THRESHOLD GRAPHS, SHIFTED COMPLEXES, AND GRAPHICAL COMPLEXES

CAROLINE J. KLIVANS

Abstract. We consider a variety of connections between threshold graphs, shifted complexes, and simplicial complexes naturally formed from a graph. These graphical complexes include the independent set, neighborhood, and dominance complexes. We present a number of structural results and relations among them including new characterizations of the class of threshold graphs.

1. Introduction

Threshold graphs are a well-studied class of graphs motivated from numerous directions. They were first introduced by Chvátal and Hammer [1] as graphs for which there exists a linear threshold function separating independent from non-independent sets. Since then many equivalent conditions have been found for threshold graphs including constructive forms and forbidden configurations. See for example [11] for nine different characterizations.

In generalizing to higher dimensions, it is then natural to consider which characterizations remain equivalent. Golumbic first considered such generalizations of threshold graphs to higher dimensions (or hypergraphs) [5]. He specifically highlighted three analogs and asked if they were in fact the same. It turns out that these three do not lead to the same class of complexes [12]. One of these analogs does give the class known as shifted simplicial complexes. We will primarily consider threshold graphs from this perspective; that they are exactly the one-dimensional shifted complexes. See also [9] which considers generalizations of threshold graphs based on degree sequence properties and [2] for a simple games/voting theory perspective.

Shifted complexes are simplicial complexes whose faces form an order ideal in the component-wise partial order. (See section 1.1 for precise definitions and examples of shifted complexes and threshold graphs.) Shifted complexes are named as such because of the existence of shifting operations. In general, a shifting operation associates a shifted complex to any simplicial complex in a way which preserves certain combinatorial properties but simplifies other structure. The original form of shifting, now known as combinatorial shifting, was first introduced by Erdös, Ko, and Rado [4] and Kleitman [7]. More recently, Kalai [6] introduced algebraic shifting and spurred new interest in shifted complexes.

We study a variety of simplicial complexes naturally formed from a simple graph. In many cases, these graphical complexes turn out to be shifted if and only if the graph is threshold. We thus further motivate shiftedness as a natural generalization of threshold graphs.

We first look at the independent set (or stable set) complex of a graph and show that it is shifted if and only if the graph is threshold. Using this, we determine a constructible form for such complexes in terms of two simple operations. Independent set complexes of graphs are also known as flag complexes. Combining this perspective and the construction, it is shown that pure shifted flag complexes are the same as pure shifted balanced complexes.

Next we consider a generalized procedure to form the independent set complex of an arbitrary simplicial complex as in [3]. This construction again yields a shifted complex if and only if we start with a shifted complex. Finally, we end with a result which shows that the dominance complex of a graph equals the neighborhood complex if and only if the graph is threshold.

1.1. **Definitions and Preliminaries.**

Definition 1. *A simplicial complex on n vertices is* shifted *if there exists a labeling of the vertices by one through n such that for any face $\{v_1, v_2, \ldots, v_k\}$, replacing any v_i by a vertex with a smaller label results in a collection which is also a face.*

An equivalent formulation of shifted complexes is in terms of order ideals. An order ideal I of a poset P is a subset of P such that if x is in I and y is less than x then y is in I. Let P_s be the partial ordering on strings of increasing integers given by $x = (x_1 < x_2 < \cdots < x_k)$ is less than $y = (y_1 < y_2 < \cdots < y_k)$ if $x_i \le y_i$ for all i and $x \neq y$. Shifted complexes are exactly the order ideals of P_s. We also allow comparisons of strings of various lengths by considering the shorter string to have the necessary number of initial zeros (slightly abusing that we are otherwise comparing strictly increasing strings). For example the string 24 is taken to be less than the string 1356 by considering 24 as 0024.

Example 1. A simplicial complex which includes the face $\{24\}$ must also have the face $\{14\}$ in order to be shifted (see Figure 1).

One-dimensional shifted complexes are known to be the same as threshold graphs [6]. Threshold graphs are graphs that can be given a vertex weighting which differentiates between independent and non-independent sets. An *independent set* of a graph is a collection of vertices no two of which are connected.

Definition 2. *A graph is threshold if for all $v \in V$ there exists weights $w(v)$, and $t \in \mathbb{R}$ such that the following condition holds: $w(U) \le t$ if and only if U is an independent set, where $w(U) = \sum_{v \in U} w(v)$ (see Figure 2).*

One of the many characterizations of threshold graphs is constructive. The construction is in terms of two basic operations; starring a vertex and adding a disjoint vertex. Starring a vertex v onto a graph $G = (V, E)$ forms the new graph:

$$G \text{ star } v \ = \ (V \cup \{v\}, E \cup \{\{x, v\} : x \in V\}).$$

Starring adds a new vertex adjacent to all previous vertices.

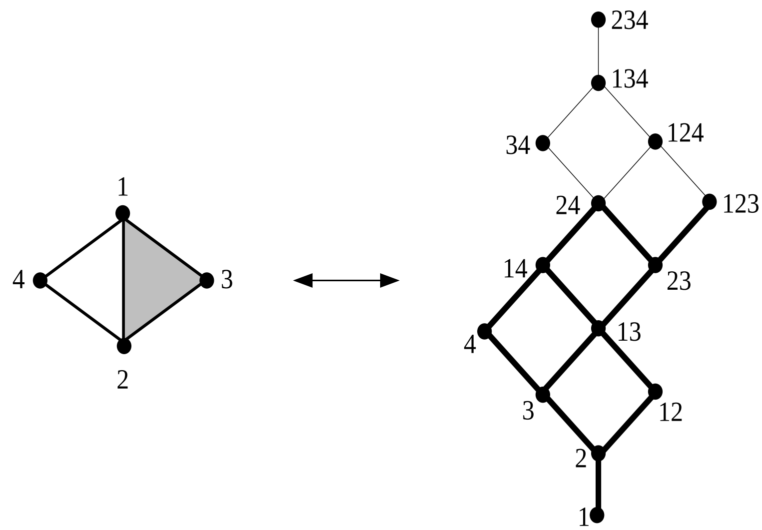

FIGURE 1. An example of a shifted complex.

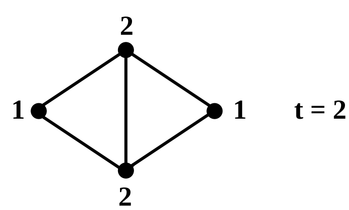

FIGURE 2. A threshold graph with threshold 2.

Theorem 1 ([11] Theorem 1.2.4)**.** *A graph is threshold if and only if it can be constructed from the one-vertex graph by repeatedly adding a disjoint vertex or a starred vertex.*

We want to extend the notion of starring a vertex to arbitrary dimensions. Namely, we will say a vertex v is starred in dimension d onto a complex K by forming the complex:

$$K \text{ star}_d\ v \ = \ K \cup \{v \cup f \mid f \in K \text{ and } |f| \leq d\}.$$

Note that this operation is not the same as coning. Coning corresponds to the special case of starring a vertex in dimension one more than the dimension of the complex. Coning will always increase the dimension of a complex whereas starring does not necessarily increase the dimension. For example, let K be the two-dimensional simplex $\{123\}$. K star$_2$ 4 is the two-dimensional boundary complex of the three-dimensional simplex. On the other hand, K star$_3$ 4 is the three-dimensional simplex $\{1234\}$.

We will represent complexes generated by these two operations as strings of Ds (disjoint), Ss (starring), and vertical lines | (for dimension increase).

Example 2. Consider the string $DDSS|SSD|S$. This represents the complex formed as follows: place two disjoint vertices, star two vertices in dimension 1, star two vertices in dimension 2, add a disjoint vertex, and star one vertex in dimension 3.

Note that the string $DDSS|SS|DS$ would give the same complex. For consistency, we will always place a vertical bar only immediately preceding an S. Also note that it does not matter whether we begin a string with an S or a D. Again for consistency, we will always start a string with a D.

Given a complex represented by such a string, a shifted labeling can easily be obtained. Suppose K is a complex with n vertices, k of which were added by starring. Label the vertices corresponding to S operations by 1 through k from right to left along the string. Label the vertices represented by D operations by $k+1$ through n from left to right along the string. The string above would give:

$$\frac{D}{6}\frac{D}{7}\frac{S}{5}\frac{S}{4}\frac{|}{}\frac{S}{3}\frac{S}{2}\frac{D}{8}\frac{|}{}\frac{S}{1}.$$

While all complexes formed this way are shifted, not all shifted complexes can be constructed by repeated application of these two operations. For example the complex of Figure 1 does not have this form.

2. Independence Complex of a Graph

Recall that an *independent set* (or stable set) of a graph is a collection of vertices no two of which are connected by an edge. Let $I(G)$ denote the independence complex of a graph G. This complex is formed by taking the collection of independent sets of G. Clearly removing a node from an independent set results in an independent set so this collection is a simplicial complex.

Theorem 1. *$I(G)$ is shifted if and only if G is a threshold graph.*

Proof. Let G be a threshold graph. Then we know G is shifted. Let l be a shifted labeling of the vertices of G. Consider any face $F = \{v_1, v_2 \dots v_k\}$ of $I(G)$ and a vertex w such that $l(w) > l(v_i)$ for some i. We will show that $F' = \{v_1, v_2, \dots, \hat{v}_i, w, \dots, v_k\}$ is a face of $I(G)$. If not, then w must be connected to some v_j $(j \neq i)$ in G. Because w has a larger label than v_i and $\{wv_j\} \in E(G)$, $\{v_iv_j\}$ must be an edge of G in order for G to be shifted. But this contradicts F being a face of $I(G)$. Hence $I(G)$ is shifted under the reverse ordering of l.

Now let $I(G)$ be shifted and l a shifted labeling. Consider any edge, $\{v_1v_2\}$ of G and a vertex w such that $l(w) > l(v_2)$. We will show that $\{v_1w\}$ is an edge of G and hence G is shifted again under the reverse ordering of l. If not, then $\{v_1w\}$ is an independent set of G and hence a face of $I(G)$. $I(G)$ is shifted and v_2 had a smaller label than w which means $\{v_1v_2\}$ must be a face of $I(G)$ and not an edge of G, again a contradiction.

□

2.1. Flag complexes.

Flag complexes. Independent set complexes of graphs are also known as flag complexes. A *flag* complex is defined as a simplicial complex such that every minimal non-face has exactly two elements [13]. By the previous result, all shifted flag complexes are formed from threshold graphs. Using both perspectives allows us to further determine the form of these complexes.

Theorem 2. *Shifted flag complexes are the complexes formed by the operations D and S with exactly one S in each dimension.*

Proof. Every shifted flag complex arises as the independence complex of a threshold graph. Every threshold graph can be represented as a string of Ds and Ss. Consider mapping this string under the following rules: $D \to |S$ and $S \to D$. Namely, switch every S to a D and switch every D to an S and also increase the dimension with every such switch.

Example 3. $DDSDSDSSD \to S|SD|SD|SDD|S$

First, we want to determine the independent sets of a threshold graph from its string of Ds and Ss. The maximal independent sets are the set of all Ds and all collections which consist of a single S and all Ds that come after it.

Next, given the image of the string, we want to determine its facets. They are the set of all Ss and all collections which consist of a D and all Ss that come after it. In particular they are exactly the independent sets of G.

This procedure is invertible showing that all strings of Ds and Ss with exactly one S in each dimension are flag complexes.

□

2.2. **Balanced complexes.** A d-dimensional simplicial complex is *balanced* if its vertices can be colored with $d+1$ colors such that within any face all vertices have different colors.

Proposition 1. *All shifted flag complexes are balanced.*

Proof. Let K be a d-dimensional shifted flag complex. We give an explicit balanced labeling. K can be represented as a string of Ds and Ss with exactly one star operation per dimension. Label the vertices with $d+1$ colors as shown below:

$$\underbrace{DD\ldots D}_{1}\underbrace{SD\ldots D}_{2}|\ldots|\underbrace{SD\ldots D}_{d}|\underbrace{SD\ldots D}_{d+1}.$$

Every face of K consists of one initially placed disjoint vertex and a set of starred vertices which come after it, all of which have been given a different color.

□

The converse of the proposition above is false: not all shifted balanced complexes are flag complexes. A simple example is the complex on 4 vertices with maximal faces $\{123, 14, 24\}$ (see Figure 1). Notice that this complex is not pure.

A pure shifted flag complex has a very simple form:

$$DD\ldots DS|S|S\ldots|S|S.$$

This yields a "pencil of facets". Namely, a d-dimensional pure shifted flag complex on n vertices consists of $n-d$ facets all sharing a common $d-1$ face.

Theorem 3. *A pure shifted complex is balanced if and only if it is a flag complex.*

Proof. We already know flag implies balanced. We will show any pure shifted balanced complex is also a "pencil of facets". Let K be a d-dimensional pure, shifted, and balanced complex with a shifted labeling of its vertices. Shiftedness implies that $\{1, 2, \ldots, d+1\} \in K$. Let $\{x_1, x_2, \ldots, x_{d+1}\}$ $(x_1 < x_2 \cdots < x_{d+1})$ be some other facet. Suppose $x_d > d$. Then x_{d+1} must be greater than $d+1$ and vertex $d+2$ must be adjacent to $d+1$. But then shiftedness implies that the complete graph on $d+2$ vertices is in the 1-skeleton of K which contradicts K being balanced. Hence x_d must equal d and all facets have the form $\{1, 2, \ldots, d, x\}$ $d < x \leq n$. Thus K has the same form as a pure shifted flag complex.

□

2.3. **Shifting.** Recall that in general, a shifting operation associates a shifted complex to any simplicial complex in a way which preserves certain combinatorial properties but simplifies other structure. In particular, both combinatorial and algebraic shifting preserve the f-vector.

A conjecture due to Kalai asks if any f-vector of a flag complex can also be realized as the f-vector of a balanced complex [13]. We note here for completeness the relationship between shifting and the properties of flag and balanced. The two main variants of algebraic shifting unfortunately do not preserve flag or balanced complexes. For definitions and much more on these shifting operations see [6]. Consider the complete bipartite graph $K_{3,3}$. It is easy to check that this is both a flag and balanced complex. Symmetric shifting yields the complex generated by top face $\{26\}$ and exterior shifting yields the complex generated by top faces $\{25\}$ and $\{34\}$. In both graphs, the collection $\{123\}$ is a minimal non-face showing it is not a flag complex and not balanced.

Moreover, no shifting operation which preserves the f-vector could preserve these properties. The graph $K_{3,3}$ has 6 vertices, 9 edges, and no faces of dimension 2 or greater. But any order ideal in the shifted partial order on 6 vertices with 9 one-dimensional faces will include the edges $\{12\}$, $\{13\}$, and $\{23\}$. Hence the graph will not be balanced and since we can not add any two-dimensional faces, this will generate a minimal non-face with three elements.

3. Generalized Independence Complex

In [3], forming the independence complex of a graph is generalized to arbitrary simplicial complexes. For a simplicial complex K, define $I(K)$ by declaring the facets of K to be the minimal non-faces of $I(K)$. (The independent set complex in [3] is defined in greater generality, allowing for K to be a set system which is not necessarily a simplicial complex.)

We start by considering the independent set complex of shifted simplicial complexes. The general statement that K is shifted if and only if $I(K)$ is shifted is false in both directions. It is not hard to construct counter-examples using non-pure complexes. For example, let K be the simplicial complex on 5 vertices with maximal faces $\{123, 14, 24, 15\}$. K is shifted but $I(K)$ which has maximal faces $\{235, 345, 12, 13\}$ is not. The induced subcomplex on vertices $\{1, 2, 4, 5\}$ is a path of length three which is an obstruction to shiftedness in dimension one. We can continue to apply the procedure to disprove the other direction. $I(I(\Delta))$ is generated by $\{245, 234, 145, 35\}$ which is also not shifted. $I(I(I(\Delta)))$ generated by $\{123, 124, 125, 134, 45\}$ is however shifted (mapping 3 to 4 gives a shifted labeling).

Restricting to the pure case is actually a more natural generalization of the independence complex of a graph. The generalized procedure only restricts to the same procedure on graphs if the graph is connected (i.e. pure). For example, if we have a graph with disjoint vertices, under the generalized procedure they would be minimal non-faces of $I(K)$. On the other hand, a disjoint vertex is in all maximal faces of the independence complex of the graph. In the pure case, we come to the following result:

Theorem 4. *For K pure, K is shifted if and only if $I(K)$ is shifted.*

Proof. Suppose K is shifted but $I(K)$ is not shifted. Then there exists $x, y, f_1, f_2 \in I(K)$ such that $xf_1, yf_2 \in I(K)$ and $yf_1, xf_2 \notin I(K)$, where x and y are vertices and f_1 and f_2 are faces, see [8]. Since yf_1 and xf_2 are not in $I(K)$, they must be facets or contain facets of K. First we note that the facets involved here must not be strictly contained in f_1 and f_2, or xf_2 and yf_1 could not be in $I(K)$.

Suppose yf_1 and xf_2 are facets of K. Let l be a shifted labeling for K and without loss of generality, let $l(x) < l(y)$. Since K is shifted, we have that $xf_1 \in K$. But, $|xf_1| = |yf_1|$ which implies xf_1 is a facet of K and can not be in $I(K)$ - a contradiction.

Suppose at least one of yf_1 and xf_2 is not a facet of K. They still must contain a facet. Let $g_1 \subseteq f_1$, $g_2 \subseteq f_2$, and xg_2, yg_1 be facets of K. They will not be in $I(K)$, but $yg_2 \subseteq yf_2 \in I(K)$ and $xg_1 \subseteq xf_1 \in I(K)$ so we are back in the first case.

Now suppose $I(K)$ is shifted but K is not shifted. Then there exists x, y, f_1, f_2 such that $xf_1, yf_2 \in K$ and $yf_1, xf_2 \notin K$. Because K is pure, we may take xf_1 and yf_2 to be maximal faces; in particular this gives that $|xf_1| = |yf_2|$. Now since xf_1 and yf_2 are facets of K, they are not in $I(K)$. Next consider xf_2 and yf_1. For these faces not to be in $I(K)$, they must contain facets. However, $|xf_2| = |yf_2| = |xf_1| = |yf_1|$ so if they contained a facet it would be of smaller size, and this can not be because K is pure. Hence xf_2 and yf_1 are in $I(K)$, which contradicts $I(K)$ being shifted.

□

3.1. **Neighborhood and Dominance.** A *dominating* set of a graph is a set of vertices D such that all vertices are either in D or adjacent to a vertex in D. The *dominance complex* $D(G)$ is the collection of **complements** to dominating sets [3]. Note that this is because dominating sets are closed under superset as opposed to subsets.

In [3] the dominance complex is studied for specific graphs and it is observed that $D(G)$ is the independent set complex of the collection of closed neighborhoods $N[v]$ of G. (The closed neighborhood of a vertex v is the usual neighborhood $N(v)$ union v itself). If we define the *closed neighborhood complex* $N[G]$ to be the simplicial complex with facets equal to the minimum (under inclusion) sets of the collection of closed neighborhoods of G, then $I(N[G])$ as we have defined $I(K)$ matches that of [3].

By the previous result, we would hope to show that G is threshold if and only if $N[G]$ is shifted, and hence $D(G) = I(N[G])$ is shifted if and only if G is threshold. This is unfortunately not the case.

We do however offer a curious relationship between these and the usual neighborhood complex of Lovasz [10]. Let $N(G)$ be the collection of sets of vertices which share a common neighbor.

Theorem 5. *$N(G) = D(G)$ (and therefore $I(N[G])$) if and only if G is threshold*

Proof. First we claim that $N(G) \subseteq D(G)$ for any graph. Suppose $\{x_1, x_2, \ldots, x_k\} \in N(G)$. Let v be their common neighbor. Now $v \in V \setminus \{x_1, x_2, \ldots, x_k\}$ so all vertices are either in the complement or adjacent to a vertex in the complement, hence it dominates.

Next we show that $D(G) \subseteq N(G)$ if G is threshold. Let G be threshold with a shifted labeling l and let $\{x_1, x_2, \ldots, x_k\} \in D(G)$. We need to show that the $\{x_i\}$ have a common neighbor. Without loss of generality let x_k have the largest label among the x_i. Because $V \setminus \{x_1, x_2, \ldots, x_k\}$ dominates, every x_i is adjacent to some vertex in $V \setminus \{x_1, x_2, \ldots, x_k\}$. Let x_k be adjacent to some vertex v. Because x_k has the largest label and G is threshold, v is a common neighbor to all the xs.

Finally, we show that if $N(G) = D(G)$ then G is threshold. We will do this by showing that G is constructed by repeatedly starring or adding a disjoint vertex. Let G be such that $N(G) = D(G)$.

First, at most one connected component of G has an edge. Suppose more than one connected component had an edge. The complement to any minimal dominating set must contain vertices from different connected components. Hence the complement can not have a common neighbor. Note that a component which is a single vertex is fine, this vertex will be in all dominating sets.

Next consider the connected component with at least one edge, if there is no such component then the graph is a collection of disjoint vertices which is shifted. Otherwise, we claim it has a star vertex. Suppose not, then any minimal dominating set has size at least two. Let $D = \{x_1, x_2, \ldots, x_k\}$ $(k \geq 2)$ be a dominant set of minimal size. Then $V \setminus D$ is a maximal element of $N(G)$ and hence the neighborhood, $N(v)$, of some vertex v. Note that $v \in D$ or else both v and $N(v)$ are in the complement and D could not be dominating. Without loss of generality let $v = x_1$. Consider another vertex y such that x_1y and x_2y are edges of G where x_2 has also been taken without loss of generality. Such a y must exist because we are working in a connected component and the x_is can not be adjacent to each other because D is minimal. y can not be in an edge with any other x_i or else $D \setminus \{x_2, x_i\} \cup \{y\}$ would be a smaller dominating set. Therefore $D' = D \setminus \{x_2\} \cup \{y\}$ is another minimal dominating set. Hence the complement of D' must be the neighborhood of some vertex, say w. Now, w can not equal any x_i or else $\{x_2x_i\}$ is an edge which contradicts the minimality of D. And, $w \neq y$ or else $D' \setminus x_1$ would be dominating which contradicts the minimality of D'. Hence we've reached a contradiction since $w \in D'$ must hold. Therefore the connected component of G with at least one edge has a star vertex.

To finish the proof, we only need to show that if $N(G) = D(G)$ then $N(G \setminus v) = D(G \setminus v)$ for v a star vertex. (Removing any disjoint vertices does not affect either complex). Clearly, $N(G \setminus v) = \{f \setminus v \mid f \in N(G), v \in f\}$. Note that the only maximal face of $N(G)$ which does not contain v is $V \setminus v$.

Similarly, moving from $D(G)$ to $D(G \setminus v)$ we lose the one facet of $D(G)$ corresponding to all vertices except v. Any minimal dominating set for G (other than the set $\{v\}$) is dominating for $G \setminus v$ as well. Therefore $D(G \setminus v)$ also equals $\{f \setminus v \mid f \in D(G), v \in f\}$.

Because threshold graphs are exactly those graphs which can be constructed by repeatedly adding a disjoint vertex and a star vertex, G is threshold.

$\square$

References

[1] V. Chvátal and P. Hammer, Aggregation of inequalities in integer programming, Annals of Discrete Mathematics, vol. 1, 145-162, 1977.

[2] P.H. Edelman and P. Fishburn, Initial segments of linear qualitative probabilities, Preprint, 2006.

[3] R. Ehrenborg and G. Hetyei, The topology of the independence complex, European Journal of Combinatorics, vol.27, 906-923, 2006.

[4] P. Erdös, C. Ko, and R. Rado, Intersection theorems for systems of finite sets, Quar. J. Math. Oxford Ser. (2), vol. 12, 313-320, 1961.

[5] M. C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Computer Science and Applied Mathematics, Academic Press Inc., 1980.

[6] G. Kalai, Algebraic shifting, Computational Commutative Algebra and Combinatorics. Advanced Studies in Pure Mathematics, vol. 33, 121-163, 2002.

[7] D. Kleitman, On a conjecture of Milner on k-graphs with non-disjoint edges,Journal of Combinatorial Theory, vol. 5, 153-156, 1968.

[8] C. Klivans, Obstructions to shiftedness, Discrete and Computational Geometry, vol 33 no. 3 535-545, 2005.

[9] C. Klivans and V. Reiner, On shifted set families, degree sequences, and plethysm, Preprint, 2006.

[10] L. Lovasz, Kneser's conjecture, chromatics number and homotopy, Journal of Combinatorial Theory Series A, vol. 25, 241-251.

[11] N.V.R. Mahadev and U.N. Peled, Threshold Graphs and Related Topics, Annals of Discrete Mathematics, vol. 56, North Holland Press, 1995.

[12] J. Reiterman, V. Rödl, E. Sinajova and M. Tuma, Threshold hypergraphs, Discrete Mathematics, vol 54, no. 2, 1985.

[13] R. Stanley, Combinatorics and Commutative Algebra second edition, Progress in Mathematics, Vol. 41, Birkauser, 1996.

Depts. of Mathematics and Computer Science, The University of Chicago