac{2}{3}$. Then

$$E(Y_1 Y_2) = \int_0^1 \int_0^1 4y_1^2 y_2^2 \, dy_1 \, dy_2 = \int_0^1 \frac{4}{3} y_2^2 \, dy_2 = \frac{4}{9}$$

$$\text{Cov}(Y_1, Y_2) = \frac{4}{9} - \frac{4}{9} = 0.$$

No, this is not surprising since $Y_1$ and $Y_2$ are independent.

**5.80** $\text{Cov}(U_1, U_2) = E\{(Y_1 + Y_2)(Y_1 - Y_2 - [E(Y_1) + E(Y_2)][E(Y_1) - E(Y_2)]\}$

$$= E(Y_1Y_2) + E(Y_1^2) - E(Y_1Y_2) - E(Y_2^2) - [E(Y_1)]^2 - E(Y_1)E(Y_2) + E(Y_1)E(Y_2) + [E(Y_2)]^2$$

$$= \sigma_1^2 - \sigma_2^2$$

Now

$$V(U_1) = E[U_1^2] - [E(U_1)]^2$$
$$= E(Y_1^2 + 2Y_1Y_2 + Y_2^2) - [(EY_1)^2 + 2(EY_1)(EY_2) + (EY_2)^2]$$
$$= V(Y_1) + V(Y_2) + 2[E(Y_1Y_2) - (EY_1)EY_2)]$$
$$= \sigma_1^2 + \sigma_2^2 + 2\text{Cov}(Y_1, Y_2)$$
$$= \sigma_1^2 + \sigma_2^2$$

since $Y_1$ and $Y_2$ are uncorrelated. A similar calculation yields $V(U_2) = \sigma_1^2 + \sigma_2^2$. Hence

$$\rho = \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{(\sigma_1^2 + \sigma_2^2)(\sigma_1^2 + \sigma_2^2)}} = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

**5.81** The marginal distributions for $Y_1$ and $Y_2$ are shown in the accompanying tables.

| $y_1$ | $p_1(y_1)$ |
|---|---|
| $-1$ | $\frac{1}{3}$ |
| 0 | $\frac{1}{3}$ |
| 1 | $\frac{1}{3}$ |

| $y_2$ | $p_2(y_2)$ |
|---|---|
| 0 | $\frac{2}{3}$ |
| 1 | $\frac{1}{3}$ |

Since, for example, $p(-1, 0) \neq p(-1)p(0)$, $Y_1$ and $Y_2$ are not independent. However,

$$E(Y_1) = -1\left(\tfrac{1}{3}\right) + 0\left(\tfrac{1}{3}\right) + 1\left(\tfrac{1}{3}\right) = 0$$
$$E(Y_1Y_2) = (-1)(0)\left(\tfrac{1}{3}\right) + (0)(1)\left(\tfrac{1}{3}\right) + (1)(0)\left(\tfrac{1}{3}\right) = 0$$

so that $\text{Cov}(Y_1, Y_2) = 0$.

**5.87** Refer to Theorem 5.12.

$$E(3Y_1 + 4Y_2 - 6Y_3) = 3(2) + 4(-1) - 6(4) = -22$$
$$V(3Y_1 + 4Y_2 - 6Y_3) = 9(4) + 16(6) + 36(8) + (2)(3)(4)(1) + (2)(3)(-6)(1) + 2(4)(-6)(0) = 480$$

5.92 [redacted] $f_1(y_1) = y_1 e^{-y_1}$, which is a gamma distribution with $\alpha = 2$, $\beta = 1$. Hence $E(Y_1) = 2(1) = 2$ and $V(Y_1) = \alpha\beta^2 = 2$.

$$f_2(y_2) = \int_{y_2}^{\infty} e^{-y_1}\, dy_1 = -e^{-y_1}\Big]_{y_2}^{\infty} = e^{-y_2}$$

which has a gamma distribution with $\alpha = \beta = 1$. Hence $E(Y_2) = V(Y_2) = 1$. Finally,

$$E(Y_1Y_2) = \int_0^{\infty}\int_0^{y_1} y_1 y_2 e^{-y_1}\, dy_2\, dy_1 = \int_0^{\infty} \frac{y_1^3}{2} e^{-y_1}\, dy_1 = \frac{\Gamma(4)1^4}{2} = 3$$

$$\text{Cov}(Y_1, Y_2) = 3 - (1)(2) = 1 \qquad E(Y_1 - Y_2) = 2 - 1 = 1$$

$$V(Y_1 - Y_2) = 2 + 1 - 2(1) = 1$$

Note: $f_1(y_1) = \int_0^{y_1} e^{-y_1}\, dy_2 = y_1 e^{-y_1}$

It is unlikely that a customer would spend more than 4 minutes at the service window because this is 3 standard deviations above the mean.

5.98 Let $Y = X_1 + X_2$, the total sustained load on the footing.

**a.** Since $X_1$ and $X_2$ have gamma distributions, $E(X_1) = \alpha_1\beta_1 = 100$ and $E(X_2) = \alpha_2\beta_2 = 40$. Also, $V(X_1) = \alpha_1\beta_1^2 = 200$ and $V(X_2) = \alpha_2\beta_2^2 = 80$. Thu

$$E(Y) = E(X_1 + X_2) = 100 + 40 = 140.$$

Since $X_1$ and $X_2$ are independent,

$$V(Y) = V(X_1 + X_2) = V(X_1) + V(X_2) = 200 + 80 = 280.$$

**b.** Consider Tchebysheff's theorem with $k = 4$, $P(|Y - \mu| \geq 4\sigma) \leq \frac{1}{16}$. The corresponding interval is $\left(140 - 4\sqrt{280}, 140 + 4\sqrt{280}\right)$ or (73.07, 206.93). Thu the sustained load will exceed 206.93 with a probability less than $\frac{1}{16}$.