Problem Set X

9.62 Since $\mu_1' = \lambda$, we equate $\overline{Y} = \Sigma \frac{Y_i}{n}$ to μ_1' and obtain $\hat{\lambda} = \overline{Y}$.

9.65 Notice for this problem we have one hypergeometric observation, Y. Notice $E(Y) = n\theta/N$. This implies our method of moments estimator is $\hat{\theta} = YN/n$.

9.70 $E(Y) = \int_0^3 \frac{\alpha y^\alpha}{3^\alpha}\, dy = \left(\frac{\alpha}{3^\alpha}\right)\left(\frac{y^{\alpha+1}}{\alpha+1}\right)\Big]_0^3 = \frac{3\alpha}{\alpha+1} = \overline{Y}$.

$\Rightarrow 3\alpha = \alpha\overline{y} + \overline{y}$

$\Rightarrow (3 - \overline{y})\,\alpha = \overline{y}$

$\Rightarrow \hat{\alpha} = \frac{\overline{Y}}{3-\overline{Y}}$.

9.76 a. As this exercise is a special case of exercise 9.77 **a** (with $\alpha = 2$) we will refer to its results.

$$\hat{\theta} = \left(\frac{\overline{Y}}{2}\right) = \frac{378}{3(2)} = 63.$$

b. From Exercise 9.69 **b**,

$$E(\hat{\theta}) = \theta \qquad V(\hat{\theta}) = \frac{\theta^2}{n\alpha} = \frac{\theta^2}{3(2)} = \frac{\theta^2}{6}$$

c. The bound on the error of estimation is

$$2\sqrt{V(\hat{\theta})} = 2\sqrt{\frac{\theta^2}{6}} = 2\sqrt{\frac{(130)^2}{6}} = 106.14$$

d. The variance of Y is $2\theta^2$. The MLE of θ was found in part a to be $\hat{\theta} = 63$. Therefore, the MLE for the variance is $2(63)^2 = 7938$.

9.77 a. The likelihood function, defined as the joint density of $Y_1, Y_2, \ldots, Y_n$ evaluated at $y_1, y_2, \ldots, y_n$, is given by

$$L = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} y_i^{\alpha-1} e^{-y_i/\theta} = \frac{1}{[\Gamma(\alpha)]^n \theta^{n\alpha}} e^{-\Sigma y_i/\theta} \prod_{i=1}^{n} y_i^{\alpha-1} = K\left(\frac{1}{\theta^{n\alpha}}\right) e^{-\Sigma y_i/\theta}$$

where K is a constant, independent of θ. Then $\ln L = \ln K - n\alpha \ln\theta - \left(\frac{\Sigma y_i}{\theta}\right)$, and if α is known,

$$\frac{d}{d\theta} \ln L = \frac{\Sigma y_i}{\theta^2} - \frac{n\alpha}{\theta}$$

Equating the derivative to 0, we obtain $\hat{\theta}$.

$$\frac{\Sigma y_i}{\hat{\theta}^2} - \frac{n\alpha}{\hat{\theta}} = 0 \qquad \text{or} \qquad \hat{\theta} = \frac{\Sigma Y_i}{n\alpha} = \frac{\overline{Y}}{\alpha}$$

b. Taking expectations and recalling that $E(Y_i) = \alpha\theta$ and $V(Y_i) = \alpha\theta^2$, we have

$$E(\hat{\theta}) = \frac{\sum_{i=1}^{n} E(Y_i)}{n\alpha} = \frac{n\alpha\theta}{n\alpha} = \theta \qquad \text{and} \qquad V(\hat{\theta}) = \frac{\sum_{i=1}^{n} V(Y_i)}{n^2\alpha^2} = \frac{n\alpha\theta^2}{n^2\alpha^2} = \frac{\theta^2}{n\alpha}$$

c. By the law of large numbers, we know that $\overline{Y}$ is a consistent estimator of $\mu = \alpha\theta$. That is, $\overline{Y}$ converges in probability to $\alpha\theta$. Then, by Theorem 9.2, the quantity $\frac{\overline{Y}}{\alpha} = \hat{\theta}$ converges in probability to $\frac{\mu}{\alpha} = \theta$, so that $\hat{\theta}$ must be a consistent estimator of θ.

d. Using Lehmann and Scheffe's method, we have

$$\frac{L(x_1, x_2, \ldots, x_n|\alpha, \theta)}{L(y_1, y_2, \ldots, y_n|\alpha, \theta)} = \frac{(\Pi x_i)^{\alpha-1} e^{\Sigma x_i/\theta}}{(\Pi y_i)^{\alpha-1} e^{\Sigma y_i/\theta}}$$

In order for this ratio to be free of θ, we need $\Sigma x_i = \Sigma y_i$ so that ΣY_i is the minimal sufficient statistic.

e. Let $U = \sum_{i=1}^{n} Y_i$. The moment-generating function of U is

$$m_U(t) = \prod_{i=1}^{n} m_{Y_i}(t) = \frac{1}{(1-\theta t)^{n\alpha}} = \frac{1}{(1-\theta t)^{10}}.$$

A random variable that possesses a χ^2 distribution is one whose moment-generating function is $\frac{1}{(1-2t)^k}$, where $2k$ are the degrees of freedom. It is necessary to transform U to obtain a random variable with such a moment-generating function. Consider $X = \frac{2U}{\theta}$, with

$$m_X(t) = m_U\left(\frac{2t}{\theta}\right) = \frac{1}{(1-2t)^{10}}$$

Hence $X = \frac{2U}{\theta}$ has a χ^2 distribution with $2(10) = 20$ degrees of freedom. Using X as a pivotal statistic, write

$$P\left(\chi^2_{.05,\,20} < \frac{2U}{\theta} < \chi^2_{.95,\,20}\right) = .90$$

or

$$P\left(\frac{2U}{\chi^2_{.95,\,20}} < \theta < \frac{2U}{\chi^2_{.05,\,20}}\right) = .90.$$

Then the 90% confidence interval is $\left[\frac{2\sum_{i=1}^{n} Y_i}{31.41}, \frac{2\sum_{i=1}^{n} Y_i}{10.85}\right]$.

(not assigned)

9.88 It was shown in Example 9.15 that the maximum-likelihood estimator of σ^2 is s^2. Using this result, we give as the maximum-likelihood estimator for σ to be s.

7.23 The Central Limit Theorem (Theorem 7.4) states that $Y_n = \frac{\sqrt{n}(\overline{X}-\mu)}{\sigma}$ converges in distribution to a standard normal random variable, which is denoted by Z. For this exercise, $n = 100$, $\sigma = 2.5$, and the approximation is

$$P\left(|\overline{X}-\mu| \le .5\right) = P\left(-.5 \le \overline{X}-\mu \le .5\right) = P\left[\frac{-.5(10)}{2.5} \le Z \le \frac{.5(10)}{2.5}\right]$$
$$= P(-2 \le Z \le 2) = 1 - 2(.0228) = .9544$$

7.37 Let X_i be the length of life for the i^{th} heat lamp, $i = 1, 2, \dots, 25$. It is given that the X_i's are independent, each with mean 50 and standard deviation 4. Then by the Central Limit Theorem, the random variable

$$Y_n = \frac{\sqrt{n}(\overline{X}-\mu)}{\sigma} = \frac{n(\overline{X}-\mu)}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \frac{\Sigma X_i - n\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)}$$

converges in distribution to a standard normal random variable. Hence, since the lifetime of the lamp system is represented by $V = \sum_{i=1}^{25} X_i$, the probability of interest is

$$P\left(\sum_{i=1}^{25} X_1 \ge 1300\right) = P\left(\frac{\Sigma X_i - n\mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} \ge \frac{1300-1250}{\sqrt{400}}\right) = P(Z > 2.5) = .0062$$

7.38 It is given that $X_1, X_2, \dots, X_n$ are independent and identically distributed with $E(X_i) = \mu_1$ and $V(X_i) = \sigma_1^2$. Similarly, $Y_1, Y_2, \dots, Y_n$ are independent and identically distributed with $E(Y_i) = \mu_2$ and $V(Y_i) = \sigma_2^2$. Consider $d_i = X_i - Y_i$, for $i = 1, 2, \dots, n$. The d_i's are independent and identically distributed with $E(d_i) = E(X_i) - E(Y_i) = \mu_1 - \mu_2$ and $V(d_i) = V(X_i) + V(Y_i) = \sigma_1^2 + \sigma_2^2 < \infty$. Hence, applying Theorem 7.4 to the set $d_1, d_2, \dots, d_n$, we have

$$Y_n = \frac{[\overline{d}-(\mu_1-\mu_2)]\sqrt{n}}{\sqrt{\sigma_1^2+\sigma_2^2}} = \frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2+\sigma_2^2}{n}}}$$

which converges in distribution to a standard normal random variable.

7.39 Use the results of Exercise 7.38. It is given that $n = 50$, $\sigma_1 = \sigma_2 = 2$, and $\mu_1 = \mu_2$. Let $\overline{X}$ be the mean for operator B and $\overline{Y}$ be the mean for operator A. Then the probability of interest is

$$P(\overline{X}-\overline{Y} > 1) = P\left[\frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2+\sigma_2^2}{n}}}\right] > \frac{1-0}{\sqrt{\frac{4+4}{50}}} = P(Z > 2.5) = .0062$$

7.42 Let X_i represent the time to process the i^{th} person's order, where $i = 1, 2, \dots, 100$. X_i has $\mu = 2.5$ minutes and $\sigma = 2$ minutes. Since 4 hours = 240 minutes, we consider

$$P\left(\sum_{i=1}^{100} X_i > 240\right) = P\left(\overline{X} > \tfrac{240}{100}\right) = P\left(\overline{X} > 2.4\right)$$
$$= P\left(Z > \frac{2.4-2.5}{2}\right) = .6915.$$

8.36 Use the fact that $Z = \frac{Y-\mu}{\sigma} = Y - \mu$ has a standard normal distribution.

a. The 95% confidence interval for μ is $(Y - 1.96, Y + 1.96)$ since

$$P(-1.96 \le Z \le 1.96) = .95$$
$$P(-1.96 \le Y - \mu \le 1.96) = .95$$
$$P(Y - 1.96 \le \mu \le Y + 1.96) = .95$$

b. Since

$$P(Z \le -1.645) = .05$$
$$P(Y - \mu \le -1.645) = .05$$
$$P(\mu \ge Y + 1.645) = .05$$

Hence $Y + 1.645$ is the 95% upper limit for μ.

c. Similarly, $Y - 1.645$ is the 95% lower limit for μ.

8.37 a. $.95 = P\left(\chi^2_{.975} \le \frac{Y^2}{\sigma^2} \le \chi^2_{.025}\right) = P\left(.0009821 \le \frac{Y^2}{\sigma^2} \le 5.02389\right)$

$= P\left(\frac{Y^2}{5.02389} \le \sigma^2 \le \frac{Y^2}{.0009821}\right)$

b. $.95 = P\left(\chi^2_{.95} \le \frac{Y^2}{\sigma^2}\right) = P\left(\sigma^2 \le \frac{Y^2}{.0039321}\right)$

c. $.95 = P\left(\frac{Y^2}{\sigma^2} \le \chi^2_{.05}\right) = P\left(\sigma^2 \ge \frac{Y^2}{3.84146}\right)$

8.42 a. $\hat{p} = \frac{268}{500} = .536$. Therefore, an approximate 98% confidence interval for p is

$$\hat{p} \pm z_{.01}\sqrt{\frac{\hat{p}\hat{q}}{n}} = .536 \pm 2.33\sqrt{\frac{(.536)(.464)}{500}} = .536 \pm .052 \text{ or } (.484, .588).$$

b. Since the interval does include $p = .51$, we cannot conclude that there is a difference in the graduation rates before and after Proposition 48.

8.46 We are given that $n = 75$, $\bar{y} = 4.2$, $s = 1.5$, and $\alpha = .05$. Then $z_{.025} = 1.96$ and hence a 95% confidence interval for the average biomass for North America's northern forests is

$$\bar{y} \pm z_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right) = 4.2 \pm 1.96\left(\frac{1.5}{\sqrt{75}}\right) = 4.2 \pm .34 = (3.86, 4.54).$$

8.48 The approximate 99% confidence interval is, since $z_{.005} = 2.575$,

$$(\bar{y}_1 - \bar{y}_2) \pm 2.575\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = (24.8 - 21.3) \pm 2.575\sqrt{\frac{(7.1)^2}{34} + \frac{(8.1)^2}{41}}$$
$$= 3.5 \pm 4.52 \quad \text{or} \quad (-1.02, 8.02).$$

The difference in mean molt time for "Normal" males versus those "split" from their mates is $(-1.030, 8.030)$ with 99% confidence.