Problem Set V

3.10 $E(Y) = \Sigma y p(y) = 1(.4) + 2(.3) + 3(.2) + 4(.1) = 2.0$

$E\left(\frac{1}{Y}\right) = \Sigma \frac{1}{y} p(y) = 1(.4) + \frac{1}{2}(.3) + \frac{1}{3}(.2) + \frac{1}{4}(.1) = .6417$

$E(Y^2 - 1) = E(Y^2) - 1 = [1(.4) + 4(.3) + 9(.2) + 16(.1)] - 1 = 5 - 1 = 4$

Using Theorem 3.6,

$$V(Y) = E(Y^2) - [E(Y)]^2 = 5 - (2)^2 = 1$$

3.16 Since the die is fair, the probability distribution for Y is

$$p(y) = \frac{1}{6} \qquad y = 1, 2, 3, 4, 5, 6$$

Then

$$E(Y) = \Sigma y p(y) = \frac{1}{6}(1 + 2 + \ldots + 6) = \frac{21}{6} = 3.5$$

$$E(Y^2) = \Sigma y^2 p(y) = \frac{1}{6}(1 + 4 + 9 + \ldots + 36) = \frac{91}{6} = 15.1667$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = 15.1667 - (3.5)^2 = 2.9167$$

3.17 Define G to be the gain to a person in drawing one card. G can take on only three values, \$15, \$5, or \$$-4$, with probabilities as shown in the accompanying table.

G	$p(G)$
15	$\frac{2}{13}$
5	$\frac{2}{13}$
-4	$\frac{9}{13}$

Then $E(G) = \Sigma G p(G) = 15\left(\frac{2}{13}\right) + 5\left(\frac{2}{13}\right) - 4\left(\frac{9}{13}\right) = \frac{4}{13} = .31$

The expected gain is \$.31.

3.21 Let Y be a random variable representing the payout on an individual policy. $P(Y = 85{,}000) = P(\text{total loss}) = .001$, $P(Y = 45{,}000) = P(50\%\ \text{loss}) = .01$, and $P(Y = 0) = 1 - .001 - .01 = .989$. Let C represent the premium the insurance company charges. Then the company's net gain or loss for this policy is given by $C - y$. To yield a long term average loss of 0 the company should choose C such that: $E(C - Y) = 0$, or $C = E(Y)$. Then we have:

$$\begin{aligned} E(Y) &= \sum_y y p(y) \\ &= (85{,}000)(.001) + (42{,}500)(.01) + (0)(.989) \\ &= 510 = C. \end{aligned}$$

3.115 Using the binomial theorem, $(a+b)^n = \sum_{y=0}^{n} \binom{n}{y} a^y b^{n-y}$, we have:

$$m(t) = E(e^{ty}) = \sum_{y=0}^{n} \binom{n}{y} (pe^t)^y q^{n-y} = (pe^t + q)^n.$$

3.116 Refer to Exercise 3.115.

$$E(Y) = \frac{d}{dt} m(t)\Big|_{t=0} = (pe^t + q)^{n-1} npe^t\Big|_{t=0} = n(p+q)^{n-1}p = np.$$

$$\begin{aligned} E(Y^2) &= \frac{d^2}{dt^2} m(t)\Big|_{t=0} \\ &= npe^t(n-1)pe^t(pe^t + q)^{n-2} + n(pe^t + q)^{n-1}pe^t\Big|_{t=0} \\ &= np^2(n-1) + np \end{aligned}$$

$$V(Y) = E(Y^2) - n^2p^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq$$

3.117 Recall that $\sum_{y=1}^{\infty} q^{y-1} p = 1$ or equivalently $\sum_{y=1}^{\infty} q^{y-1} = \frac{1}{p}$ for $0 < q < 1$. Now we have

$$m(t) = E(e^{ty}) = \sum_{y=1}^{\infty} pe^{ty}q^{y-1} = pe^t \sum_{y=1}^{\infty} e^{t(y-1)}q^{y-1} = pe^t \sum_{y=1}^{\infty} (qe^t)^{y-1} = \frac{pe^t}{1-qe^t}$$

if $qe^t < 1$ or equivalently, $t < -\ln q$.

3.118 Refer to Exercise 3.117.

$$E(Y) = \frac{d}{dt} m(t)\Big|_{t=0} = \frac{(1-qe^t)(pe^t) - pe^t(-qe^t)}{(1-qe^t)^2}\Bigg|_{t=0} = \frac{pe^t}{(1-qe^t)^2}\Bigg|_{t=0}$$
$$= \frac{p}{(1-q)^2} = \frac{1}{p}$$
$$E(Y^2) = \frac{d}{dt}\,\frac{pe^t}{(1-qe^t)^2}\Bigg|_{t=0} = \frac{(1-qe^t)^2 pe^t - 2pe^t(-qe^t)(1-qe^t)}{(1-qe^t)^4}\Bigg|_{t=0}$$
$$= \frac{p^3+2pq^2}{p^4} = \frac{p+2q}{p^2} = \frac{1+q}{p^2}$$

Finally,

$$V(Y) = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

3.123 **a.** Differentiate $m(t)$ to find the necessary moments.

$$E(Y) = \frac{d}{dt} m(t)\Big|_{t=0} = \frac{1}{6}e^t + \frac{4}{6}e^{2t} + \frac{9}{6}e^{3t}\Big|_{t=0} = \frac{14}{6} = \frac{7}{3}$$

b. $E(Y^2) = \frac{d^2}{dt^2} m(t)\Big|_{t=0} = \frac{1}{6} + \frac{8}{6} + \frac{27}{6} = 6,\ V(Y) = 6 - \left(\frac{7}{3}\right)^2 = \frac{5}{9}.$

c. Since $m(t) = E\left(e^{ty}\right)$, Y must take only the values $Y = 1, 2$, and 3, with

3.132 The random variable Y has a binomial distribution with $p = .35$ and $n = 2300$.

a. $E(Y) = np = (2300)(.35) = 805$

b. $V(Y) = npq = (2300)(.35)(.65) = 523.25 = \sigma^2$

$\sigma = \sqrt{523.25} = 22.875$

c. The interval is $(805 - 2(22.875), 805 + 2(22.875)) = (759.25, 850.75)$

d. The observation $Y = 249$ is 24.3 standard deviations below the mean value 805 $\left(\frac{805-249}{22.875} = 24.3\right)$. This value $(Y = 249)$ is not consistent with a rate of 35% as Tchebyscheff's theorem tells us $P(|Y - 805| > (24.3)(22.875)) \leq \frac{1}{24.3^2} \approx .002$.

3.133 **a.**
$$E(Y) = (-1)p(-1) + (0)p(0) + (1)p(1)$$
$$= -1\left(\tfrac{1}{18}\right) + 0\left(\tfrac{16}{18}\right) + 1\left(\tfrac{1}{18}\right)$$
$$= 0.$$

$$E\left(Y^2\right) = (-1)^2p(-1) + (0)^2p(0) + (1)^2p(1)$$
$$= (-1)^2\left(\tfrac{1}{18}\right) + (0)^2\left(\tfrac{16}{18}\right) + (1)^2\left(\tfrac{1}{18}\right)$$
$$= \tfrac{1}{9}$$
$$V(Y) = E\left(Y^2\right) - (E(Y))^2$$
$$= \tfrac{1}{9} - 0 = \tfrac{1}{9}$$

b. $\sigma = \sqrt{V(Y)} = \sqrt{\frac{1}{9}} = \frac{1}{3}$.

By Tchebysheff's theorem,

$$P\left(|y - \mu| \geq 3\sigma\right) \leq \frac{1}{3^2} = \frac{1}{9}.$$

According to the probability distribution of Y,

$$P\left(|y - \mu| \geq 3\sigma\right) = P\left(|y| \geq 1\right)$$
$$= p(-1) + p(1)$$
$$= \tfrac{1}{18} + \tfrac{1}{18} = \tfrac{1}{9}$$

so that the bound is attained when $k = 3$.

c. Let x have the probability distribution

$$p(-1) = \tfrac{1}{8} \qquad p(0) = \tfrac{6}{8} \qquad \text{and} \qquad p(1) = \tfrac{1}{8}$$

so that $E[x] = 0$ and $V(x) = E\left[x^2\right] = \frac{1}{4}$.

It follows that

$$P\left(|X - \mu_x| \geq 2\sigma_x\right) = P\left(|x| \geq 1\right) = p(-1) + p(1) = \tfrac{1}{4},$$

as desired.

d. Letting all the probability mass be on values $-1, 0, 1$, $E(W) = 0$ if

$$p(-1) = p \qquad p(0) = 1 - 2p \qquad \text{and} \qquad p(1) = p$$

for some probability p. We want $k\sigma_W = 1$ so that $\sigma_W = \frac{1}{k}$ and $\sigma_W^2 = \frac{1}{k^2}$. With $E(W) = 0$, the $V(W) = E(W^2) = 2p$. Setting $2p = \frac{1}{k^2}$ gives $p = \frac{1}{2k^2}$. Therefore for any specified $k > 1$, $P(|W - \mu_W| \geq k\sigma_W) = \frac{1}{k^2}$ if

$$p(-1) = \frac{1}{2k^2} \qquad p(0) = 1 - \frac{1}{k^2} \qquad \text{and} \qquad p(1) = \frac{1}{2k^2}.$$

Alternatively, we can show the same result using complements. We want, with $k\sigma_W = 1$,

$$P(W = 0) = P(|W - \mu_W| < k\sigma_W) = 1 - \frac{1}{k^2}$$

Then, in order for $E(W) = 0$, we must have the same distribution as above.

3.134 Similar to Exercise 3.131 **a.** The interval .48 to .52 is the interval $|Y - .50| \leq 2\sigma$. Hence the lower bound is $1 - \left(\frac{1}{k^2}\right) = 1 - \left(\frac{1}{4}\right) = \frac{3}{4}$. The expected number of coins is then at least $\left(\frac{3}{4}\right)(400) = 300$.

3.139 Let Y be the number of fatalities. Then Y is binomial with $p = .0006$ and $n = 40{,}000$.

a. $E(Y) = np = 40{,}000(.0006) = 24$

b. $V(Y) = npq = 24(.9994) = 23.9856 = \sigma^2$

$$\sigma = \sqrt{23.9856} = 4.898$$

c. No. The value 40 is $\frac{40-24}{4.898} = 3.26$ standard deviations above the mean, and both Tchebyscheff's theorem and the empirical rule suggest this is unlikely. (Tchebyscheff's theorem is probably more relevant in this situation, see 3.120).

3.141 The mean of C is $E(C) = \$50 + \$3\,E(Y) = \$50 + \$3(10) = \$80$. The variance is $V(C) = V(50 + 3Y) = 9V(Y) = 9(10) = 90$, so that $\sigma = \sqrt{90} = 9.487$. Using Tchebysheff's theorem with $k = 2$, we have $P(|Y - 80| < 2(9.487)) \geq .75$ so that the required interval is $(80 - 2(9.487), 80 + 2(9.487))$ or $(61.03, 98.97)$.

3.160 Write

$$(q + pe^t)^n \left[q + p\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right)\right]^n = \left(q + p + pt + p\,\frac{t^2}{2!} + p\,\frac{t^3}{3!} + \ldots\right)^n$$
$$= \left(1 + pt + p\,\frac{t^2}{2!} + p\,\frac{t^3}{3!} + \ldots\right)^n$$

The terms that are of interest in this exercise are only those terms that contain either t or t^2, since we are interested in obtaining only μ_1' and μ_2', the coefficients of t and $\frac{t^2}{2!}$. Hence we need only to expand the above multinomial to show the first few terms. Then

$$(q + pe^t)^n = \left[1^n + n(pt)(1)^{n-1} + n\left(p\frac{t^2}{2!}\right)(1)^{n-1} + \frac{n(n-1)}{2}(pt)^2(1)^{n-2}\right.$$
$$\left. + \text{(terms involving } t^3 \text{ and higher powers)}\right].$$

Recall that the multinomial coefficient was given in Chapter 2 as

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

where n_i represents the exponent given in the i^{th} member of the multinomial sum in the particular term we wish to evaluate. For example, the fourth term in the above expansion is actually

$$\frac{n!}{2!\,(n-2)!\,0!\,0!\cdots}(1)^{n-2}(pt)^2\left(p\,\frac{t^2}{2!}\right)^0\left(p\,\frac{t^3}{3!}\right)^0\cdots = \frac{n(n-1)}{2}\,p^2t^2$$

Thus the coefficient of t is $np(1)^{n-1} = np$. The term involving t^2 is

$$np(1)^{n-1}\,\frac{t^2}{2} + n(n-1)p^2\,\frac{t^2}{2}$$

so that the coefficient of $\frac{t^2}{2}$ is $np + n(n-1)p^2$, which agrees with the results of Exercise 3.100.

(4)

3.162 Let W = # of drivers who wish to park, and W' = # of cars, which is Poisson with mean λ. We consider

$$\begin{aligned}
P(W=k) &= \sum_{n=k}^{\infty} P(W=k \cap W'=n) = \sum_{n=k}^{\infty} P(W=k|W'=n)\,P(W'=n) \\
&= \sum_{n=k}^{\infty} \left\{ \left[\frac{n!}{k!\,(n-k)!}\right] p^k(1-p)^{n-k} \right\} \left[e^{-\lambda}\left(\frac{\lambda^n}{n!}\right)\right] \\
&= \lambda^k e^{-\lambda}\left(\frac{p^k}{k!}\right) \sum_{n=k}^{\infty} \left[\frac{(1-p)^{n-k}}{(n-k)!}\right] \lambda^{n-k} \\
&= \left(\frac{\lambda^k p^k e^{-\lambda}}{k!}\right) \sum_{j=0}^{\infty} \frac{[(1-p)\lambda]^j}{j!} = \left[\frac{(\lambda p)^k}{k!}\right] e^{-\lambda} e^{(1-p)\lambda} \\
&= \left[\frac{(\lambda p)^k}{k!}\right] e^{-\lambda p}.
\end{aligned}$$

Where $j = n - k$, in line 4 of the previous equation.

a. If W = # of drivers who wish to park, then the probability that a space will still be available when you reach the lot = $P(W = 0)$. Using the information that was derived above,

$$P(W=0) = \left[\frac{(\lambda p)^0}{0!}\right] e^{-\lambda p} = e^{-\lambda p}.$$

b. Using the information derived above, we see that the probability distribution for W is Poisson with mean λp.