Sample Exam Problems, AM165, 6 Nov 2001: This first seven problems below constitute a midterm exam that was given in a two hour period in the evening at about this point in the course syllabus. The last five problems make up another midterm exam that was also given in a two hour period in the evening.

In order for this to be a meaningful practice for the midterm on November 13, we will *not* make sample solutions available, but we will gladly answer any questions you may have on these problems.

A1. A random variable Y has probability density function, $f(y)$, given by

$$f(y) = \begin{cases} c(1-y^2) & |y| < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is c?

(b) What is the cumulative distribution function of Y, $F(y)$?

(c) What are the mean and variance of Y?

A2. Suppose we flip a fair coin 100 times and let Y be the number of heads that come up. An upcoming highlight of this course advises us to approximate the distribution of Y as a normal distribution with the same mean and variance as Y.

(a) Write an expression for the exact probability of getting between (and including) 43 and 57 heads?

(b) Using the suggested approximation, what is this probability?

(c) What does Tchebychev's theorem say about this probability?

A3. Compute $E[X]$:

(a)

$$F(x) = P(X \le x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{1}{4}x^2 & \text{if} \quad 0 \le x \le 2 \\ 1 & \text{if} \quad x > 2 \end{cases}$$

(b)

$$F(x) = P(X \le x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 2 - \frac{2}{x} & \text{if} \quad 1 \le x \le 2 \\ 1 & \text{if} \quad x > 2 \end{cases}$$

A4. $X_1, X_2, \ldots, X_{100}$ are iid random variables with unknown distribution, but with known mean and variance:

$$E[X_1] = 1 \quad (1)$$

$$Var(X_1) = 1. \quad (2)$$

What can you say about

$$P(80 \leq \sum_{i=1}^{100} X_i \leq 120)?$$

Can you say two things?

A5. X ia a random variable with uniform distribution on the interval $[-\frac{1}{2}, \frac{1}{2}]$. If $Y = X^2$,

(a) What is the density of Y?

(b) What is $E[Y]$?

A6. A random variable X has density of the form

$$f(x) = \begin{cases} cx^{3/2}e^{-x} & \text{for} \quad 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

where c is a normalizing constant. Compute $E[X^2]$.

A7. A deck of cards has 13 different ranks each of which appears in 4 different suits making a total of 52 cards. Two decks are shuffled and placed on a table. Let Y be the number of positions where the i^{th} card in the first deck is the same suit as the i^{th} card in the second deck. What is $E(Y)$?

hint: remember the matching problem

B1. At a theatrical production the program notes instruct us that, after the opening curtain, we will wait Y hours before Godot appears on stage. The notes also advise us that Y has an exponential distribution with mean 1/2. The play lasts one hour and Godot never appears.

(a) What is the probability of this?

(b) Someone says that if the play had continued 15 more minutes, she is *sure* Godot would have appeared. What is the probability that she is right, in view of what we have already experienced?

B2. A unit of a certain product is considered to be of high quality if the deviation of its dimensions from the standard does not exceed 3.45 mm in absolute value. The random deviation X in this dimension is normally distributed with standard deviation 3 mm and with zero mean. Determine:

(a) The probability that a randomly selected unit of this product is of high quality, and

(b) The average number of units of high quality from a total of four units produced.

B3. Suppose that a particular state allows individuals filing tax returns to itemize deductions only if the total of all itemized deductions is at least \$5,000. Let X (in thousands of dollars) be the total of itemized deductions on a randomly chosen form. Assume that X has the density function

$$f_\alpha(x) = \begin{cases} k/x^\alpha & \text{for} \quad x \geq 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of k. What restriction on α is necessary?

(b) What is the cdf of X?

(c) What is the expected deduction on a randomly chosen form? What restriction on α is necessary for $E[X]$ to be finite?

(d) Find

$$P(\ln(X/5) \leq x).$$

B4.

(a) The probability of occurrence of an event A in one trial of an experiment is $1/2$. Can you be absolutely certain that with probability greater than 0.97 the number of occurrences of A in 1000 independent trials will within the limits 400 to 600?

(b) In order to calibrate a scales, an experiment is performed to determine the mean μ of the *measured weight* X of a standard 10 lb weight. Independent (iid) measurements $X_1, X_2, \ldots, X_n$ will be made. ¿From prior experience, it is known that the standard deviation of X_i is $\sigma = 0.1$ lb. How many measurements need to be made (n) to be absolutely certain that $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ will be within 0.01 of μ (in absolute value) with probability (at least) 0.96?

B5. Suppose that your waiting time for a bus in the morning is uniformly distributed on $[0, 5]$, while waiting time in the evening is uniformly distributed on $[0, 10]$ independent of morning waiting time.

(a) If you take the bus each morning and evening for five consecutive days (a working week), what is your total expected waiting time?

(b) What is the variance of your total waiting time during the working week?

(c) What are the expected value and variance of the difference between morning and evening waiting time on a given day?

(d) What are the expected value and variance of the difference between total morning waiting time and total evening waiting time for a particular working week?