PROBLEM SET 7

4.69 Let Y = magnitude of earthquake. Y is exponential with $\beta = 2.4$.

a. $P(Y > 3) = \int_3^{\infty} \left(\frac{1}{2.4}\right) e^{-y/2.4}\, dy = -e^{-y/2.4}\Big]_3^{\infty} = e^{-3/2.4} = .2865$

b. $P(2 < Y < 3) = \int_2^{3} \left(\frac{1}{2.4}\right) e^{-y/2.4}\, dy = -e^{-y/2.4}\Big]_2^{3} = .1481$

4.71 **a.** Let Y = demand for water. Y is exponential with $\beta = 100$.

$$P(Y > 200) = \int_{200}^{\infty} \left(\frac{1}{100}\right) e^{-y/100}\, dy = -e^{-y/100}\Big]_{200}^{\infty} = .1353$$

b. Let C = capacity. $P(Y > c) = \int_c^{\infty} \left(\frac{1}{100}\right) e^{-y/100}\, dy = -e^{-y/100}\Big]_c^{\infty} = e^{-c/100} = .01$.

so that $c = -100\, ln\,(0.01) = 460.52$ cfs.

4.73 **a.** $P(Y \le 31) = \int_0^{31} \left(\frac{1}{44}\right) e^{-y/44}\, dy = 1 - e^{-31/44} = .5057$

b. $V(Y) = \beta^2 = (44)^2 = 1936$

4.74 **a.** $P(Y > 9) = \int_9^{\infty} \left(\frac{1}{3.6}\right) e^{-y/3.6}\, dy = -e^{-y/3.6}\Big]_9^{\infty} = e^{-9/3.6} = .0821$.

b. $P(Y > 9) = \int_9^{\infty} \left(\frac{1}{2.5}\right) e^{-y/2.5}\, dy = -e^{-y/2.5}\Big]_9^{\infty} = e^{-9/2.5} = .0273$.

4.87 From Theorem 4.8, a gamma-type random variable with $\alpha = 3$ and $\beta = 2$ has $E(Y) = \alpha\beta = 6$ and $E(Y^2) = V(Y) + (E(Y))^2 = \alpha(\alpha+1)\beta^2 = 3(4)(4) = 48$. Hence

$$E(L) = 30E(Y) + 2E(Y^2) = 30(6) + 2(48) = 276$$

Since

$V(L) = E(L^2) - [E(L)]^2 = E(L^2) - 76{,}176 = E(900Y^2 + 120Y^3 + 4Y^4) - 76{,}176$

we need the third and fourth moments about the origin.

$$E(Y^3) = \int_0^{\infty} \frac{y^5 e^{-y/2}}{\Gamma(3)2^3}\, dy = \frac{\Gamma(6)2^3}{\Gamma(3)} \int_0^{\infty} \frac{y^5 e^{-y/2}}{\Gamma(6)2^6}\, dy = \frac{5!2^3}{2!} = 480$$

$$E(Y^4) = \int_0^{\infty} \frac{y^6 e^{-y/2}}{\Gamma(3)2^3}\, dy = \frac{\Gamma(7)2^4}{\Gamma(3)} \int_0^{\infty} \frac{y^6 e^{-y/2}}{\Gamma(7)2^7}\, dy = \frac{6!2^4}{2!} = 5760$$

Then $V(L) = 900(48) + 120(480) + 4(5760) - 76{,}176 = 47{,}664$.

4.88 Since Y has a gamma distribution with $\alpha = 3$, $\beta = \frac{1}{2}$,

$$E(Y) = \alpha\beta = \frac{3}{2} \qquad \text{and} \qquad V(Y) = \alpha\beta^2 = 3\left(\frac{1}{4}\right) = \frac{3}{4}$$

2

4.104 **a.** Refer to Example 4.13 with $\beta = \theta$, $\alpha = 1$. The mgf for y is $m(t) = \frac{1}{(1-\beta t)^\alpha}$ $= \frac{1}{(1-\theta t)}$.

b. Differentiating with respect to t,

$E(Y) = m'(0) = \left.\frac{\theta}{(1-\theta t)^2}\right|_{t=0} = \theta$ and $E(Y^2) = m''(0) = \left.\frac{2\theta^2}{(1-\theta t)^3}\right|_{t=0} = 2\theta^2$

so that $V(Y) = 2\theta^2 - \theta^2 = \theta^2$.

4.106 **a.** From Example 4.16, $m_{Y-\mu}(t) = E\left[e^{t(Y-\mu)}\right] = e^{t^2\sigma^2/2}$. Since

$$E\left[e^{t(Y-\mu)}\right] = E\left(e^{tY-\mu t}\right) = e^{-\mu t}E\left(e^{tY}\right)$$

we have $e^{t^2\sigma^2/2} = e^{-\mu t}E\left(e^{tY}\right)$, so the mgf for Y must be $m_Y(t) = E\left(e^{tY}\right)$ $= e^{\mu t + (t^2\sigma^2/2)}$.

b. Differentiating with respect to t,

$$EY = m'(0) = \left.(\mu + t\sigma^2)\, e^{\mu t + (t^2\sigma^2/2)}\right|_{t=0} = \mu$$

Then $V(Y) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$.

4.114 The interval must include 90% of all mileage on tires he sells. Using Tchebysheff's theorem, we must have

$$P(|Y-\mu| \le k\sigma) \ge .90 = 1 - \tfrac{1}{k^2}.$$

Then $k = \sqrt{\frac{1}{.10}} = \sqrt{10} = 3.1622$. The necessary interval is then

$|Y - 25{,}000| \le 3.1622(4000)$ or $12{,}351 \le Y \le 37{,}649$

4.115 It is necessary to have $P(\,|Y-\mu| \le 1) \ge .75$. Hence, $1 - \frac{1}{k^2} = .75$ and $k = 2$. According to Tchebysheff's inequality, then, $1 = k\sigma$ and $\sigma = \frac{1}{k} = \frac{1}{2}$.

4.117 From Table A2.2, Appendix II, we find that when Y is uniform over (θ_1, θ_2)

$$E(Y) = \frac{\theta_1+\theta_2}{2} \qquad V(Y) = \frac{(\theta_2-\theta_1)^2}{12}$$

Thus,

$$2\sigma = 2\sqrt{V(Y)} = \frac{2(\theta_2-\theta_1)}{\sqrt{12}} = \frac{\theta_2-\theta_1}{\sqrt{3}}$$

The probability of interest is $P(|Y-\mu| \le 2\sigma) = P(\mu - 2\sigma \le Y \le \mu + 2\sigma)$. Now

$\theta_2 - \frac{\theta_1+\theta_2}{2} = \frac{\theta_2-\theta_1}{2} < \frac{\theta_2-\theta_1}{\sqrt{3}}$ and hence $\theta_2 < \frac{\theta_1+\theta_2}{2} + \frac{\theta_2-\theta_1}{\sqrt{3}} = \mu + 2\sigma$

Similarly,

$\frac{\theta_1+\theta_1}{2} - \theta_1 = \frac{\theta_2-\theta_1}{2} < \frac{\theta_2-\theta_1}{\sqrt{3}}$ so that $\theta_1 > \frac{\theta_1+\theta_2}{2} - \frac{\theta_2-\theta_1}{\sqrt{3}} = \mu - 2\sigma$

But θ_1 and θ_2 are the upper and lower limits on Y. Hence

$$P(\mu - 2\sigma \le Y \le \mu + 2\sigma) = P\left[\left(\frac{\theta_1+\theta_2}{2} - \frac{\theta_2-\theta_1}{\sqrt{3}}\right) \le Y \le \left(\frac{\theta_1+\theta_2}{2} + \frac{\theta_2-\theta_1}{\sqrt{3}}\right)\right]$$

$$= P(\theta_1 \le Y \le \theta_2) = 1$$

Tchebysheff's theorem is satisfied, but the approximation suggested by the empirical rule is inaccurate. This is because the probability distribution for the uniform random variable is far from mound-shaped.

(3)

4.130 The 3000 light bulbs utilized by the manufacturing plant comprise the entire population (i.e., this is not a sample from the population) whose length of life is normally distributed with mean $\mu = 500$, and standard deviation $\sigma = 50$. The objective is to find a particular value, y_0, so that $P(Y \le y_0) = .01$. That is, only 1% of the bulbs will burn out before they are replaced at time y_0. Then

$$P(Y \le y_0) = P(Z \le z_0) = .01 \quad \text{when} \quad z_0 = \frac{y_0 - 500}{50}$$

From Table 4, Appendix III, the value of z corresponding to an area (in the left tail of the distribution) of .01 is $z_0 = -2.327$ (see Figure 4.23). Solving for y_0 corresponding to $z_0 = -2.327$, we obtain

$$-2.327 = \frac{y_0 - 500}{50}$$

$$-116.35 = y_0 - 500 \quad \text{or} \quad y_0 = 383.65$$

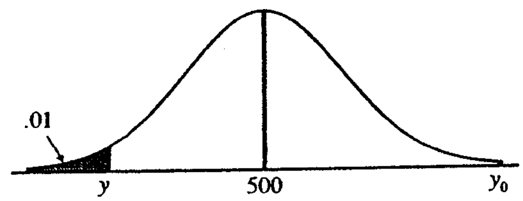

Figure 4.23

4.133 **a.** The variable factor of $f(y)$ is that of a gamma density with $\alpha = 2$ and $\beta = \frac{1}{2}$. Hence

$$c = \frac{1}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{\Gamma(2)\left(\frac{1}{2}\right)^2} = \frac{4}{1!} = 4$$

b. Since Y has a gamma distribution with $\alpha = 2$, $\beta = \frac{1}{2}$,

$$E(Y) = \alpha\beta = 1 \qquad V(Y) = \alpha\beta^2 = 2\left(\tfrac{1}{4}\right) = \tfrac{1}{2}$$

c. Recall that the moment-generating function of a gamma random variable is

$$m(t) = \frac{1}{(1-\beta t)^\alpha} \quad \text{and in this case} \quad m(t) = \frac{1}{\left[1 - \left(\frac{t}{2}\right)\right]^2} = \left(1 - \frac{t}{2}\right)^{-2}$$

4.136 The probability distribution for n, the number of arrivals in time $(0, t)$, is Poisson with mean λt, so that

$$P[n \text{ arrivals in } (0, t)] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Let T be the length of time until the first arrival, and consider the distribution function for T.

$$F(t) = P(T \le t) = 1 - P(T > t) = 1 - \mathrm{P}[n = 0 \text{ in } (0, t)] = 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = 1 -$$

Since T is a continuous random variable and $F(t)$ is its distribution function, the density function for T may be found, using Definition 4.3, to be

$$f(t) = \frac{d}{dt} F(t) = \lambda e^{-\lambda t} \qquad \text{for} \qquad t > 0$$

Note that the first derivative exists and is everywhere continuous, as is $F(t)$ itself. Hence Definitions 4.2 and 4.3 are satisfied. Moreover, $f(t)$ is the density of an Exponei random variable with mean $1/\lambda$.

4.137 Let Y be the time between the arrival of two calls, measured in hours. We require $P\left(Y > \frac{1}{4}\right)$. Since $\lambda t = 10$ and $t = 1$ (hour), $\frac{1}{\lambda} = \frac{1}{10}$, and

$$f(y) = \frac{1}{.1} e^{-y/.1} = 10e^{-10y}$$

and

$$P\left(Y > \tfrac{1}{4}\right) = \int_{1/4}^{\infty} 10e^{-10y}\, dy = -e^{-10y}\Big]_{1/4}^{\infty} = e^{-2.5} = .082$$

4.152 For $m = 2$,

$$E(Y) = \int_0^{\infty} \frac{2y^2 e^{-y^2/\alpha}}{\alpha}\, dy$$

Let $z = y^2$. Then $dz = 2y\, dy$ and

$$E(Y) = \int_0^{\infty} \frac{\sqrt{z}\, e^{-z/\alpha}}{\alpha}\, dz = \int_0^{\infty} \frac{z^{1/2} e^{-z/\alpha}}{\alpha}\, dz$$

which, when the proper constant is added, will be the integral of the density function of a gamma random variable with parameters $\frac{3}{2}$ and α. Then

$$E(Y) = \frac{\alpha^{3/2}\Gamma\left(\frac{3}{2}\right)}{\alpha} \int_0^{\infty} \frac{z^{1/2} e^{-z/\alpha}}{\Gamma\left(\frac{3}{2}\right)\alpha^{3/2}}\, dz = \frac{\alpha^{3/2}\Gamma\left(\frac{3}{2}\right)}{\alpha} = \alpha^{1/2}\Gamma\left(\tfrac{3}{2}\right) = \alpha^{1/2}\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{(\alpha\pi)^{1/2}}{2}$$

where $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ is shown in exercise 4.162.

Again using the transformation $z = y^2$, we find

$$E\left(Y^2\right) = \int_0^{\infty} \frac{2y^3 e^{-y^2/\alpha}}{\alpha}\, dy = \int_0^{\infty} \frac{z e^{-z/\alpha}}{\alpha}\, dz = \frac{\Gamma(2)\alpha^2}{\alpha} \int_0^{\infty} \frac{z e^{-z/\alpha}}{\Gamma(2)\alpha^2}\, dz = \alpha$$

so that $V(Y) = \alpha - \left[\alpha^{1/2}\Gamma\left(\frac{3}{2}\right)\right]^2 = \alpha\left\{1 - \left[\Gamma\left(\frac{3}{2}\right)\right]^2\right\} = \alpha\left[1 - \frac{\pi}{4}\right]$

See Exercise 4.162.