SHIFTED MATROID COMPLEXES

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Abstract. We study the class of matroids whose independent set complexes are shifted simplicial complexes. We prove two characterization theorems, one of which is constructive. In addition, we show this class is closed under taking minors and duality. Finally, we give results on shifted broken circuit complexes.

1. Introduction

We want to consider those matroids whose complexes of independent sets are shifted. This class of *shifted matroids* has been significantly investigated before, in a variety of guises. They seem to have been first used in [5] by Crapo. He used them to show that there are at least 2^n non-isomorphic matroids on n elements. They were later investigated in [11] as a particular kind of transversal matroid. This work includes a characterization in terms of forbidden minors. Shifted matroids appeared again in [2] and [3] as a kind of lattice path matroid. Mostly closely related to the presentation here is [1], where these matroids arise in the context of Catalan matroids. We refer the reader to [3] for a more complete history of these matroids.

Our approach is to consider the matroid complexes specifically from a shifted perspective. We provide new proofs utilizing the structure of a shifted complex to show the class is constructible, closed under taking minors and duality, and exactly the set of principal order ideals in the shifted partial ordering.

2. Shifted complexes

A simplicial complex on n vertices is *shifted* if there exists a labeling of the vertices by one through n such that for any face $\{v_1, v_2, \ldots, v_k\}$, replacing any v_i by a vertex with a smaller label results in a collection which is also a face.

An equivalent formulation of shifted complexes is in terms of order ideals. An order ideal I of a poset P is a subset of P such that if x is in I and y is less than x then y is in I. Let P_s be the partial ordering on strings of increasing integers given by $x = (x_1 < x_2 < \cdots < x_k)$ is less than $y = (y_1 < y_2 < \cdots < y_k)$ if $x_i \leq y_i$ for all i and $x \neq y$. Shifted complexes are exactly the order ideals of P_s. Note that we allow comparisons of strings of various lengths by considering the shorter string to have the necessary number of initial zeros (abusing that we are otherwise comparing strictly increasing strings). For example the string 24 is taken to be less than the string 1356 by considering 24 as 0024.

Example: A simplicial complex which includes the face 24 must also have the face 14 in order to be shifted (see Figure 1).

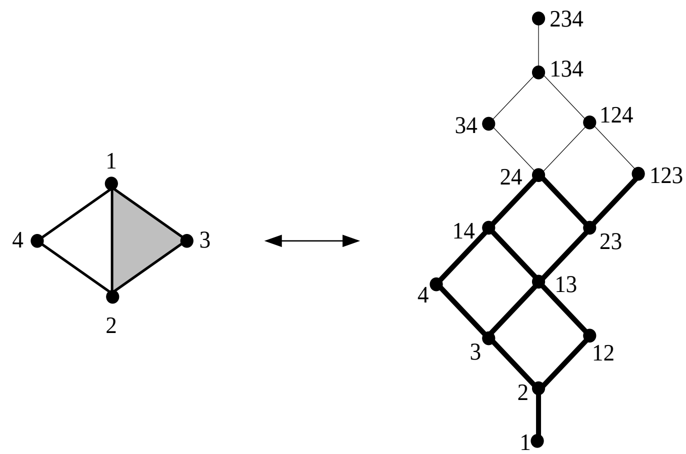

Figure 1. An example of a shifted complex.

3. Independent Set Complexes of Matroids

We first note that any matroid with a shifted independent set complex can have at most one parallel class of size greater than one. Suppose we had two parallel classes $\{a, b\}$ and $\{c, d\}$ in a matroid M. Let K_M be the corresponding complex. Then we would have $ac, bd \in K_M$ and $ab, cd \notin K_M$. Consider a labeling of the points of M. With out loss of generality, let a have the smallest label among all four points. $bd \in K$ implies both ab and ad must be faces in order for K_M to be shifted. Hence this complex can not be shifted and our matroids will have at most one parallel class of size greater than one.

For the remainder of this section, we will consider all matroids in terms of their affine diagrams.

3.1. **Ranks 1 and 2.**

Rank one matroids have independent sets of size at most one. In terms of the complex of independent sets this is just a collection of disjoint vertices, which is shifted.

In terms of the affine diagram, there are only two cases of rank two matroids; two points or a line. It is easy to see that both of these cases with at most one multiplicity give shifted matroids. In fact, we can see how to construct these complexes which are just graphs. The only non-adjacencies are between the members of the parallel class of size greater than one. (If we have no multiplicities, then we have a complete graph.) We can form any such graph by starting with a collection of disjoint vertices and then starring on all other vertices. By starring we mean connecting the new vertex to all previous vertices in the graph. Let D stand for adding a disjoint vertex and S for starring a vertex. Both operations preserve shiftedness. Hence complexes formed by successive applications of these two operations are shifted.

Proposition 1. *Rank two shifted matroid independent set complexes are exactly those complexes of the form $DD \cdots DSS \cdots S$.*

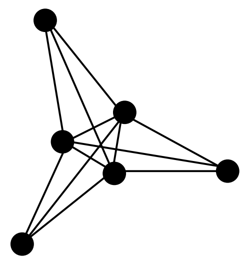

FIGURE 2. A rank two shifted matroid.

We want to extend the notion of starring a vertex to arbitrary dimensions. Namely, we will say a vertex v is starred in dimension d onto a complex K by forming the complex $K \operatorname{star}_d v = K \cup \{v \cup f \mid f \in K \text{ and } |f| \leq d\}$. Note that this operation is not the same as coning. Coning corresponds to the special case of starring a vertex in dimension one more than the dimension of the complex. As an example of starring, let K be the two dimensional triangle $\{123\}$. $K \operatorname{star}_2 4$ is the complex with top faces $\{123, 124, 134, 234\}$. On the other hand, $K \operatorname{star}_3 4$ is the complex with top face $\{1234\}$ which is the same as coning by 4. We will represent these complexes as strings of Ds, Ss, and | - for dimension increase. Thus in the examples above, $K \operatorname{star}_2 4$ would be represented by $DS|SS$ and $K \operatorname{star}_3 4$ would be represented by $DS|S|S$.

3.2. **Rank 3.** For rank three matroids, we quickly see that we cannot have more than one line in the affine diagram for a shifted complex. Let $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$ be two not necessarily disjoint lines (for example c_1 and c_2 could be the same point) of the affine diagram of a matroid M with complex K_M. This would give us $(a_1b_1c_1), (a_2b_2c_2) \notin K_M$ and $(a_1b_2c_1), (a_2b_1c_2) \in K_M$ showing K_M is not shifted. Therefore the affine diagram of a shifted rank 3 matroid can have at most one line and possibly disjoint points. Moreover, the parallel class of size greater than one, if it exists, lies on this line. To see this, let $\{a, b, c\}$ be the unique line of the affine diagram, and suppose $\{x, y\}$ $(\neq a, b, c)$ are parallel elements. Then we would have $(abc), (axy) \notin K_M$ and $(abx), (acy) \in K_M$, again an obstruction to shiftedness.

Hence rank 3 shifted matroids have the structure of the rank 2 case with additional disjoint points. Again, it is easy to see that all matroids of this form do have shifted matroid complexes. The extra disjoint points correspond to vertices starred in dimension 2. Therefore we started with a shifted one dimensional complex and starred vertices in the second dimension, which leaves us with a shifted complex.

Proposition 2. *Rank 3 shifted matroid independent set complexes are exactly those complexes of the form $DD\cdots DSS\cdots S|SS\cdots S$.*

3.3. **General Rank.** The same argument as above shows that we cannot have more than one maximal dimensional surface in the affine diagram of a shifted rank n matroid. Moreover, the affine diagram contains at most one flat per dimension and they are nested. Let F_1 and F_2 be flats of the affine diagram and suppose one is not contained in the other. Let $x_1 \in F_1, x_2 \in F_2$ and $x_1 \notin F_2, x_2 \notin F_1$. Now take maximally independent sets I_1 of F_1 s.t. $x_1 \notin I_1$ and I_2 of F_2 s.t. $x_2 \notin I_2$. We can do this because F_1 is a flat of the affine diagram and so $\text{rank}(F_1) = \text{rank}(F_1 - x_1)$, and similarly $\text{rank}(F_2) = \text{rank}(F_2 - x_2)$. Clearly $I_1 \cup x_1 \notin K_M$ and $I_2 \cup x_2 \notin K_M$. But, $I_1 \cup x_2 \in K_M$ otherwise x_2 would be in the closure of I_1 and hence in F_1. Similarly, $I_2 \cup x_1 \in K_M$ and K_M would not be shifted. Therefore our only option again becomes the rank $n-1$ case with disjoint points added. As before this results in a shifted complex since the disjoint vertices simply contribute by starring in the top dimension.

Theorem 1. *Rank n shifted matroid independent set complexes are exactly those complexes of the form $DD\cdots DSS\cdots S|SS\cdots S|\cdots|SS\cdots S$ (with exactly $n-2$ vertical bars).*

4. Principal Order Ideals

Next we characterize shifted matroids within the class of general shifted complexes. Recall that shifted complexes can be defined as order ideals in the shifted partial ordering.

Theorem 2. *An order ideal in the shifted partial ordering corresponds to a shifted matroid iff it is a principal order ideal.*

Proof. Suppose we have a principal order ideal with top element $(x_1, x_2, \dots, x_n)$. Then we claim the corresponding shifted complex has the form:

$$\underbrace{DD\dots D}_{x_n - x_{n-1}}\,\underbrace{SS\dots S}_{x_{n-1}-x_{n-2}}\,|\dots|\,\underbrace{SS\dots S}_{x_2 - x_1}\,|\,\underbrace{SS\dots S}_{x_1}.$$

The faces of the shifted complex will be all those strings coordinate-wise smaller than $(x_1, x_2, \dots, x_n)$. Hence all vertices with labels between x_{n-1} and x_n cannot appear in the same face since they all have label smaller than only one vertex in the defining face. Let us form a complex by initially adding $x_n - x_{n-1}$ disjoint vertices. This takes care of all the non-faces of size two. Next we consider the non-faces of size three. These will be the triples of vertices with smallest label larger than x_{n-2}. In order to not form these two dimensional faces, we star on all vertices between x_{n-2} and x_{n-1} in dimension one. Similarly, to avoid the three dimensional non-faces we star the appropriate vertices in dimension two. Again these will be those vertices with label between x_{n-3} and x_{n-4}. Continuing in this manner will produce the complex claimed above.

In the other direction, suppose we have an order ideal with at least two top elements, $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ with X lexicographically smaller than Y. Also let us say we have an incomparability of the form $x_i < y_i$ and $x_j > y_j$ with i the smallest index such that $x_i < y_i$. Thus i is also the smallest index such that $x_i \neq y_i$. Consider the induced subcomplex on all the xs and ys except x_i. In this subcomplex, $(y_1, y_2, \ldots, y_n)$ forms a face of size n and $(x_1, \ldots \hat{x_i}, \ldots, x_n)$ forms a face of size $n-1$. This subcomplex must be pure for us to have a matroid. Hence $(x_1, \ldots \hat{x_i}, \ldots, x_n)$ must be contained in a face of size n, so it must form a face with y_l for some l. For $l < i$, $y_l = x_l$. Therefore l must be greater than or equal to i and y_l must be strictly greater than x_i. We know there exists some such y not equal to any x because $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ were taken to be incomparable elements in the shifted partial ordering. But then the face $\{(x_1, \ldots \hat{x_i}, \ldots, x_n) \cup y_l\}$ is greater than $(x_1, x_2, \ldots, x_n)$ in the shifted partial order contradicting the maximality of $(x_1, x_2, \ldots, x_n)$.

□

In [1], a class of matroids SM is defined and shown to be a class of shifted matroids. In our language, this class is precisely the set of principal order ideals in the shifted partial ordering. Therefore, the previous theorem shows that not only are the elements of SM shifted matroids, but they are *all* the shifted matroids.

Corollary 1. *There are $\binom{n}{k}$ shifted matroids of rank k on n vertices.*

5. Minors, Duality, and Shifting

Theorem 3. *The class of shifted matroids is closed under taking minors.*

Proof. Let M be a shifted matroid on the base set E and e be any element of E. First we consider $M \backslash e$, whose independent sets are the independent sets of M which do not involve e. In terms of the complex of independent sets, this is equivalent to the geometric deletion of e from the complex. Since the class of shifted complexes is closed under deletion, we have that $M \backslash e$ is a shifted matroid.

Next we look at M/e. First we note that $M \backslash (E - e)$ is the matroid on just one element, e. Therefore, the independent sets of M/e are those subsets which form an independent set with e. Again, in terms of the complex of independent sets, this is simply the link of e and the link of any vertex in a shifted complex is shifted. Therefore M/e is a shifted matroid. □

Theorem 4. *The class of shifted matroids is closed under duality.*

Proof. Consider any shifted matroid as a principal order ideal in the shifted partial ordering with unique top element $(x_1, x_2, \ldots, x_k)$. The bases of the matroid are $\{(y_1, y_2, \ldots, y_k) \,|\, y_i \leq x_i \text{ and } y_i \neq 0 \,\forall i\}$. Clearly, if we have two bases B_1 and B_2 such that B_1 is less than B_2 in the shifted partial ordering, then $E - B_1$ is greater than $E - B_2$ in the shifted partial ordering. Taking the complement of all bases gives a principal filter in the shifted partial ordering. This is equivalent to a principal order ideal since we may simply reverse the labeling. Therefore the dual of a shifted matroid is a shifted matroid. □

We remark here that algebraically shifting a complex which is a matroid does not necessarily result in a matroid. See [6] for a survey on algebraic shifting. Consider the simplicial complex which is the boundary complex of the octahedron. It is not hard to check that this is a matroid. For both symmetric and exterior shifting, the result is the shifted complex with top faces 136 and 234 and hence not a matroid. Also, we can combinatorial shift the boundary complex of an octahedron to the shifted complex generated by 145 and 136.

We can observe that matroids are not preserved under any shifting procedure which preserves the f-vector. A shifted matroid is always a principal order ideal in the shifted partial ordering, and so we are limited by the size of such ideals. For the octahedron, we have eight two dimensional faces on six vertices. But there are no principal order ideals in the shifted partial ordering on six vertices with eight two dimensional elements.

6. Broken Circuit Complex

Next we investigate shifted broken circuit complexes. Given a matroid M on the base set E and a linear ordering of the base set, a *broken circuit* of M is a subset $C - \{x_i\}$ where C is a circuit and x_i is the smallest element of C with respect to the linear ordering.

The *broken circuit complex* $BC(M)$ of a matroid M on a base set E is defined by:

$$BC(M) = \{S \subseteq M : S \text{ contains no broken circuit}\}$$

Now let us consider the case where we have a rank n shifted matroid M with a shifted labeling of the vertices.

Theorem 5. *Broken circuit complexes of shifted matroids are shifted and inherit a shifted labeling.*

Proof. Let $x = (x_1, \ldots, x_n)$ be a face of the matroid complex and not be a face of BC(M). We need any greater face, $y = (y_1, \ldots, y_n)$, to also not be a face of $BC(M)$. x must contain a broken circuit, say $(x_{i_1}, \ldots, x_{i_d})$ of some circuit $(a, x_{i_1}, \ldots, x_{i_d})$. If $(y_{i_1}, \ldots, y_{i_d})$ is not a face of the matroid complex then we are done. Otherwise we need it to be a broken circuit. Now since $(x_{i_1}, \ldots, x_{i_d})$ is a circuit, it had to be a non-face of the complex of independent sets. But then $(y_{i_1}, \ldots y_{i_d})$ must also be a non-face or the matroid would not be shifted. Hence $(y_{i_1}, \ldots y_{i_d})$ is also a broken circuit, and $BC(M)$ is shifted under the initial shifted labeling of M.

□

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