4.1 By definition, \( F(y) = P(Y \leq y) \) for \( y = 1, 2, 3, \ldots \)
Then
\[
P(Y = y) = P(Y \leq y) - P(Y \leq y - 1)
= F(y) - F(y - 1)
\]
\( y = 2, 3, \ldots \).
Also, \( P(Y = 1) = P(Y \leq 1) - F(1) \).

4.2 a. \( F(i) = P(Y \leq i) = \sum_{k=0}^{i} q^k \frac{1}{p} \quad i = 0, 1, 2, \ldots \)
\[
= p \sum_{k=0}^{i-1} q^k = p \left( \frac{1 - q^i}{1 - q} \right) = \frac{p(1 - q^i)}{p} = 1 - q^i.
\]
Because \( Y \) is a discrete random variable, the only changes in \( F(y) \) are at the positive integers. The result follows.

b. 1. \( F(y) = 0 \) for \( y < 0 \). Hence,
\[
\lim_{y \to -\infty} F(y) = 0.
\]

2. \( F(y) = 1 - q^1 \quad i \leq y < i + 1 \) where \( i = 0, 1, 2, 1, \ldots \),
Then \( \lim_{y \to \infty} F(y) = 1 - \lim_{i \to \infty} q^1 \) for \( i \) an integer and \( 0 < q < 1 \)
\[
= 1
\]

3. Suppose \( i \leq y_1 < y_2 < i + 1 \) for \( i = 0, 1, 2, \ldots \).
Then \( F(y_1) = 1 - q^{i-1} \quad 1 - q^i = F(y_2) \).
On the other hand, suppose
\( \quad 1 \leq y_1 < t \leq y_2 < i + 1 \). Then
\[
F(y_1) = 1 - q^{i-1} < 1 - q^i = F(y_2).
\]

4.8 a. \( f(y) \)

b. For \( y < 0 \), \( F(y) = 0 \).
For \( y > 2 \), \( F(y) = 1 \).
For \( 0 \leq y \leq 1 \),
\[
F(y) = \int_{0}^{y} t \ dt = \frac{y^2}{2}
\]
For \( 1 \leq y \leq 2 \),
\[
F(y) = \int_{0}^{1} t \ dt + \int_{1}^{y} (2 - t) \ dt = \left[ \frac{1}{2} - 2y + \frac{y^2}{2} \right]_1^y = 2 - y^2 - \frac{y^2}{2} - 1
\]

c. \( P(8 \leq Y \leq 1.2) = F(1.2) - F(8) = (2.4 - .72 - 1) - .32 = 36 \)

d. \( P(Y > 1.5 \mid Y > 1) = \frac{P(Y > 1.5)}{P(Y > 1)} = \frac{1 - (3 - 1) \frac{125 - 1}{2}}{5} = \frac{175}{5} = 25 \)
4.11 a. \[ F(\infty) = \int_{-\infty}^{\infty} f(y) \, dy = \int_{-\infty}^{\infty} \left( c y^2 + y \right) \, dy = c \left[ \frac{y^3}{3} \right]_0^\infty + \left[ \frac{y^2}{2} \right]_0^\infty = \frac{\xi^3}{3} + \frac{\xi^2}{2} = 1 \]

Hence \( \frac{\xi}{3} = -\frac{1}{2} \) and \( c = \frac{3}{2} \).

b. \[ F(y) = \int_{-\infty}^{y} f(t) \, dt = \int_{-\infty}^{y} \left( \frac{3}{2} t^2 + t \right) \, dt - \left[ \frac{t^3}{3} + \frac{t^2}{2} \right]_0^y = \frac{y^3}{3} + \frac{y^2}{2} \]

for \( 0 < y \leq 1 \)

and \( F(y) = 0 \) for \( y < 0 \), \( F(y) = 1 \) for \( y > 1 \).

c. The graphs of \( F(y) \) and \( f(y) \) are shown in Figures 4.6 and 4.7.

\[ \text{Figure 4.6} \]
\[ \text{Figure 4.7} \]

d. \( F(-1) = 0 \) since \( y < 0 \); \( F(0) = 0 \); \( F'(1) = \frac{1}{2} + \frac{1}{2} = 1 \)

e. \( P(0 \leq Y \leq .5) = F(.5) - F(0) = \left( \frac{3}{2} \right)^2 + \frac{1}{2} - 0 = \frac{1}{16} + \frac{1}{8} = \frac{3}{16} \)

f. \( P \left( Y > \frac{1}{2} \right) \rightarrow \left( Y > \frac{1}{4} \right) = P \left( \frac{Y > \frac{1}{2}}{Y > \frac{1}{4}} \right) = \frac{1 - \frac{1}{4}}{1 - \frac{1}{8}} = \frac{\frac{3}{4}}{\frac{7}{8}} = \frac{12}{14} = .8571 \)

4.15 Refer to Exercise 4.11.

\[ E(Y) = \int_{0}^{1} \left( \frac{3}{2} y^3 + y^2 \right) \, dy = \left[ \frac{3}{8} y^4 + \frac{y^2}{3} \right]_0^1 - \frac{1}{8} + \frac{1}{3} = \frac{9 + 4}{8} = \frac{13}{12} = .708 \]

\[ E(Y^2) = \int_{0}^{1} \left( \frac{3}{2} y^4 + y^2 \right) \, dy = \left[ \frac{31}{10} y^5 + \frac{1}{4} y^4 \right]_0^1 = \frac{3}{10} + \frac{1}{4} = .55 \]

so that \( V(Y) = E(Y^2) - (EY)^2 = .55 - (.708)^2 = .0487 \).

4.24 a. \[ E(Y) = \int_{0}^{1} y(2y) \, dy = \left[ \frac{2y^2}{3} \right]_0^1 = \frac{1}{3} \]

\[ V(Y) = E(Y^2) - [E(Y)]^2 - \left[ \int_{0}^{1} y^2(2y) \, dy \right] - \left( \frac{1}{3} \right)^2 \]

\( (\frac{1}{3})^2 = (\frac{1}{3}) \)

b. \[ E(X) = E(200Y - 60) = 200 E(Y) - 60 = 200 \left( \frac{1}{3} \right) - 60 = \frac{200}{2} \]

\[ V(X) = V(200Y - 60) = V(200Y) = (200)^2 V(Y) = 40,000 \left( \frac{1}{18} \right) = \frac{20,000}{9} \]

c. Recall Tchebycheff's theorem from exercise 1.24.

\( P(\mu \pm k\sigma) \geq 1 - \left( \frac{1}{k^2} \right) \) where \( 1 - \left( \frac{1}{k^2} \right) = \frac{3}{4} \). Solving, \( k = 2 \).

The desired interval is \( \left[ \frac{120}{3} \pm 2 \sqrt{\frac{20,000}{9}} \right] = (-20.948, 167.614) \)
4.29 Since the parachutist is landing at a random point in the interval \((A, B)\), the point of landing is a continuous random variable \(Y\), with a uniform distribution over \((A, B)\). Hence

\[
f(y) = \frac{1}{B-A}, \quad A \leq y \leq B
\]

\[\begin{array}{c}
\includegraphics[width=0.5\textwidth]{figure10.png}
\end{array}\]

\textbf{Figure 4.10}

a. Refer to Figure 4.10. If he lands closer to \(A\) than to \(B\), he has landed in the interval \((A, \frac{A+B}{2})\). The probability is

\[
\int_A^{(B+A)/2} \frac{1}{B-A} \, dy = \left(\frac{A+B}{2}\right) - A = \frac{1}{2}
\]

b. The point at which the distance to \(A\) is exactly 3 times the distance to \(B\) is the point \(H - \left(\frac{1}{3}\right)(B - A) = \frac{3B + A}{4}\). Then

\[
P(\text{distance to } A \text{ is more than 3 times distance to } B) = P(\frac{3B + A}{4} \leq Y \leq B)
\]

\[
= \frac{B - \frac{3B + A}{4}}{B - A} - \frac{\frac{3B + A}{4} - A}{B - A} = \frac{1}{4}
\]

4.31 Recall Theorem 4.6.

\[
V(Y) = E(Y^2) - (E(Y))^2 = \left[\int_y y^2 \left(\frac{1}{\theta_2 - \theta_1} \, dy\right)\right] - \left(\frac{\theta_1 + \theta_2}{2}\right)^2
\]

\[
= \left[\frac{1}{3(\theta_2 - \theta_1)} y^3\right]_{\theta_1}^{\theta_2} - \left(\frac{1}{4}\right)(\theta_2 - \theta_1)^2
\]

\[
= \left[\frac{1}{3(\theta_2 - \theta_1)} \right] (\theta_2 - \theta_1) (\theta_2^2 + \theta_1 \theta_2 + \theta_1^2) - \left(\frac{1}{4}\right)(\theta_2 + 2\theta_1 \theta_2 + \theta_2^2)
\]

\[
= \left(\frac{4}{12}\right)(\theta_2^2 + \theta_1 \theta_2 + \theta_1^2) - \left(\frac{1}{12}\right)(\theta_2^2 + 2\theta_1 \theta_2 + \theta_2^2)
\]

\[
= \frac{\theta_2^2 - 7\theta_1 \theta_2 + \theta_1^2}{12} = \left(\frac{\theta_2 - \theta_1}{2}\right)^2
\]

4.39 Let \(Y\) = cycle time. Then

\[
f(y) = \frac{1}{70-50} = \frac{1}{20} \text{ for } 50 \leq y \leq 70
\]

and

\[
F(y) = \int_50^y \frac{1}{20} \, dt = \frac{y - 50}{20} \text{ for } 50 \leq y \leq 70;
\]

\[
F(y) = 0 \text{ for } y < 50;
\]

\[
F(y) = 1 \text{ for } y > 70.
\]

Thus

\[
P(Y > 65|Y > 55) = \frac{P(Y > 65)}{P(Y > 55)} = \frac{1 - \left(\frac{65 - 50}{20 - 5}\right)^2}{1 - \left(\frac{55 - 50}{20 - 5}\right)^2} = \frac{70 - 15}{20 - 5} = \frac{1}{3}
\]
The next few exercises are designed to provide practice for the student in evaluating areas under the normal curve. The following notes may be of some assistance.

1. Table 4 tabulates the area under the standard normal curve to the right of a specified value \( z_0 \). See Figure 1. Denote the area obtained by indexing \( z = z_0 \) in Table 4 by \( A(z_0) \) and the desired area by \( A \).

2. Because of the symmetry of the normal distribution, and since the total area under the curve is 1, the total area lying on one side of 0 will be \( .5 \). Thus in order to calculate the area between 0 and \( z_0 \) (when \( z_0 > 0 \)) we index \( z_0 \), which gives us \( A(z_0) \). We then subtract \( A(z_0) \) from \( .5 \). That is, \( A - .5 - A(z_0) \).

3. Notice that \( Z \) is actually a random variable that may take on an infinite number of values, both positive and negative. However, since the standardized normal curve is symmetric about 0, a left-hand area (i.e., an area corresponding to a negative value of \( z \)) may be evaluated by indexing the corresponding positive value in Table 4.

   (a) The area between \( z = 0 \) and \( z = 1.2 \) is
   \[ A_1 = .5 - A(1.2) = .5 - .1151 = .3849. \]
   See Figure 2.

   (b) The area between \( z = 0 \) and \( z = -.9 \) is
   \[ A_2 = .5 - A(-.9) = .5 - A(.9) \]
   \[ = .5 - .1841 = .3159. \]

   (c) The desired area is \( A_1 \), as shown in Figure 3. Note that \( A(.3) = .3821 \) and \( A(1.56) = .0594 \).
   \[ A_1 = A(.3) - A(1.56) = .3821 - .0594 = .3227. \]

   (d) The desired area is \( A_1 + A_2 \)
   \[ = .5 - A(-.2) + .5 - A(.2) \]
   \[ = 1 - 2(.4207) = .1586. \] See Figure 4.

   (e) The desired area is \( A(-.2) - A(-1.56) \)
   \[ = .4207 - .0594 = .3613. \]
4.41 Let $Y =$ time the defective board is detected.

a. $P(0 < Y < 1) = \int_{0}^{1} \left( \frac{1}{3} \right) \, dx = \frac{1}{3}$

b. $P(7 < Y < 8) = \int_{7}^{8} \left( \frac{1}{5} \right) \, dx = \frac{1}{5}$

c. $P(4 < Y < 5|Y > 4) = \frac{\int_{4}^{5} (\frac{1}{4}) \, dx}{\int_{4}^{\infty} (\frac{1}{4}) \, dx} = \frac{1}{4}$

4.42 Let $Y =$ amount of measurement error. $Y$ is $U(-.05, .05)$.

a. $P(-.01 < Y < .01) = \int_{-.01}^{.01} \left( \frac{1}{2} \right) \, dx = .2$

b. $E(Y) = \frac{-05 + 05}{2} = 0$

$V(Y) = \frac{(0.05 + 0.05)^2}{12} = \frac{0.01}{12} = 0.00083$

4.50 This normal distribution has $\mu = 400$ and $\sigma = 20$. Probabilities associated with any normal random variable $Y$ can be obtained by converting the necessary values of $y$ to their corresponding $z$ values. This conversion is made by using the formula $z = \frac{y - \mu}{\sigma}$. Note that $z$ is the distance from the mean, $y - \mu$, measured in units of $\sigma$. In this case, the desired probability is $P(Y > y_1) = 450$. The $z$ value corresponding to $y_1 = 450$ is $z_1 = \frac{450 - 400}{20} = 2.5$. Then

$P(Y > 450) = P(Z > 2.5) = A(2.5) = .0062$

4.54 The fraction of students with grade point averages greater than 3.0 is given by $A_1 = P(Y > 3.0)$ (shown in Figure 4.19). Then the $z$ value corresponding to the point $y = 3.0$ is

$z = \frac{y - \mu}{\sigma} = \frac{3.0 - 2.4}{.8} = .75$

Hence $A_1 = P(Y > 3.0) = P(Z > .75) = A(.75) = .2266$

4.55 The probability of interest is $P(Y < 1.9)$ with corresponding $z$ value

$z = \frac{y - \mu}{\sigma} = \frac{1.9 - 2.4}{.8} = -.625$

(Recall that a negative value of $z$ implies a value to the left of the mean.) Then

$A_2 = P(Y < 1.9) = P(Z < -.625) = P(Z > .625) = A(.625) = .2660$

(after interpolating).

4.56 Let $X$ be the number of students with a grade point average in excess of 3.0 when 3 students are randomly selected. Then $X$ has a binomial distribution with $n = 3$ and $p = P$(student's GPA exceeds 3.0) = .2266, from Exercise 4.54. The probability of interest is

$P(X = 3) = {3 \choose 3} p^3 q^0 = (.2266)^3 = .0116$