PROBLEM SET VI

4.1 By definition, $F(y) = P(Y \le y)$ for $y = 1, 2, 3, \ldots$

Then

$$P(Y = y) = P(Y \le y) - P(Y \le y - 1)$$
$$= F(y) - F(y-1) \qquad y = 2, 3, \ldots .$$

Also, $P(Y = 1) = P(Y \le 1)$
$$= F(1).$$

4.2 **a.** $F(i) = P(Y \le i) = \sum\limits_{k=1}^{i} q^{k-1}p \qquad i = 0, 1, 2, \ldots$

$$= p\sum_{k=0}^{i-1} q^k = p\left(\frac{1-q^i}{1-q}\right) = \frac{p(1-q^i)}{p}$$
$$= 1 - q^i.$$

Because Y is a discrete random variable, the only changes in $F(y)$ are at the positive integers. The result follows.

b. 1. $F(y) = 0$ for $y < 0$. Hence,
$$\lim_{y \to -\infty} F(y) = 0.$$

2. $F(y) = 1 - q^i \qquad i \le y < i + 1$ where $i = 0, 1, 2, 1\ldots$.
Then $\lim\limits_{y \to \infty} F(y) = 1 - \lim\limits_{i \to \infty} q^i \qquad$ for i an integer and $0 < q < 1$
$$= 1$$

3. Suppose $i \le y_1 < y_2 < i + 1 \qquad$ for $i = 0, 1, 2, \ldots$.
Then $F(y_1) = 1 - q^i = F(y_2)$.
On the other hand, suppose
$i - 1 \le y_1 < i \le y_2 < i + 1$. Then
$F(y_1) = 1 - q^{i-1} < 1 - q^i = F(y_2)$.

4.8 **a.** $f(y)$

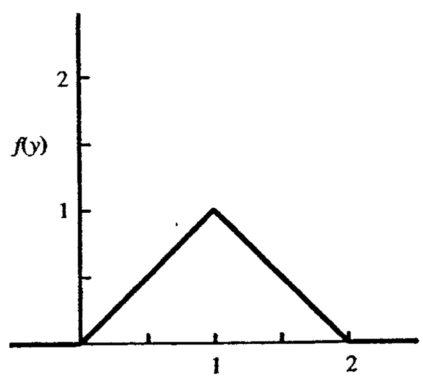

Figure 4.3

b. For $y < 0$, $F(y) = 0$.
For $y > 2$, $F(y) = 1$.
For $0 \le y \le 1$,

$$F(y) = \int_0^y t\,dt = \frac{y^2}{2}$$

For $1 \le y \le 2$,

$$F(y) = \int_0^1 t\,dt + \int_1^y (2-t)\,dt = \frac{1}{2} + \left[2t - \frac{t^2}{2}\right]_1^y = 2y - \frac{y^2}{2} - 1$$

c. $P(.8 \le Y \le 1.2) = F(1.2) - F(.8) = (2.4 - .72 - 1) - .32 = .36$

d. $P(Y > 1.5 \mid Y > 1) = \frac{P(Y>1.5)}{P(Y>1)} = \frac{1-(3-1.125-1)}{\frac{1}{2}} = \frac{.125}{.5} = .25$

4.11 a. $F(\infty) = \int_{-\infty}^{\infty} f(y)\,dy = \int_0^1 (cy^2+y)\,dy = c\left[\frac{y^3}{3}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 = \frac{c}{3}+\frac{1}{2} = 1$

Hence $\frac{c}{3} = \frac{1}{2}$ and $c = \frac{3}{2}$.

b. $F(y) = \int_{-\infty}^{y} f(t)\,dt = \int_0^y \left(\frac{3}{2}t^2 + t\right)dt = \frac{t^3}{2}\Big]_0^y + \frac{t^2}{2}\Big]_0^y = \frac{y^3}{2} + \frac{y^2}{2}$ for $0 \le y \le 1$

and $F(y) = 0$ for $y < 0$, $F(y) = 1$ for $y > 1$.

c. The graphs of $F(y)$ and $f(y)$ are shown in Figures 4.6 and 4.7.

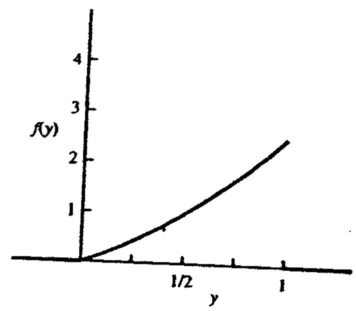

Figure 4.6

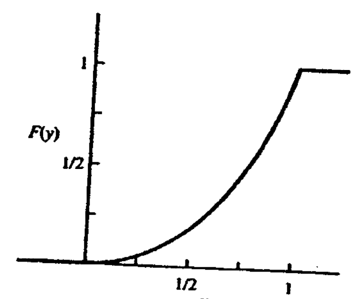

Figure 4.7

d. $F(-1) = 0$ since $y < 0$; $F(0) = 0$; $F(1) = \frac{1}{2} + \frac{1}{2} = 1$

e. $P(0 \le Y \le .5) = F(.5) - F(0) = \frac{(.5)^3}{2} + \frac{(.5)^2}{2} - 0 = \frac{1}{16} + \frac{1}{8} = \frac{3}{16}$

f. $P\left(Y > \frac{1}{2} \middle| Y > \frac{1}{4}\right) = \frac{P(Y>\frac{1}{2})}{P(Y>\frac{1}{4})} = \frac{1-(\frac{3}{16})}{1-(\frac{1}{128}+\frac{1}{32})} = \frac{\frac{13}{16}}{\frac{123}{128}} = \frac{104}{123}$

4.15 Refer to Exercise 4.11.

$$E(Y) = \int_0^1 \left(\frac{3}{2}y^3 + y^2\right)dy = \left[\frac{3}{8}y^4 + \frac{y^3}{3}\right]_0^1 = \frac{3}{8} + \frac{1}{3} = \frac{9+8}{24} = \frac{17}{24} = .708$$

$$E(Y^2) = \int_0^1 \left(\frac{3}{2}y^4 + y^3\right)dy = \left[\frac{3}{10}y^5 + \frac{1}{4}y^4\right]_0^1 = \frac{3}{10} + \frac{1}{4} = .55$$

so that $V(Y) = E(Y^2) - (EY)^2 = .55 - (.708)^2 = .0487$.

4.24 a. $E(Y) = \int_0^1 y(2y)\,dy = \frac{2y^3}{3}\Big|_0^1 = \frac{2}{3}$

$V(Y) = E(Y^2) - [E(Y)]^2 = \left[\int_0^1 y^2(2y)\,dy\right] - \left(\frac{2}{3}\right)^2 = \left(\frac{1}{2}\right)y^4\Big]_0^1 - \left(\frac{4}{9}\right) = \frac{1}{18}$

b. $E(X) = E(200Y - 60) = 200E(Y) - 60 = 200\left(\frac{2}{3}\right) - 60 = \frac{220}{3}$

$V(X) = V(200Y - 60) = V(200Y) = (200)^2V(Y) = 40{,}000\left(\frac{1}{18}\right) = \frac{20{,}000}{9}$

c. Recall Tchebysheff's theorem from exercise 1.24.

$P(\mu_x \pm k\sigma_{\bar{x}}) \ge 1 - \left(\frac{1}{k^2}\right)$ where $1 - \left(\frac{1}{k^2}\right) = \frac{3}{4}$. Solving, $k = 2$.

The desired interval is $\left(\left[\frac{220}{3}\right] \pm 2\sqrt{\frac{20{,}000}{9}}\right) = (-20.948, 167.614)$

4.29 Since the parachutist is landing at a random point in the interval (A, B), the point of landing is a continuous random variable Y, with a uniform distribution over (A, B). Hence

$$f(y) = \frac{1}{B-A} \qquad A \le y \le B$$

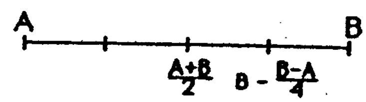

Figure 4.10

a. Refer to Figure 4.10. If he lands closer to A than to B, he has landed in the interval $\left(A, \frac{A+B}{2}\right)$. The probability is

$$\int_{A}^{(B+A)/2} \frac{1}{B-A}\, dy = \frac{\left(\frac{A+B}{2}\right)-A}{B-A} = \frac{1}{2}$$

b. The point at which the distance to A is exactly 3 times the distance to B is the point $B - \left(\frac{1}{4}\right)(B-A) = \frac{3B+A}{4}$. Then

P(distance to A is more than 3 times distance to B) $= P\left(\frac{3B+A}{4} \le Y \le B\right)$

$$= \frac{B-\left(\frac{3B+A}{4}\right)}{B-A} = \frac{\left(\frac{B-A}{4}\right)}{B-A} = \frac{1}{4}$$

4.31 Recall Theorem 4.6.

$$V(Y) = E(Y^2) - (E(Y))^2 = \left[\int_{\theta_1}^{\theta_2} y^2 \left(\frac{1}{\theta_2-\theta_1}\, dy\right)\right] - \left(\frac{\theta_2+\theta_1}{2}\right)^2$$

$$= \left[\frac{1}{3(\theta_2-\theta_1)}\right] y^3\Big]_{\theta_1}^{\theta_2} - \left(\frac{1}{4}\right)(\theta_2-\theta_1)^2$$

$$= \left[\frac{1}{3(\theta_2-\theta_1)}\right](\theta_2-\theta_1)(\theta_2^2+\theta_1\theta_2+\theta_1^2) - \left(\frac{1}{4}\right)(\theta_2^2+2\theta_1\theta_2+\theta_1^2)$$

$$= \left(\frac{4}{12}\right)(\theta_2^2+\theta_1\theta_2+\theta_1^2) - \left(\frac{3}{12}\right)(\theta_2^2+2\theta_1\theta_2+\theta_1^2)$$

$$= \frac{\theta_2^2-2\theta_1\theta_2+\theta_1^2}{12} = \frac{(\theta_2-\theta_1)^2}{12}.$$

4.39 Let Y = cycle time. Then

$$f(y) = \frac{1}{70-50} = \frac{1}{20} \text{ for } 50 \le y \le 70$$

and

$$F(y) = \int_{50}^{y} \frac{1}{20}\, dt = \frac{y-50}{20} \text{ for } 50 \le y \le 70;$$

$$F(y) = 0 \text{ for } y < 50;$$

$$F(y) = 1 \text{ for } y > 70.$$

Thus

$$P(Y > 65 | Y > 55) = \frac{P(Y>65)}{P(Y>55)} = \frac{1-\left(\frac{65-50}{20}\right)}{1-\left(\frac{55-50}{20}\right)} = \frac{20-15}{20-5} = \frac{1}{3}$$

4.46 The next few exercises are designed to provide practice for the student in evaluating areas under the normal curve. The following notes may be of some assistance.

(1) Table 4 tabulates the area under the standard normal curve to the right of a specified value z_0. See Figure [redacted] 1. Denote the area obtained by indexing $z = z_0$ in Table 4 by $A(z_0)$ and the desired area by A.

(2) Because of the symmetry of the normal distribution, and since the total area under the curve is 1, the total area lying on one side of 0 will be .5. Thus in order to calculate the area between 0 and z_0 (when $z_0 > 0$) we index z_0, which gives us $A(z_0)$. We then subtract $A(z_0)$ from .5. That is, $A = .5 - A(z_0)$.

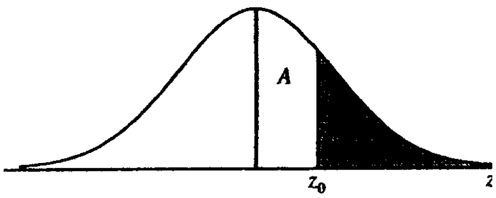

Figure [redacted] 1

(3) Notice that Z is actually a random variable that may take on an infinite number of values, both positive and negative. However, since the standardized normal curve is symmetric about 0, a left-hand area (i.e., an area corresponding to a negative value of z) may be evaluated by indexing the corresponding positive value in Table 4.

(a) The area between $z = 0$ and $z = 1.2$ is $A_1 = .5 - A(1.2) = .5 - .1151 = .3849$. See Figure [redacted] 2.

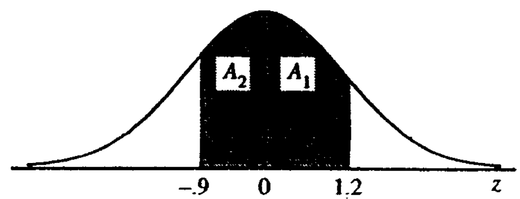

Figure [redacted] 2

(b) The area between $z = 0$ and $z = -.9$ is $A_2 = .5 - A(-.9) = .5 - A(.9)$ $.5 - .1841 = .3159$.

(c) The desired area is A_1, as shown in Figure [redacted] 3. Note that $A(.3) = .3821$ and $A(1.56) = .0594$. $A_1 = A(.3) - A(1.56) = .3821 - .0594$ $= .3227$.

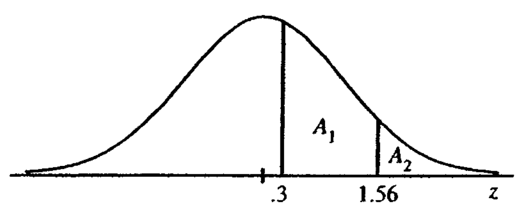

Figure [redacted] 3

(d) The desired area is $A_1 + A_2$ $= .5 - A(-.2) + .5 - A(.2)$ $= 1 - 2(.4207) = .1586$. See Figure [redacted] 4.

(e) The desired area is $A(-.2) - A(-1.56)$ $= .4207 - .0594 = .3613$.

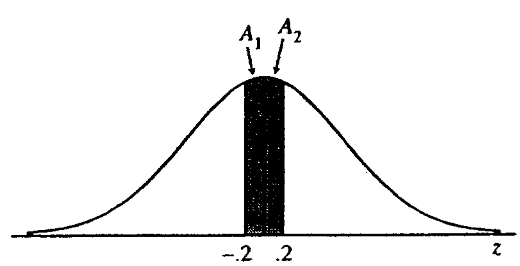

Figure [redacted] 4

4.41 Let Y = time the defective board is detected.

a. $P(0 < Y < 1) = \int_0^1 \left(\frac{1}{8}\right) dx = \frac{1}{8}$

b. $P(7 < y < 8) = \int_7^8 \left(\frac{1}{8}\right) dx = \frac{1}{8}$

c. $P(4 < Y < 5|Y > 4) = \dfrac{\int_4^5 \left(\frac{1}{8}\right) dx}{\int_4^8 \left(\frac{1}{8}\right) dx} = \dfrac{\left(\frac{1}{8}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{4}$

4.42 Let Y = amount of measurement error. Y is $U(-.05, .05)$.

a. $P(-.01 < Y < .01) = \int_{-.01}^{.01} \left(\frac{1}{.1}\right) dx = .2$

b. $E(Y) = \dfrac{-.05 + .05}{2} = 0$

$V(Y) = \dfrac{(.05 + .05)^2}{12} = \dfrac{.01}{12} = 0.00083$

4.50 This normal distribution has $\mu = 400$ and $\sigma = 20$. Probabilities associated with any normal random variable Y can be obtained by converting the necessary values of y to their corresponding z values. This conversion is made by using the formula $z = \frac{y-\mu}{\sigma}$. Note that z is the distance from the mean, $y - \mu$, measured in units of σ. In this case, the desired probability is $P(Y > y_1) = 450$. The z value corresponding to $y_1 = 450$ is $z_1 = \frac{450-400}{20} = 2.5$. Then

$$P(Y > 450) = P(Z > 2.5) = A(2.5) = .0062$$

4.54 The fraction of students with grade point averages greater than 3.0 is given by $A_1 = P(Y > 3.0)$ (shown in Figure 4.19). Then the z value corresponding to the point $y = 3.0$ is

$$z = \frac{y-\mu}{\sigma} = \frac{3.0-2.4}{.8} = .75$$

Hence

$$A_1 = P(Y > 3.0) = P(Z > .75) = A(.75) = .2266$$

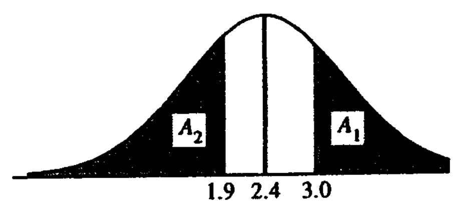

Figure 4.19

4.55 The probability of interest is $P(Y < 1.9)$ with corresponding z value

$$z = \frac{y-\mu}{\sigma} = \frac{1.9-2.4}{.8} = -.625$$

(Recall that a negative value of z implies a value to the left of the mean.) Then

$$A_2 = P(Y < 1.9) = P(Z < -.625) = P(Z > .625) = A(.625) = .2660$$

(after interpolating).

4.56 Let X be the number of students with a grade point average in excess of 3.0 when 3 students are randomly selected. Then X has a binomial distribution with $n = 3$ and $p = P(\text{student's GPA exceeds 3.0}) = .2266$, from Exercise 4.54. The probability of interest is

$$P(X = 3) = \binom{3}{3} p^3 q^0 = (.2266)^3 = .0116$$