RELATIONS ON GENERALIZED DEGREE SEQUENCES

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Abstract. We study degree sequences for simplicial posets and polyhedral complexes, generalizing the well-studied graphical degree sequences. Here we extend the more common generalization of vertex-to-facet degree sequences by considering arbitrary face-to-flag degree sequences. In particular, these may be viewed as natural refinements of the flag f-vector of the poset. We investigate properties and relations of these generalized degree sequences, proving linear relations between flag degree sequences in terms of the composition of rank jumps of the flag. As a corollary, we recover an f-vector inequality on simplicial posets first shown by Stanley.

1. Introduction

Degree sequences of graphs, which record how many edges contain each vertex, have been studied extensively, and enjoy a rich literature. One popular chapter of this literature is the characterization of when an integer sequence can be a degree sequence; for example, see [6].

In higher dimensions, the notion of degree sequence for simplicial complexes has received considerably less attention. For example, although the behavior of the vertex-to-facet degree sequence for pure simplicial complexes has been studied in [1, 3, 5], little is known about the intrinsic properties or characterizations for this, possibly most natural, extension of graphical degree sequences.

In this paper, we formulate a higher dimensional and more general analogue of graphical degree sequences, and we study the nature of these sequences for simplicial posets and for more general polyhedral complexes. In particular, we define the face-to-flag degree sequence, recording how many flags with prescribed rank jumps contain each face of a given rank. More precisely, let $\mathcal{P}$ be a pure, rank-k simplicial poset, and $\sigma = (\sigma_1, \ldots, \sigma_m)$ a composition of k. The *face-to-flag degree sequence* $d^\sigma(\mathcal{P})$ is a sequence indexed by the faces F_i of rank σ_1, with the corresponding entry in $d^\sigma(\mathcal{P})$ counting the number of flags

$$\{F_i \subset X_2 \subset \cdots \subset X_m : \mathrm{rk}(X_j) = \sigma_1 + \sigma_2 + \cdots + \sigma_j\},$$

which are the flags containing F_i with rank jumps $\sigma_1, \ldots, \sigma_m$. We give linear relations between such sequences, proving a majorization result in the case of simplicial posets, in terms of the relative sizes of the rank jumps of the flags. Namely, for $\pi = (\pi_1, \ldots, \pi_m)$ a permutation of σ, if $\pi_1 \geq \sigma_1$, then

$$d^\sigma \trianglerighteq d^\pi.$$

Furthermore, we obtain a weak majorization result for face-to-flag degree sequences analogously defined for complexes whose maximal faces are simple polytopes.

Our majorization result in the simplicial case yields results on f-vectors and flag f-vectors of pure simplicial posets because these sequences naturally refine the flag f-vector of the poset. In particular, we recover a result of Stanley: for a rank-k simplicial poset, the f-vector satisfies the inequality $f_i \leq f_{k-i}$, when $i \leq k - i$.

In Section 2, we review graphical degree sequences as a basis and motivation for studying generalized degree sequences, also recalling the definition of vertex-to-facet degree sequences and a few preliminary results. In Section 3, we introduce the face-to-flag degree sequences, and prove our main result (Theorem 3.5), giving a total ordering (via majorization) of the face-to-flag degree sequences of a simplicial poset. As a corollary, we recover the aforementioned result of Stanley. In Section 4 we consider non-simplicial posets. First, Theorem 3.5 is extended to the setting of complexes all of whose maximal faces are simple polytopes. Finally, we give an elementary example showing that these results may not be extended to arbitrary polyhedral complexes.

2. Degree sequences, motivation and preliminaries

2.1. Graphical degree sequences.

Definition 2.1. For a simple graph $G = (V, E)$ with $|V| = n$, the *graphical degree sequence* of G is the sequence

$$d(G) = (d_1, d_2, \ldots, d_n),$$

where

$$d_i = |\{j : \{i, j\} \in E\}|,$$

and the vertices are labeled so that $d_1 \geq d_2 \geq \ldots \geq d_n$.

Majorization (or dominance) order is often the natural choice for comparing graphical degree sequences, as in [4, Chapter 7].

Definition 2.2. Given two sequences of nonnegative integers $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_m)$ having the same sum, a *majorizes* b, written $a \trianglerighteq b$, if the following system of inequalities hold:

$$a_1 \geq b_1$$

$$a_1 + a_2 \geq b_1 + b_2$$

$$\vdots$$

$$a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1}.$$

If one weakens the sum requirement to the inequality $\sum_i a_i \geq \sum_i b_i$, then a is said to *weakly majorize* b, written $a \succeq b$. Majorization forms a partial ordering on integer sequences.

The *conjugate* of a sequence $d = (d_1, \ldots, d_n)$ of nonnegative integers is the sequence d^T given by

$$d_i^T = |\{j : d_j \geq i\}|$$

for $i \geq 1$. This agrees with the transpose when considering d as an integer partition. An elementary relation on graphical degree sequences compares a sequence and its conjugate.

Proposition 2.3 ([4, Chapter 7, Section D])**.** *The degree sequence of any simple graph G is majorized by its conjugate:*

$$\big(d(G)\big)^T \trianglerighteq d(G).$$

The problem of characterizing graphical degree sequences, that is, deciding which sequences arise as the degree sequence of some graph, is well understood. For example, [6] lists seven criteria, all of which are linear relations on the degree sequence or between the degree sequence and its conjugate.

2.2. **Vertex-to-facet degree sequences.** Moving to higher dimensions, we first consider *pure, rank-k* simplicial complexes; that is, complexes in which every maximal face is of rank k, where we define the rank of a face to be one more than its dimension.

For a pure, rank-k simplicial complex, the "vertex-to-facet" degree sequence, counting the number of facets containing each vertex, has received the most attention as a generalization of graphical degree sequences. For example, these sequences are treated in [1], [3], and [5], where they arise in connection with Laplacian eigenvalues, plethysms, and zonotopes respectively.

Little is known about the intrinsic properties of such integer sequences. In general, the vertex-to-facet sequence does not share many of the nice properties of the graphical case. For example, Proposition 2.3 no longer holds.

Example 2.4. Let Δ be the complete, rank-3 complex on five vertices. Then the vertex-to-facet degree sequence is $(6, 6, 6, 6, 6)$, and $(6, 6, 6, 6, 6)^T = (5, 5, 5, 5, 5, 5)$. Proposition 2.3 does not hold here, since the majorization occurs in the wrong direction: $(6, 6, 6, 6, 6) \trianglerighteq (5, 5, 5, 5, 5, 5)$.

To match the notation for generalized degree sequences introduced in the upcoming Definition 3.2, let $d^{(\sigma_1,\sigma_2)}(\Delta)$ be the sequence counting, for each rank-σ_1 face in a simplicial complex Δ, the number of rank-$(\sigma_1 + \sigma_2)$ faces containing that face. Duval, Reiner, and Dong consider face-to-facet sequences, and give the following relation for a rank-k complex.

Proposition 2.5 ([1, Proposition 8.4])**.** *For any pure, rank-k simplicial complex Δ,*

$$\Big(d^{(i,k-i)}(\Delta)\Big)^T \trianglerighteq d^{(k-i,i)}(\Delta).$$

In fact, a more direct relationship exists between $d^{(i,k-i)}$ and $d^{(k-i,i)}$, and between more general "face-to-flag" degree sequences where the compositions of rank jumps for the flags

are permutations of each other. These generalized degree sequences, indexed by compositions of rank jumps, are defined in the subsequent section.

3. Generalized degree sequences

3.1. Face-to-flag degree sequences.

For the remainder of the paper, we will assume that all simplicial complexes and posets are pure. That is, we assume that all maximal faces have the same rank.

Definition 3.1. A *simplicial poset* is a poset with a minimal element $\hat{0}$ in which every principal order ideal $[\hat{0}, x]$ is a boolean algebra.

In parallel to the terminology for the more commonly studied vertex-to-facet degree sequences of simplicial complexes, we will also use the term *face* to denote an element of the poset, and *flag* to denote a nested sequence of faces (that is, a chain) in the poset. Additionally, when no confusion can arise, we will move freely between these two interpretations.

Definition 3.2. For a pure, rank-k simplicial poset $\mathcal{P}$, and $\sigma = (\sigma_1, \ldots, \sigma_m)$ a composition of k, the *face-to-flag degree sequence* $d^\sigma(\mathcal{P})$ is

$$d^\sigma(\mathcal{P}) = (d^\sigma(F_1), d^\sigma(F_2), \ldots, d^\sigma(F_s)),$$

where $\{F_1, F_2, \ldots, F_s\}$ are the rank-σ_1 faces of $\mathcal{P}$, and

$$d^\sigma(F_i) = |\{F_i \subset X_2 \subset \cdots \subset X_m : \mathrm{rk}(X_j) = \sigma_1 + \sigma_2 + \cdots + \sigma_j\}|.$$

The sequence $d^\sigma(\mathcal{P})$ records the degrees of rank-σ_1 faces to flags with rank jumps $\sigma_1, \ldots, \sigma_m$. As with graphical degree sequences, for a fixed σ, the faces F_i are indexed so that $d^\sigma(F_1) \geq d^\sigma(F_2) \geq \ldots \geq d^\sigma(F_s)$. We assume that all flags end at the top rank, since otherwise one could take the appropriate truncation of the poset.

Note that there is an abuse of notation in Definition 3.2: the function d^σ is defined both on a simplicial poset and on a face, which itself could be considered as a simplicial poset. Throughout this article, the context of the usage should be clear.

Example 3.3. Let Δ be the pure, rank-4 simplicial complex with facets $\{1234, 1246, 1256\}$. Consider the following three degree sequences recording vertex-to-edge/tetrahedron flags, vertex-to-triangle/tetrahedron flags, and edge-to-triangle/tetrahedron flags, respectively.

$$\begin{aligned} d^{(1,1,2)}(\Delta) &= (9,9,6,6,3,3) \\ d^{(1,2,1)}(\Delta) &= (9,9,6,6,3,3) \\ d^{(2,1,1)}(\Delta) &= (6,4,4,4,4,2,2,2,2,2,2,2) \end{aligned}$$

From here it is easy to see that

$$d^{(1,1,2)}(\Delta) = d^{(1,2,1)}(\Delta) \trianglerighteq d^{(2,1,1)}(\Delta).$$

To formalize the majorization behavior observed in Example 3.3, comparing face-to-flag degree sequences for a given complex, fix $\mathcal{P}$ a rank-k simplicial poset, and $\sigma = (\sigma_1, \ldots, \sigma_m)$ a composition of k. Additionally, let $\sum d^\sigma(\mathcal{P})$ denote the sum of the entries in $d^\sigma(\mathcal{P})$.

Let $\{F_1, \ldots, F_s\}$ be the rank-σ_1 elements of $\mathcal{P}$. Observe that

$$d^\sigma(\mathcal{P}) = \binom{k-\sigma_1}{\sigma_2, \ldots, \sigma_m} \left(d^{(\sigma_1, k-\sigma_1)}(F_1), \ldots, d^{(\sigma_1, k-\sigma_1)}(F_s) \right). \tag{1}$$

Thus, the sum of the entries in $d^\sigma(\mathcal{P})$ is

$$\begin{aligned} \sum d^\sigma(\mathcal{P}) &= \binom{k-\sigma_1}{\sigma_2, \ldots, \sigma_m} \sum_{l=1}^{s} d^{(\sigma_1, k-\sigma_1)}(F_l) = \binom{k-\sigma_1}{\sigma_2, \ldots, \sigma_m} f_{\sigma_1, k}(\mathcal{P}) \\ &= \binom{k-\sigma_1}{\sigma_2, \ldots, \sigma_m} f_k(\mathcal{P}) \binom{k}{\sigma_1} = f_k(\mathcal{P}) \binom{k}{\sigma_1, \ldots, \sigma_m}, \end{aligned} \tag{2}$$

where $f_k(\mathcal{P})$ is the number of rank-k elements of $\mathcal{P}$, and

$$f_{\sigma_1, k}(\mathcal{P}) = |\{X_1 \subset X_2 : \mathrm{rk}(X_1) = \sigma_1, \mathrm{rk}(X_2) = k\}|$$

is an entry in the flag-f vector of $\mathcal{P}$. Additionally, since the right side of equation (2) only depends on the entries of σ, we see that $\sum d^\sigma(\mathcal{P}) = \sum d^\pi(\mathcal{P})$ for any permutation π of the composition σ.

Lemma 3.4. *Fix $\sigma = (\sigma_1, \ldots, \sigma_m)$ a composition of k and π a permutation of σ, with $\pi_1 > \sigma_1$. If F is a rank-σ_1 element of a pure, rank-k simplicial poset $\mathcal{P}$, and $G > F$ is a rank-π_1 element, then $d^\sigma(F) \geq d^\pi(G)$.*

Proof. Let $\pi_1 = \sigma_l$ for some $1 < l \leq m$.

$$\begin{aligned} d^\sigma(F) &= d^{(\sigma_1, k-\sigma_1)}(F) \binom{k-\sigma_1}{\sigma_2, \ldots, \sigma_m} = \sum_{\substack{H : \mathrm{rk}(H) = k \\ F < H}} \binom{k-\sigma_1}{\pi_1} \binom{k-\sigma_1-\pi_1}{\sigma_2, \ldots, \widehat{\sigma_l}, \ldots, \sigma_m} \\ &\geq \sum_{\substack{H : \mathrm{rk}(H) = k \\ G < H}} \binom{k-\sigma_1}{\pi_1} \binom{k-\sigma_1-\pi_1}{\sigma_2, \ldots, \widehat{\sigma_l}, \ldots, \sigma_m} \\ &= \frac{\binom{k-\sigma_1}{k-\sigma_1-\pi_1}}{\binom{k-\pi_1}{k-\pi_1-\sigma_1}} \sum_{\substack{H : \mathrm{rk}(H) = k \\ G < H}} \binom{k-\pi_1}{\sigma_1} \binom{k-\sigma_1-\pi_1}{\sigma_2, \ldots, \widehat{\sigma_l}, \ldots, \sigma_m} \\ &\geq \sum_{\substack{H : \mathrm{rk}(H) = k \\ G < H}} \binom{k-\pi_1}{\pi_2, \ldots, \pi_m} = d^{(\pi_1, k-\pi_1)}(G) \binom{k-\pi_1}{\pi_2, \ldots, \pi_m} = d^\pi(G). \end{aligned}$$

The last inequality is due to the fact that $\pi_1 > \sigma_1$, and $\{\sigma_1, \ldots, \widehat{\sigma_l}, \ldots, \sigma_m\}$ and $\{\pi_2, \ldots, \pi_m\}$ are equal as sets. □

Theorem 3.5. *Fix a pure, rank-k simplicial poset $\mathcal{P}$, a composition $\sigma = (\sigma_1, \ldots, \sigma_m)$ of k, and $\pi = (\pi_1, \ldots, \pi_m)$ a permutation of σ. If $\pi_1 \geq \sigma_1$, then*

$$d^\sigma(\mathcal{P}) \trianglerighteq d^\pi(\mathcal{P}).$$

Proof. We recall that $\sum d^\sigma(\mathcal{P})$ depends only on the entries of σ, and therefore $\sum d^\sigma(\mathcal{P}) = \sum d^\pi(\mathcal{P})$. We divide the rest of the proof into two cases: $\pi_1 = \sigma_1$ and $\pi_1 > \sigma_1$.

First, consider the case $\pi_1 = \sigma_1$. Here we are considering elements of the same rank compared to different flags. By equation (1), we see that $d^\sigma(\mathcal{P}) = d^\pi(\mathcal{P})$.

Now, suppose $\pi_1 > \sigma_1$. Let $\{F_1, F_2, \ldots, F_s\}$ be the rank-σ_1 elements of $\mathcal{P}$, and let $\{G_1, G_2, \ldots, G_t\}$ be the rank-π_1 elements. Furthermore, assume they are labeled so that

$$d^\sigma(F_i) \geq d^\sigma(F_j) \text{ and } d^\pi(G_i) \geq d^\pi(G_j)$$

for all $i < j$.

By Lemma 3.4, if $F < G$, then $d^\sigma(F) \geq d^\pi(G)$. It remains to prove that

$$\sum_{l=1}^{r} d^\pi(G_l) \leq \sum_{l=1}^{r} d^\sigma(F_l) \text{ for all } r \leq s. \tag{3}$$

We prove this by induction on r. Suppose $r = 1$, and consider the smallest j such that $F_j < G_1$. Then we have

$$d^\sigma(F_1) \geq d^\sigma(F_j) \geq d^\pi(G_1).$$

For $r > 1$, we consider two subcases.

Suppose there exists $F_j < G_l$ with $j > r$ and $l \leq r$. As above, we have

$$d^\sigma(F_r) \geq d^\sigma(F_j) \geq d^\pi(G_l) \geq d^\pi(G_r).$$

In this case, the rth terms in the sums satisfy the necessary inequalities themselves, hence the rth partial sum, equation (3), holds by induction.

Now assume there does not exist $F_j < G_l$ with $j > r$ and $l \leq r$. We prove this case directly. First, note that if we sum the degrees of all rank-π_1 elements which are greater than a given rank-σ_1 element, we count the degree of the fixed rank-σ_1 element $\binom{\pi_1}{\sigma_1}$ times:

$$\sum_{G_l : F_j < G_l} d^\pi(G_l) = \binom{\pi_1}{\sigma_1} d^\sigma(F_j).$$

This implies that

$$\begin{aligned}
\binom{\pi_1}{\sigma_1} \sum_{j=1}^{r} d^\sigma(F_j) &= \sum_{j=1}^{r} \sum_{G_l : F_j < G_l} d^\pi(G_l) \\
&\geq \sum_{j=1}^{r} \sum_{\substack{G_l : F_j < G_l \\ l \leq r}} d^\pi(G_l) \\
&= \binom{\pi_1}{\sigma_1} \sum_{j=1}^{r} d^\pi(G_j).
\end{aligned}$$

□

As a corollary, we obtain an inequality for the f-vector of a simplicial poset. This result was first shown by Stanley (see [2] for a brief history of this inequality).

Corollary 3.6. *For $\mathcal{P}$ a pure simplicial poset of rank k, with $i \leq k - i$,*

$$f_i \leq f_{k-i},$$

where f_i is the ith entry of the f-vector of $\mathcal{P}$, indexed by rank.

Proof. If a and b are nonnegative sequences with $a \trianglerighteq b$, then the number of non-zero entries of a must be less than or equal to the number of non-zero entries of b. For the sequence $d^{(i,k-i)}$, the number of (non-zero) entries is equal to the number of rank-i elements of $\mathcal{P}$, because $\mathcal{P}$ is pure of rank k. □

Remark 3.7. Note also that:

$$\sum_{l=1}^{s} d^{(i,j-i)}(F_l) = \sum_{l=1}^{s} d^{(j-i,i)}(F_l) = f_{i,j} = f_{j-i,j}$$

where the $f_{i,j}$ are entries in the flag f-vector of $\mathcal{P}$, indexed by ranks. Hence, we can view generalized degree sequences as refinements of flag f-vectors. The theorem above states that for all pure simplicial posets, and for $i < j - i$, the contributions which make up $f_{j-i,j}$ are "more spread out" than those which make up $f_{i,j}$.

The definition of face-to-flag degree sequences can easily be extended to a notion of flag-to-flag degree sequences which record the degrees of flags with specified rank jumps to flags with continuing rank jumps. The flag-to-flag degree sequence can be computed in terms of face-to-flag degree sequences. An analogous majorization result holds in this case where the necessary condition is in terms of the relative sizes of the sums of rank jumps of the initial flags.

4. Non-simplicial degree sequences

The definition of face-to-flag degree sequences is not specific to simplicial posets, and so we carry over that definition to the setting of a pure polyhedral complex. In particular, if all maximal faces of our complex are *simple* polytopes, then many of the results from the simplicial case also hold. For a discussion of simple polytopes, see [7].

Definition 4.1. A polytope is *simple* if every proper upper interval in its face lattice, $[x, \hat{1}]$ with $x \neq \hat{0}$, is a boolean algebra.

As a starting point, we observe that Lemma 3.4 holds in this case.

Lemma 4.2. *Let $\sigma = (\sigma_1, \ldots, \sigma_m)$ be a composition of k, and $\pi = (\pi_1, \ldots, \pi_m)$ a permutation of σ with $\sigma_1 < \pi_1$. If F is a rank-σ_1 face of a pure, rank-k complex in which every maximal face is a simple polytope, and if $G \supset F$ is a rank-π_1 face, then $d^{\sigma}(F) \geq d^{\pi}(G)$.*

Proof. Lemma 4.2 follows as in Lemma 3.4. In particular, the initial multinomial expression

$$d^{\sigma}(F) = d^{(\sigma_1, k-\sigma_1)}(F)\binom{k-\sigma_1}{\sigma_2, \ldots, \sigma_m}$$

is easy to see, since every upper interval of the face lattice ending at rank k is boolean. □

The simple case differs from the simplicial case in that the sums of the degree sequences $\sum d^{\sigma}(\mathcal{C})$ and $\sum d^{\pi}(\mathcal{C})$ are not necessarily equal for a complex $\mathcal{C}$ with simple maximal faces. However, if every maximal face of $\mathcal{C}$ is the *same* simple polytope, then we establish an inequality on the sums which implies weak majorization, as demonstrated in the following analogue to Theorem 3.5.

Theorem 4.3. *Fix a pure, rank-k complex $\mathcal{C}$ in which every maximal face is the same simple polytope, a composition $\sigma = (\sigma_1, \ldots, \sigma_m)$ of k, and $\pi = (\pi_1, \ldots, \pi_m)$ a permutation of σ. If $\pi_1 \geq \sigma_1$, then*

$$d^{\sigma}(\mathcal{C}) \succeq d^{\pi}(\mathcal{C}).$$

Proof. The proof is identical to that of Theorem 3.5, with Lemma 4.2 replacing Lemma 3.4, except that equality of the total sums is no longer guaranteed.

Let $\{F_1, F_2, \ldots, F_s\}$ be the rank-σ_1 faces of $\mathcal{C}$. Then we have

$$d^{\sigma}(\mathcal{C}) = \binom{k-\sigma_1}{\sigma_2, \ldots, \sigma_m}\left(d^{(\sigma_1, k-\sigma_1)}(F_1), \ldots, d^{(\sigma_1, k-\sigma_1)}(F_s)\right).$$

By the same argument as in Theorem 3.5, if $\sigma_1 < \pi_1$, then

$$\sum_{i=1}^{r} d^{\pi}(G_i) \leq \sum_{i=1}^{r} d^{\sigma}(F_i) \text{ for all } r \leq s,$$

where $\{G_1, \ldots, G_t\}$ are the rank-π_1 faces in $\mathcal{C}$.

Now,

$$\begin{aligned} \sum d^{\sigma}(\mathcal{C}) &= \binom{k-\sigma_1}{\sigma_2, \ldots, \sigma_m} f_{\sigma_1, k}(\mathcal{C}) \\ &= \binom{k-\sigma_1}{\sigma_2, \ldots, \sigma_m} f_k(\mathcal{C}) N_{\sigma_1}, \end{aligned}$$

where N_{σ_1} is the number of rank-σ_1 faces contained in a given rank-k face.

Comparing this to the sum $\sum d^{\pi}(\mathcal{C})$, we find

$$\frac{\sum d^{\sigma}(\mathcal{C})}{\sum d^{\pi}(\mathcal{C})} = \frac{N_{\sigma_1}}{N_{\pi_1}} \cdot \frac{\binom{k-\sigma_1}{\pi_1}}{\binom{k-\pi_1}{\sigma_1}}. \tag{4}$$

This ratio must be greater than or equal to 1 for weak majorization to hold.

To establish this inequality, we restrict our attention to a single maximal face and consider the number of pairs of rank-σ_1 faces contained in rank-π_1 faces. Since the maximal face is

simple, each rank-σ_1 face is contained in $\binom{k-\sigma_1}{\pi_1-\sigma_1}$ rank-π_1 faces. Each rank-π_1 face contains at least $\binom{\pi_1}{\sigma_1}$ rank-σ_1 faces, so

$$\binom{k-\sigma_1}{\pi_1-\sigma_1}N_{\sigma_1} \geq \binom{\pi_1}{\sigma_1}N_{\pi_1}.$$

This implies that the expression in equation (4) is at least 1. Therefore, if $\sigma_1 < \pi_1$, then

$$d^{\sigma}(\mathcal{C}) \succeq d^{\pi}(\mathcal{C}).$$

Furthermore if $\sigma_1 = \pi_1$, then

$$d^{\sigma}(\mathcal{C}) = d^{\pi}(\mathcal{C}).$$

□

In fact, the result of Theorem 4.3 would hold for suitably defined "polyhedral posets" as well. For example, just as a simplicial poset is one in which every principal order ideal is a boolean algebra, a poset is *cubical* if every principal order ideal is isomorphic to the face lattice of a cube, and such posets also satisfy the results of the theorem.

4.1. **A polyhedral counterexample.**

Theorem 3.5 breaks down, even in the case of weak majorization, as we diverge from the simplicial and simple cases. For example, in a 3-polytope, the sum of the entries of $d^{(1,3)}$ is equal to f_1, the number of vertices of the polytope, while the sum of the entries of $d^{(3,1)}$ is f_3, the number of facets. Thus, any 3-polytope with more facets than vertices results in $d^{(3,1)}$ weakly majorizing $d^{(1,3)}$. The following example describes one such polyhedral complex; in fact, a Platonic solid.

Example 4.4. Let Δ be the octahedron. We have

$$\begin{aligned} d^{(1,3)}(\Delta) &= (1,1,1,1,1,1), \text{ and} \\ d^{(3,1)}(\Delta) &= (1,1,1,1,1,1,1,1). \end{aligned}$$

Although $(3,1) \trianglerighteq (1,3)$, the (weak) majorization of these degree sequences is in the direction opposite to that in the results for the simplicial and simple cases: $d^{(3,1)}(\Delta) \succeq d^{(1,3)}(\Delta)$.

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