CUTS AND FLOWS OF CELL COMPLEXES

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Abstract. We study the vector spaces and integer lattices of cuts and flows associated with an arbitrary finite CW complex, and their relationships to group invariants including the critical group of a complex. Our results extend to higher dimension the theory of cuts and flows in graphs, most notably the work of Bacher, de la Harpe and Nagnibeda. We construct explicit bases for the cut and flow spaces, interpret their coefficients topologically, and give sufficient conditions for them to be integral bases of the cut and flow lattices. Second, we determine the precise relationships between the discriminant groups of the cut and flow lattices and the higher critical and cocritical groups with error terms corresponding to torsion (co)homology. As an application, we generalize a result of Kotani and Sunada to give bounds for the complexity, girth, and connectivity of a complex in terms of Hermite's constant.

1. Introduction

This paper is about vector spaces and integer lattices of cuts and flows associated with a finite cell complex. Our primary motivation is the study of critical groups of cell complexes and related group invariants. The critical group of a graph is a finite abelian group whose order is the number of spanning forests. The definition was introduced independently in several different settings, including arithmetic geometry [24], physics [10], and algebraic geometry [2] (where it is also known as the Picard group or Jacobian group). It has received considerable recent attention for its connections to discrete dynamical systems, tropical geometry, and linear systems of curves; see, e.g., [3, 4, 7, 19].

In previous work [15], the authors extended the definition of the critical group to a cell complex Σ of arbitrary dimension. To summarize, the critical group $K(\Sigma)$ can be calculated using a reduced combinatorial Laplacian, and its order is a weighted enumeration of the cellular spanning trees of Σ. Moreover, the action of the critical group on cellular $(d-1)$-cochains gives a

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model of discrete flow on Σ, generalizing the chip-firing and sandpile models; see, e.g., [4, 10].

Bacher, de la Harpe, and Nagnibeda first defined the lattices $\mathcal{C}$ and $\mathcal{F}$ of integral cuts and flows for a graph [2]. By regarding a graph as an analogue of a Riemann surface, they interpreted the discriminant groups $\mathcal{C}^\sharp/\mathcal{C}$ and $\mathcal{F}^\sharp/\mathcal{F}$ respectively as the Picard group of divisors and as the Jacobian group of holomorphic forms. In particular, they showed that the critical group $K(G)$ is isomorphic to both $\mathcal{C}^\sharp/\mathcal{C}$ and $\mathcal{F}^\sharp/\mathcal{F}$. Similar definitions and results appear in the work of Biggs [4].

In the present paper, we define the *cut and flow spaces* and *cut and flow lattices* of a cell complex Σ by

$$\begin{aligned} \mathrm{Cut}(\Sigma) &= \mathrm{im}_{\mathbb{R}}\,\partial^*, & \mathrm{Flow}(\Sigma) &= \ker_{\mathbb{R}}\,\partial, \\ \mathcal{C}(\Sigma) &= \mathrm{im}_{\mathbb{Z}}\,\partial^*, & \mathcal{F}(\Sigma) &= \ker_{\mathbb{Z}}\,\partial, \end{aligned}$$

where ∂ and ∂^* are the top cellular boundary and coboundary maps of Σ. In topological terms, cut- and flow-vectors are cellular coboundaries and cycles, respectively. Equivalently, the vectors in $\mathrm{Cut}(\Sigma)$ support sets of facets whose deletion increases the codimension-1 Betti number, and the vectors in $\mathrm{Flow}(\Sigma)$ support nontrivial rational homology classes. In the language of matroid theory, cuts and flows correspond to cocircuits and circuits, respectively, of the *cellular matroid* represented by the columns of ∂.

In the higher-dimensional setting, the groups $\mathcal{C}^\sharp/\mathcal{C}$ and $\mathcal{F}^\sharp/\mathcal{F}$ are not necessarily isomorphic to each other. Their precise relationship involves several other groups: the critical group $K(\Sigma)$, a dually defined *cocritical group* $K^*(\Sigma)$, and the *cutflow group* $\mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F})$. We show that the critical and cocritical groups are respectively isomorphic to the discriminant groups of the cut lattice and flow lattice, and that the cutflow group mediates between them with an "error term" given by homology. Specifically, if $\dim\Sigma = d$, then we have the short exact sequences

$$\begin{aligned} 0 \to \mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F}) \to K(\Sigma) \cong \mathcal{C}^\sharp/\mathcal{C} \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z})) \to 0, \\ 0 \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z})) \to \mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F}) \to K^*(\Sigma) \cong \mathcal{F}^\sharp/\mathcal{F} \to 0 \end{aligned}$$

(Theorems 7.6 and 7.7) where $\mathbf{T}$ denotes the torsion summand. The sizes of these groups are then given by

$$\begin{aligned} |\mathcal{C}^\sharp/\mathcal{C}| = |K(\Sigma)| = \tau(\Sigma) \quad &= \tau^*(\Sigma)\cdot\mathbf{t}^2, \\ |\mathcal{F}^\sharp/\mathcal{F}| = |K^*(\Sigma)| = \tau^*(\Sigma) \quad &= \tau(\Sigma)/\mathbf{t}^2, \\ |\mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F})| = \tau(\Sigma)/\mathbf{t} &= \tau^*(\Sigma)\cdot\mathbf{t}, \end{aligned}$$

(Theorems 8.1 and 8.2), where $\mathbf{t} = |\mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z}))|$ and $\tau(\Sigma)$ and $\tau^*(\Sigma)$ are the weighted enumerators

$$\tau(\Sigma) = \sum_{\Upsilon} |\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})|^2, \qquad \tau^*(\Sigma) = \sum_{\Upsilon} |\tilde{H}_d(\Omega,\Upsilon;\mathbb{Z})|^2,$$

where both sums run over all cellular spanning forests $\Upsilon \subseteq \Sigma$ (see equation (3)) and Ω is an acyclization of Υ (see Definition 7.3).

Before proving these results, we study the cut space (Section 4), the flow space (Section 5), and the cut and flow lattices (Section 6) in some detail. We obtain bases of the cut and flow spaces using the theory of cellular spanning forests, developed by the authors in [13, 14] for cell complexes satisfying a connectedness condition and extended here (in Section 3) to arbitrary complexes (Theorem 3.5). For every cellular spanning forest $\Upsilon \subseteq \Sigma$, the fundamental bonds of Υ give rise to a basis of the cut space (Theorem 4.7). Likewise, the fundamental circuits of Υ induce a basis of the flow space (Theorem 5.5). Moreover, under certain conditions on Υ, these bases are in fact integral bases for the cut and flow lattices (Theorems 6.1 and 6.2).

In the graphic case, these basis vectors are signed characteristic vectors of fundamental bonds and fundamental circuits; in particular, their entries are all 0 and ± 1. In higher dimension, the supports of basis elements are still fundamental bonds and fundamental circuits, but there is the additional subtlety that their entries may not scale to ± 1, making it less clear how to associate a canonical cut- or flow-vector with a bond or circuit. We prove that the characteristic vectors which form the constructed bases of both the cut and flow spaces can be scaled so that their entries are torsion coefficients of homology groups of certain subcomplexes (Theorems 4.10 and 5.3).

The idea of studying cuts and flows of matroids goes back to Tutte [28]. More recently, Su and Wagner [27] define cuts and flows of a regular matroid (i.e., one represented by a totally unimodular matrix M); when M is the boundary matrix of a cell complex, this is the case where the torsion coefficients are all trivial. Su and Wagner's definitions coincide with ours; their focus, however, is on recovering the structure of a matroid from the metric data of its flow lattice.

In the final section of the paper, we generalize a theorem of Kotani and Sunada [22], who observed that a classical inequality for integer lattices, involving Hermite's constant (see, e.g., [23]), could be applied to the flow lattice of a graph to give a bound for girth and complexity. We prove the corresponding result for cell complexes (Theorem 9.2), where "girth" means the size of a smallest circuit in the cellular matroid (or, topologically, the minimum number of facets supporting a nonzero homology class) and "complexity" is the torsion-weighted count of cellular spanning trees.

2. Preliminaries

In this section we review the tools needed throughout the paper: cell complexes, cellular spanning trees and forests, integer lattices, and matroids.

2.1. **Cell complexes.** We assume that the reader is familiar with the basic topology of cell complexes; for a general reference, see [20].

Throughout the paper, Σ will denote a finite CW complex (which we refer to simply as a cell complex) of dimension d. We adopt the convention that Σ

has a unique cell of dimension -1 (as though it were an abstract simplicial complex); this will allow our results to specialize correctly to the case $d = 1$ (i.e., that Σ is a graph). We write Σ_i for the set of i-dimensional cells in Σ, and $\Sigma_{(i)}$ for the i-dimensional skeleton of Σ, i.e., $\Sigma_{(i)} = \Sigma_i \cup \Sigma_{i-1} \cup \cdots \cup \Sigma_0$. A cell of dimension d is called a *facet*.

Unless otherwise stated, every d-dimensional subcomplex $\Gamma \subseteq \Sigma$ will be assumed to have a full codimension-1 skeleton, i.e., $\Gamma_{(d-1)} = \Sigma_{(d-1)}$. Accordingly, for simplicity of notation, we will often make no distinction between the subcomplex Γ itself and its set Γ_d of facets.

The symbol $C_i(\Sigma) = C_i(\Sigma; R)$ denotes the group of i-dimensional cellular chains with coefficients in a ring R. The i-dimensional cellular boundary and coboundary maps are respectively $\partial_i(\Sigma; R) \colon C_i(\Sigma; R) \to C_{i-1}(\Sigma; R)$ and $\partial_i^*(\Sigma; R) \colon C_{i-1}(\Sigma; R) \to C_i(\Sigma; R)$; we will write simply ∂_i and ∂_i^* whenever possible.

The i^{th} reduced cellular homology and cohomology groups of Σ are respectively $\tilde{H}_i(\Sigma; R) = \ker \partial_i / \operatorname{im} \partial_{i+1}$ and $\tilde{H}^i(\Sigma; R) = \ker \partial_{i+1}^* / \operatorname{im} \partial_i^*$. We say that Σ is *R-acyclic in codimension one* if $\tilde{H}_{d-1}(\Sigma; R) = 0$. For a graph ($d = 1$), both $\mathbb{Q}$- and $\mathbb{Z}$-acyclicity in codimension one are equivalent to connectedness. The i^{th} reduced Betti number is $\tilde{\beta}_i(\Sigma) = \dim \tilde{H}_i(\Sigma; \mathbb{Q})$, and the i^{th} torsion coefficient $\mathbf{t}_i(\Sigma)$ is the cardinality of the torsion subgroup $\mathbf{T}(\tilde{H}_i(\Sigma; \mathbb{Z}))$.

A pair of complexes $\Gamma \subseteq \Sigma$ induces a relative complex (Σ, Γ), with relative homology and cohomology $\tilde{H}_i(\Sigma, \Gamma; R)$ and $\tilde{H}^i(\Sigma, \Gamma; R)$ and torsion coefficients $\mathbf{t}_i(\Sigma, \Gamma) = |\mathbf{T}(\tilde{H}_i(\Sigma, \Gamma; \mathbb{Z}))|$.

We will frequently use the fact that

$$\mathbf{T}(\tilde{H}_{d-1}(\Sigma; \mathbb{Z})) \cong \mathbf{T}(\tilde{H}^d(\Sigma; \mathbb{Z})) \tag{1}$$

which is a special case of the universal coefficient theorem for cohomology [20, p. 205, Corollary 3.3].

2.2. **Spanning Forests and Laplacians.** Our work on cuts and flows will use the theory of spanning forests in arbitrary dimensions. Define a *cellular spanning forest* (CSF) of Σ to be a subcomplex $\Upsilon \subseteq \Sigma$ such that $\Upsilon_{(d-1)} = \Sigma_{(d-1)}$ and

$$\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0, \tag{2a}$$

$$\operatorname{rank} \tilde{H}_{d-1}(\Upsilon; \mathbb{Z}) = \operatorname{rank} \tilde{H}_{d-1}(\Sigma; \mathbb{Z}), \quad \text{and} \tag{2b}$$

$$|\Upsilon_d| = |\Sigma_d| - \tilde{\beta}_d(\Sigma) + \tilde{\beta}_{d-1}(\Sigma). \tag{2c}$$

These conditions generalize the definition of a spanning forest[1] of a graph G: respectively, it is acyclic, connected, and has $n - c$ edges, where n and c are the numbers of vertices and components of G. Just as in the graphic case, any two of the conditions (2a), (2b), (2c) together imply the third. An

[1]That is, a maximal acyclic subgraph of G, not merely an acyclic subgraph containing all vertices.

equivalent definition is that a subcomplex $\Upsilon \subseteq \Sigma$ is a cellular spanning forest if and only if its d-cells correspond to a column basis for the cellular boundary matrix $\partial = \partial_d(\Sigma)$.

In the case that Σ is $\mathbb{Q}$-acyclic in codimension one, this definition specializes to our earlier definition of a cellular spanning tree [14, Definition 2.2].

There are two main reasons that enumeration of spanning forests of cell complexes is more complicated than for graphs. First, many properties of graphs can be studied component by component, so that one can usually make the simplifying assumption of connectedness; on the other hand, a higher-dimensional cell complex cannot in general be decomposed into disjoint pieces that are all acyclic in codimension one. Second, for complexes of dimension greater than or equal to two, the possibility of torsion homology affects enumeration.

Define the i^{th} *up-down*, *down-up* and *total* Laplacian operators[2] on Σ by

$$\begin{aligned} L_i^{\mathrm{ud}} &= \partial_{i+1}\partial_{i+1}^* \colon C_i(\Sigma; R) \to C_i(\Sigma; R), \\ L_i^{\mathrm{du}} &= \partial_i^*\partial_i \colon C_i(\Sigma; R) \to C_i(\Sigma; R), \\ L_i^{\mathrm{tot}} &= L_i^{\mathrm{ud}} + L_i^{\mathrm{du}}. \end{aligned}$$

Moreover, define the *complexity* of Σ as

$$\tau(\Sigma) = \tau_d(\Sigma) = \sum_{\text{CSFs } \Upsilon \subseteq \Sigma} |\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})|^2. \tag{3}$$

The *cellular matrix-tree theorem* [14, Theorem 2.8] states that if Σ is $\mathbb{Q}$-acyclic in codimension one and $L_{\bar{\Upsilon}}$ is the submatrix of $L_{d-1}^{\mathrm{ud}}(\Sigma)$ obtained by deleting the rows and columns corresponding to the facets of a $(d-1)$-spanning tree Υ, then

$$\tau(\Sigma) = \frac{|\tilde{H}_{d-2}(\Sigma; \mathbb{Z})|^2}{|\tilde{H}_{d-2}(\Upsilon; \mathbb{Z})|^2} \det L_{\bar{\Upsilon}}.$$

In Section 3, we will generalize this formula to arbitrary cell complexes (i.e., not requiring that Σ be $\mathbb{Q}$-acyclic in codimension one). If G is a connected graph, then $\tau(G)$ is just the number of spanning trees, and we recover the classical matrix-tree theorem of Kirchhoff.

2.3. **Lattices.** Starting in Section 6, we will turn our attention to lattices of integer cuts and flows. We review some of the general theory of integer lattices; see, e.g., [1, Chapter 12], [18, Chapter 14], [21, Chapter IV].

A *lattice* $\mathcal{L}$ is a discrete subgroup of a finite-dimensional vector space V; that is, it is the integer linear combinations of some basis of V. Every lattice $\mathcal{L} \subseteq \mathbb{R}^n$ is isomorphic to $\mathbb{Z}^r$ for some integer $r \leq n$, called the *rank* of $\mathcal{L}$. The elements of $\mathcal{L}$ span a vector space denoted by $\mathcal{L} \otimes \mathbb{R}$. For $\mathcal{L} \subseteq \mathbb{Z}^n$, the *saturation* of $\mathcal{L}$ is defined as $\hat{\mathcal{L}} = (\mathcal{L} \otimes \mathbb{R}) \cap \mathbb{Z}^n$. An *integral*

[2]These are discrete versions of the Laplacian operators on differential forms of a Riemannian manifold. The interested reader is referred to [16] and [12] for their origins in differential geometry and, e.g., [11, 17, 25] for more recent appearances in combinatorics.

basis of $\mathcal{L}$ is a set of linearly independent vectors $v_1, \ldots, v_r \in \mathcal{L}$ such that $\mathcal{L} = \{c_1v_1 + \cdots + c_rv_r : c_i \in \mathbb{Z}\}$. We will need the following fact about integral bases of lattices; the equivalences are easy consequences of the theory of free modules (see, e.g., [1, Chapter 12], [21, Chapter IV]):

Proposition 2.1. *For any lattice $\mathcal{L} \subseteq \mathbb{Z}^n$, the following are equivalent:*

(a) *Every integral basis of $\mathcal{L}$ can be extended to an integral basis of $\mathbb{Z}^n$.*
(b) *Some integral basis of $\mathcal{L}$ can be extended to an integral basis of $\mathbb{Z}^n$.*
(c) *$\mathcal{L}$ is a summand of $\mathbb{Z}^n$, i.e., $\mathbb{Z}^n$ can be written as an internal direct sum $\mathcal{L} \oplus \mathcal{L}'$.*
(d) *$\mathcal{L}$ is the kernel of some group homomorphism $\mathbb{Z}^n \to \mathbb{Z}^m$.*
(e) *$\mathcal{L}$ is saturated, i.e., $\mathcal{L} = \hat{\mathcal{L}}$.*
(f) *$\mathbb{Z}^n/\mathcal{L}$ is a free $\mathbb{Z}$-module, i.e., its torsion submodule is zero.*

Fixing the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, we define the *dual lattice* of $\mathcal{L}$ by

$$\mathcal{L}^\sharp = \{v \in \mathcal{L} \otimes \mathbb{R} : \langle v, w \rangle \in \mathbb{Z} \ \ \forall w \in \mathcal{L}\}.$$

Note that $\mathcal{L}^\sharp$ can be identified with the dual $\mathbb{Z}$-module $\mathcal{L}^* = \mathrm{Hom}(\mathcal{L}, \mathbb{Z})$, and that $(\mathcal{L}^\sharp)^\sharp = \mathcal{L}$. A lattice is called *integral* if it is contained in its dual; for instance, any subgroup of $\mathbb{Z}^n$ is an integral lattice. The *discriminant group* (or *determinantal group*) of an integral lattice $\mathcal{L}$ is $\mathcal{L}^\sharp/\mathcal{L}$; its cardinality can be calculated as $\det M^T M$, for any matrix M whose columns form an integral basis of $\mathcal{L}$. We will need the following facts about bases and duals of lattices.

Proposition 2.2. [18, Section 14.6] *Let M be an $n \times r$ integer matrix.*

(a) *If the columns of M form an integral basis for the lattice $\mathcal{L}$, then the columns of $M(M^TM)^{-1}$ form the corresponding dual basis for $\mathcal{L}^\sharp$.*
(b) *The matrix $P = M(M^TM)^{-1}M^T$ represents orthogonal projection from $\mathbb{R}^n$ onto the column space of M.*
(c) *If the greatest common divisor of the $r \times r$ minors of M is 1, then $\mathcal{L}^\sharp$ is generated by the columns of P.*

2.4. **The cellular matroid.** Many results of the paper may be expressed efficiently using the language of matroids. For a general reference on matroids, see, e.g., [26]. We will primarily consider cellular matroids. The *cellular matroid* of Σ is the matroid $\mathcal{M}(\Sigma)$ represented over $\mathbb{R}$ by the columns of the boundary matrix ∂. Thus the ground set of $\mathcal{M}(\Sigma)$ naturally corresponds to the d dimensional cells Σ_d, and $\mathcal{M}(\Sigma)$ records which sets of columns of ∂ are linearly independent. If Σ is a graph, then $\mathcal{M}(\Sigma)$ is its usual graphic matroid, while if Σ is a simplicial complex then $\mathcal{M}(\Sigma)$ is its simplicial matroid (see [9]).

The bases of $\mathcal{M}(\Sigma)$ are the collections of facets of cellular spanning forests of Σ. If r is the rank function of the matroid $\mathcal{M}(\Sigma)$, then for each set of facets $B \subseteq \Sigma_d$, we have $r(B) = \operatorname{rank} \partial_B$, where ∂_B is the submatrix consisting of

the columns indexed by the facets in B. Moreover, we have

$$r(\Sigma) := r(\Sigma_d) = \operatorname{rank} \mathcal{M}(\Sigma) = \operatorname{rank} \partial = |\Sigma_d| - \tilde{\beta}_d(\Sigma)$$

by the definition of Betti number.

A set of facets $B \subseteq \Sigma_d$ is called a *cut* if deleting B from Σ increases its codimension-one homology, i.e., $\tilde{\beta}_{d-1}(\Sigma \setminus B) > \tilde{\beta}_{d-1}(\Sigma)$. A cut B is a *bond* if $r(\Sigma \setminus B) = r(\Sigma) - 1$, but $r((\Sigma \setminus B) \cup \sigma) = r(\Sigma)$ for every $\sigma \in B$. That is, a bond is a minimal cut. In matroid terminology, a bond of Σ is precisely a cocircuit of $\mathcal{M}(\Sigma)$, i.e., a minimal set that meets every basis of $\mathcal{M}(\Sigma)$. Equivalently, a bond is the complement of a flat of rank $r(\Sigma) - 1$. If Υ is a cellular spanning forest (i.e., a basis of $\mathcal{M}(\Sigma)$) and $\sigma \in \Upsilon_d$ is a facet, then the *fundamental bond* of the pair (Υ, σ) is

$$\mathsf{bo}(\Upsilon, \sigma) = \sigma \cup \{\rho \in \Sigma_d \setminus \Upsilon \colon \Upsilon \setminus \sigma \cup \rho \text{ is a CSF}\}\,. \tag{4}$$

This is the fundamental cocircuit of the pair (Υ, σ) of $\mathcal{M}(\Sigma)$ [26, p. 78].

While we will frequently adopt a matroid-theoretic perspective, it is important to point out that the cut and flow spaces and lattices of a complex Σ are not matroid invariants, i.e., they are not determined by the cellular matroid $\mathcal{M}(\Sigma)$. (See [27] for more on this subject.) Below is a table collecting some of the standard terminology from linear algebra, graph theory, and matroid theory, along with the analogous concepts that we will be using for cell complexes.

Linear algebra	**Graph**	**Matroid**	**Cell complex**
Column vectors	Edges	Ground set	Facets
Independent set	Acyclic subgraph	Independent set	Acyclic subcomplex
Min linear dependence	Cycle	Circuit	Circuit
Basis	Spanning forest	Basis	CSF
Set meeting all bases	Disconnecting set	Codependent set	Cut
Min set meeting all bases	Bond	Cocircuit	Bond
Rank	# edges in spanning forest	Rank	# facets in CSF

Here "codependent" means dependent in the dual matroid.

3. Enumerating Cellular Spanning Forests

In this section, we count the cellular spanning forests of an arbitrary cell complex Σ. This will be necessary for our calculations of cut and flow vectors in Sections 4 and 5, and may be of independent interest. The main result (Theorem 3.5) generalizes the simplicial and cellular matrix-tree theorems of [13] and [14] (in which we required that Σ be $\mathbb{Q}$-acyclic in codimension one). The arguments require some tools from homological algebra, in particular the long exact sequence for relative homology and some facts about the torsion-subgroup functor. The details of the proofs are not necessary to understand the constructions of cut and flow spaces in the later sections.

Let Σ be a d-dimensional cell complex with rank r. Let $\Gamma \subseteq \Sigma$ be a subcomplex of dimension less than or equal to $d-1$ such that $\Gamma_{(d-2)} = \Sigma_{(d-2)}$.

Thus the inclusion map $i\colon \Gamma \to \Sigma$ induces isomorphisms $i_*\colon \tilde{H}_k(\Gamma;\mathbb{Q}) \to \tilde{H}_k(\Sigma;\mathbb{Q})$ for all $k < d-2$.

Definition 3.1. The subcomplex $\Gamma \subseteq \Sigma$ is called *relatively acyclic* if in fact the inclusion map $i\colon \Gamma \to \Sigma$ induces isomorphisms $i_*\colon \tilde{H}_k(\Gamma;\mathbb{Q}) \to \tilde{H}_k(\Sigma;\mathbb{Q})$ for all $k < d$.

By the long exact sequence for relative homology, Γ is relatively acyclic if and only if $\tilde{H}_d(\Sigma;\mathbb{Q}) \to \tilde{H}_d(\Sigma,\Gamma;\mathbb{Q})$ is an isomorphism and $\tilde{H}_k(\Sigma,\Gamma;\mathbb{Q}) = 0$ for all $k < d$. These conditions can occur only if $|\Gamma_{d-1}| = |\Sigma_{d-1}| - r$. This quantity may be zero (in which case the only relatively acyclic subcomplex is $\Sigma_{(d-2)}$). Two additional special cases are worth noting. First, if $d = 1$, then a relatively acyclic complex consists of one vertex in each connected component. Second, if $\tilde{H}_{d-1}(\Sigma;\mathbb{Q}) = 0$, then Γ is relatively acyclic if and only if it is a cellular spanning forest of $\Sigma_{(d-1)}$.

For a matrix M, we write $M_{A,B}$ for the restriction of M to rows indexed by A and columns indexed by B.

Proposition 3.2. *Let $\Gamma \subseteq \Upsilon \subseteq \Sigma$ be subcomplexes such that $\dim \Upsilon = d$; $\dim \Gamma = d-1$; $|\Upsilon_d| = r$; $|\Gamma_{d-1}| = |\Sigma_{d-1}| - r$; $\Upsilon_{(d-1)} = \Sigma_{(d-1)}$; and $\Gamma_{(d-2)} = \Sigma_{(d-2)}$. Also, let $R = \Sigma_{d-1} \setminus \Gamma$. Then the following are equivalent:*

(a) *The $r \times r$ square matrix $\hat{\partial} = \partial_{R,\Upsilon}$ is nonsingular.*
(b) *$\tilde{H}_d(\Upsilon,\Gamma;\mathbb{Q}) = 0$ and $\tilde{H}_{d-1}(\Upsilon,\Gamma;\mathbb{Q}) = 0$.*
(c) *$\tilde{H}_d(\Upsilon,\Gamma;\mathbb{Q}) = 0$ or $\tilde{H}_{d-1}(\Upsilon,\Gamma;\mathbb{Q}) = 0$.*
(d) *Υ is a cellular spanning forest of Σ and Γ is relatively acyclic.*

Proof. The cellular chain complex of the relative complex (Υ,Γ) is

$$0 \to C_d(\Upsilon,\Gamma;\mathbb{Q}) = \mathbb{Q}^r \xrightarrow{\hat{\partial}} C_{d-1}(\Upsilon,\Gamma;\mathbb{Q}) = \mathbb{Q}^r \to 0$$

with other terms zero. If $\hat{\partial}$ is nonsingular, then $\tilde{H}_d(\Upsilon,\Gamma;\mathbb{Q})$ and $\tilde{H}_{d-1}(\Upsilon,\Gamma;\mathbb{Q})$ are both zero; otherwise, both are nonzero. This proves the equivalence of (a), (b) and (c).

Next, note that $\tilde{H}_d(\Gamma;\mathbb{Q}) = 0$ (because Γ has no cells in dimension d) and that $\tilde{H}_{d-2}(\Upsilon,\Gamma;\mathbb{Q}) = 0$ (because $\Gamma_{(d-2)} = \Upsilon_{(d-2)}$). Accordingly, the long exact sequence for relative homology of (Υ,Γ) is

$$\begin{aligned} 0 \to \tilde{H}_d(\Upsilon;\mathbb{Q}) &\to \tilde{H}_d(\Upsilon,\Gamma;\mathbb{Q}) \\ &\to \tilde{H}_{d-1}(\Gamma;\mathbb{Q}) \to \tilde{H}_{d-1}(\Upsilon;\mathbb{Q}) \to \tilde{H}_{d-1}(\Upsilon,\Gamma;\mathbb{Q}) \\ &\to \tilde{H}_{d-2}(\Gamma;\mathbb{Q}) \to \tilde{H}_{d-2}(\Upsilon;\mathbb{Q}) \to 0. \end{aligned} \tag{5}$$

If $\tilde{H}_d(\Upsilon,\Gamma;\mathbb{Q}) = \tilde{H}_{d-1}(\Upsilon,\Gamma;\mathbb{Q}) = 0$, then $\tilde{H}_d(\Upsilon;\mathbb{Q}) = 0$ (which says that Υ is a cellular spanning forest) and the rest of (5) splits into two isomorphisms that assert precisely that Γ is relatively acyclic (recall that $\tilde{H}_{d-1}(\Upsilon;\mathbb{Q}) = \tilde{H}_{d-1}(\Sigma;\mathbb{Q})$ when Υ is a cellular spanning forest). This implication is reversible, completing the proof. □

The *torsion subgroup* of a finitely generated abelian group A is defined as the subgroup

$$\mathbf{T}(A) = \{x \in A \colon kx = 0 \text{ for some } k \in \mathbb{Z}\}.$$

Note that $A = \mathbf{T}(A)$ if and only if A is finite. The torsion functor $\mathbf{T}$ is left-exact [21, p. 179]. Moreover, if $A \to B \to C \to 0$ is exact and $A = \mathbf{T}(A)$, then $\mathbf{T}(A) \to \mathbf{T}(B) \to \mathbf{T}(C) \to 0$ is exact. We will need the following additional fact about the torsion functor.

Lemma 3.3. *Suppose we have a commutative diagram of finitely generated abelian groups*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{j'} & E' & \longrightarrow & 0 \end{array}$$

such that both rows are exact; A, A' are free; α is an isomorphism; β is surjective; and C, C' are finite. Then there is an induced commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{T}B \oplus G & \longrightarrow & \mathbf{T}C & \longrightarrow & \mathbf{T}D & \longrightarrow & \mathbf{T}E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{T}B' \oplus G & \longrightarrow & \mathbf{T}C' & \longrightarrow & \mathbf{T}D' & \longrightarrow & \mathbf{T}E' & \longrightarrow & 0 \end{array} \tag{6}$$

such that G is finite and both rows are exact. Consequently

$$|\mathbf{T}B| \cdot |\mathbf{T}C'| \cdot |\mathbf{T}D| \cdot |\mathbf{T}E'| = |\mathbf{T}B'| \cdot |\mathbf{T}C| \cdot |\mathbf{T}D'| \cdot |\mathbf{T}E|. \tag{7}$$

Proof. First, let Z be a maximal-rank free summand of D, i.e., $D = Z \oplus \mathbf{T}D$. Then $\operatorname{im}(h) \subseteq \mathbf{T}D$. By exactness at E, the map j must map Z isomorphically onto a free summand of E. Hence $\ker j \subseteq \mathbf{T}D$ as well, and replacing D, E with their torsion summands preserves exactness. The same argument implies that we can replace D', E' with $\mathbf{T}D', \mathbf{T}E'$.

Second, note that A, A', B, B' all have the same rank (since the rows are exact, C, C' are finite, and α is an isomorphism). Hence $f(A)$ is a maximal-rank free submodule of B; we can write $B = \mathbf{T}B \oplus F$, where F is a free summand of B containing $f(A)$. Likewise, write $B' = \mathbf{T}B' \oplus F'$, where F' is a free summand of B' containing $f'(A')$. Meanwhile, β is surjective, hence must restrict to an isomorphism $F \to F'$, which induces an isomorphism $F/f(A) \to F'/f'(A')$. Abbreviating this last group by G, we obtain the desired diagram (6) . Since $\ker g = \operatorname{im} f \subseteq F$, the map $g \colon \mathbf{T}B \oplus G \to \mathbf{T}C$ is injective, proving exactness of the first row; the second row is exact by the same argument. Exactness of each row implies that the alternating product of the cardinalities of the groups is 1, from which the formula (7) follows. □

Proposition 3.4. *Let Σ be a d-dimensional cell complex, let $\Upsilon \subseteq \Sigma$ be a cellular spanning forest, and let $\Gamma \subseteq \Sigma$ be a relatively acyclic $(d-1)$-subcomplex. Then*

$$|\mathbf{t}_{d-1}(\Upsilon)| \cdot |\mathbf{t}_{d-1}(\Sigma, \Gamma)| = |\mathbf{t}_{d-1}(\Sigma)| \cdot |\mathbf{t}_{d-1}(\Upsilon, \Gamma)|.$$

Proof. The inclusion $\Upsilon \subseteq \Sigma$ induces a commutative diagram

$$\begin{array}{ccccccccc}
0 \to \tilde{H}_{d-1}(\Gamma;\mathbb{Z}) & \xrightarrow{i_*} & \tilde{H}_{d-1}(\Upsilon;\mathbb{Z}) & \to & \tilde{H}_{d-1}(\Upsilon,\Gamma;\mathbb{Z}) & \to & \tilde{H}_{d-2}(\Gamma;\mathbb{Z}) & \to & \tilde{H}_{d-2}(\Upsilon;\mathbb{Z}) \to 0 \\
\downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong & & \downarrow\cong \\
0 \to \tilde{H}_{d-1}(\Gamma;\mathbb{Z}) & \xrightarrow{i_*} & \tilde{H}_{d-1}(\Sigma;\mathbb{Z}) & \to & \tilde{H}_{d-1}(\Sigma,\Gamma;\mathbb{Z}) & \to & \tilde{H}_{d-2}(\Gamma;\mathbb{Z}) & \to & \tilde{H}_{d-2}(\Sigma;\mathbb{Z}) \to 0
\end{array}$$

whose rows come from the long exact sequences for relative homology. (For the top row, the group $\tilde{H}_d(\Upsilon,\Gamma;\mathbb{Z})$ is free because $\dim \Upsilon = d$, and on the other hand is purely torsion by Proposition 3.2, so it must be zero. For the bottom row, the condition that Γ is relatively acyclic implies that i_* is an isomorphism over $\mathbb{Q}$; therefore, it is one-to-one over $\mathbb{Z}$.) The groups $\tilde{H}_{d-1}(\Upsilon,\Gamma;\mathbb{Z})$ and $\tilde{H}_{d-1}(\Sigma,\Gamma;\mathbb{Z})$ are purely torsion. The first, fourth and fifth vertical maps are isomorphisms (the last because $\Upsilon_{(d-1)} = \Sigma_{(d-1)}$) and the second is a surjection by the relative homology sequence of the pair (Σ,Υ) (since the relative complex has no cells in dimension $d-1$.) The result now follows by applying Lemma 3.3 and canceling like terms. □

As a consequence, we obtain a version of the cellular matrix-forest theorem that applies to all cell complexes (not only those that are $\mathbb{Q}$-acyclic in codimension one).

Theorem 3.5. *Let Σ be a d-dimensional cell complex, let $\Gamma \subseteq \Sigma$ be a relatively acyclic $(d-1)$-dimensional subcomplex, and let L_Γ be the restriction of $L^{\mathrm{ud}}_{d-1}(\Upsilon)$ to the $(d-1)$-cells of Γ. Then*

$$\tau_d(\Sigma) = \frac{\mathbf{t}_{d-1}(\Sigma)^2}{\mathbf{t}_{d-1}(\Sigma,\Gamma)^2} \det L_\Gamma.$$

Proof. By the Binet-Cauchy formula and Propositions 3.2 and 3.4, we have

$$\begin{aligned}
\det \hat{L} = \det \partial_\Gamma \partial_\Gamma^* &= \sum_{\Upsilon \subseteq \Sigma_d \colon |\Upsilon| = r(\Sigma)} (\det \partial_{\Gamma,\Upsilon})^2 \;=\; \sum_{\text{CSFs } \Upsilon \subseteq \Sigma_d} \mathbf{t}_{d-1}(\Upsilon,\Gamma)^2 \\
&= \frac{\mathbf{t}_{d-1}(\Sigma,\Gamma)^2}{\mathbf{t}_{d-1}(\Sigma)^2} \sum_{\text{CSFs } \Upsilon \subseteq \Sigma_d} \mathbf{t}_{d-1}(\Upsilon)^2 \;=\; \frac{\mathbf{t}_{d-1}(\Sigma,\Gamma)^2}{\mathbf{t}_{d-1}(\Sigma)^2} \tau_d(\Sigma)
\end{aligned}$$

and solving for $\tau_d(\Sigma)$ gives the desired formula. □

If $\tilde{H}_{d-1}(\Sigma;\mathbb{Z}) = \mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z}))$, then the relative homology sequence of the pair (Σ,Γ) gives rise to the exact sequence

$$0 \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z})) \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma,\Gamma;\mathbb{Z})) \to \mathbf{T}(\tilde{H}_{d-2}(\Gamma;\mathbb{Z})) \to \mathbf{T}(\tilde{H}_{d-2}(\Sigma;\mathbb{Z})) \to 0$$

which implies that $\mathbf{t}_{d-1}(\Sigma)/\mathbf{t}_{d-1}(\Sigma,\Gamma) = \mathbf{t}_{d-2}(\Sigma)/\mathbf{t}_{d-2}(\Gamma)$, so Theorem 3.5 becomes the formula $\tau_d(\Sigma) = \frac{\mathbf{t}_{d-2}(\Sigma)^2}{\mathbf{t}_{d-2}(\Gamma)^2} \det L_\Gamma$. This was one of the original versions of the cellular matrix-tree theorem [14, Theorem 2.8(2)].

4. The Cut Space

Throughout this section, let Σ be a cell complex of dimension d and rank r (that is, every cellular spanning forest of Σ has r facets). For each $i \leq d$, the *i-cut space* and *i-flow space* of Σ are defined respectively as the spaces of cellular coboundaries and cellular cycles:

$$\mathrm{Cut}_i(\Sigma) = \mathrm{im}(\partial_i^* \colon C_{i-1}(\Sigma, \mathbb{R}) \to C_i(\Sigma, \mathbb{R})),$$
$$\mathrm{Flow}_i(\Sigma) = \ker(\partial_i \colon C_i(\Sigma, \mathbb{R}) \to C_{i-1}(\Sigma, \mathbb{R})).$$

We will primarily be concerned with the case $i = d$. For $i = 1$, these are the standard graph-theoretic cut and flow spaces of the 1-skeleton of Σ.

There are two natural ways to construct bases of the cut space of a graph, in which the basis elements correspond to either (a) vertex stars or (b) the fundamental circuits of a spanning forest (see, e.g. [18, Chapter 14]). The former is easy to generalize to cell complexes, but the latter takes more work.

First, if G is a graph on vertex set V and R is a set of ("root") vertices, one in each connected component, then the rows of ∂ corresponding to the vertices $V \setminus R$ form a basis for $\mathrm{Cut}_1(G)$. This observation generalizes easily to cell complexes:

Proposition 4.1. *A set of r rows of ∂ forms a row basis if and only if the corresponding set of $(d-1)$-cells is the complement of a relatively acyclic $(d-1)$-subcomplex.*

This is immediate from Proposition 3.2. Recall that if $\tilde{H}_{d-1}(\Sigma; \mathbb{Q}) = 0$, then "relatively acyclic $(d-1)$-subcomplex" is synonymous with "spanning tree of the $(d-1)$-skeleton". In this case, Proposition 4.1 is also a consequence of the fact that the matroid represented by the rows of ∂_d is dual to the matroid represented by the columns of ∂_{d-1} [14, Proposition 6.1].

The second way to construct a basis of the cut space of a graph is to fix a spanning tree and take the signed characteristic vectors of its fundamental bonds. In the cellular setting, it is not hard to show that each bond supports a unique (up to scaling) vector in the cut space (Lemma 4.3) and that the fundamental bonds of a fixed cellular spanning forest give rise to a vector space basis (Theorem 4.7). (Recall from Section 2 that a bond in a cell complex is a minimal collection of facets whose removal increases the codimension-one homology, or equivalently a cocircuit of the cellular matroid.) The hard part is to identify the entries of these cut-vectors. For a graph, these entries are all 0 or ± 1. In higher dimension, this need not be the case, but the entries can be interpreted as the torsion coefficients of certain subcomplexes (Theorem 4.10). In Section 5, we will prove analogous results for the flow space.

4.1. A basis of cut-vectors.

Recall that the *support* of a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is the set

$$\mathrm{supp}(v) = \{i \in [n] \colon v_i \neq 0\}.$$

Proposition 4.2. [26, Proposition 9.2.4]. *Let M be a $r \times n$ matrix with rowspace $V \subseteq \mathbb{R}^n$, and let $\mathcal{M}$ be the matroid represented by the columns of M. Then the cocircuits of $\mathcal{M}$ are the inclusion-minimal elements of the family* $\operatorname{Supp}(V) := \{\operatorname{supp}(v) \colon v \in V \setminus \{0\}\}$.

Lemma 4.3. *Let B be a bond of Σ. Then the set*

$$\operatorname{Cut}_B(\Sigma) = \{0\} \cup \{v \in \operatorname{Cut}_d(\Sigma) \colon \operatorname{supp}(v) = B\}$$

is a one-dimensional subspace of $\operatorname{Cut}_d(\Sigma)$. *That is, up to scalar multiple, there is a unique cut-vector whose support is exactly B.*

Proof. Suppose that v, w are vectors in the cut space, both supported on B, that are not scalar multiples of each other. Then there is a linear combination of v, w with strictly smaller support; this contradicts Proposition 4.2. On the other hand, Proposition 4.2 also implies that $\operatorname{Cut}_B(\Sigma)$ is not the zero space; therefore, it has dimension 1. □

We now know that for every bond B, there is a cut-vector supported on B that is uniquely determined up to a scalar multiple. As we will see, there is a choice of scale so that the coefficients of this cut-vector are given by certain minors of the down-up Laplacian $L = L_d^{\mathrm{du}}(\Sigma) = \partial^*\partial$ (Lemma 4.5); these minors (up to sign) can be interpreted as the cardinalities of torsion homology groups (Theorem 4.10).

In choosing a scale, the first step is to realize the elements of $\operatorname{Cut}_B(\Sigma)$ explicitly as images of the map ∂^*. Fix an inner product $\langle \cdot, \cdot \rangle$ on each chain group $C_i(\Sigma; \mathbb{R})$ by declaring the i-dimensional cells to be an orthonormal basis. (This amounts to identifying each cell with the cochain that is its characteristic function.) Thus, for $\alpha \in C_i(\Sigma; \mathbb{R})$, we have $\operatorname{supp}(\alpha) = \{\sigma \in \Sigma_i \colon \langle \sigma, \alpha \rangle \neq 0\}$. Moreover, for all $\beta \in C_{i-1}(\Sigma; \mathbb{R})$, we have by basic linear algebra

$$\langle \partial\alpha, \beta \rangle = \langle \alpha, \partial^*\beta \rangle \,. \tag{8}$$

Lemma 4.4. *Let B be a bond of Σ and let U be the space spanned by $\{\partial\sigma \colon \sigma \in \Sigma_d \setminus B\}$. In particular, U is a subspace of* $\operatorname{im}\partial$ *of codimension one. Let V be the orthogonal complement of U in* $\operatorname{im}\partial$, *and let v be a nonzero element of V. Then* $\operatorname{supp}(\partial^* v) = B$.

Proof. First, we show that $\partial^* v \neq 0$. To see this, observe that the column space of ∂ is $U + \mathbb{R}v$, so the column space of $\partial^*\partial$ is $\partial^* U + \mathbb{R}\partial^* v$. However, $\operatorname{rank}(\partial^*\partial) = \operatorname{rank}\partial = r$, and $\dim U = r - 1$; therefore, $\partial^* v$ cannot be the zero vector. Second, if $\sigma \in \Sigma_d \setminus B$, then $\partial\sigma \in U$, so $\langle \partial^* v, \sigma \rangle = \langle v, \partial\sigma \rangle = 0$. It follows that $\operatorname{supp}(\partial^* v) \subseteq B$, and in fact $\operatorname{supp}(\partial^* v) = B$ by Proposition 4.2. □

Given a bond B, let $A = \{\sigma_1, \ldots, \sigma_{r-1}\}$ be a cellular spanning forest of $\Sigma \setminus B$. Fix a facet $\sigma = \sigma_r \in B$, so that $A \cup \sigma$ is a cellular spanning forest

of Σ. Define a vector

$$v = v_{A,\sigma} = \sum_{j=1}^{r} (-1)^j (\det L^{\mathrm{du}}_{A,A\cup\sigma\setminus\sigma_j}) \partial\sigma_j \ \in \ C_{d-1}(\Sigma;\mathbb{Z})$$

so that

$$\partial^* v = \sum_{j=1}^{r} (-1)^j (\det L^{\mathrm{du}}_{A,A\cup\sigma\setminus\sigma_j}) L^{\mathrm{du}}\sigma_j \ \in \ \mathrm{Cut}_d(\Sigma). \tag{9}$$

Lemma 4.5. *For the cut-vector $\partial^* v$ defined in equation* (9)*,*

$$\partial^* v = \sum_{\rho\in B} (\det L^{\mathrm{du}}_{A\cup\rho,A\cup\sigma})\rho.$$

In particular, $\mathrm{supp}(\partial^* v) = B$.

Proof. For each $\rho \in B$,

$$\begin{aligned}
\langle \partial^* v, \rho\rangle &= \sum_{j=1}^{r} (-1)^j \det L^{\mathrm{du}}_{A,A\cup\sigma\setminus\sigma_j} \left\langle L^{\mathrm{du}}\sigma_j, \rho \right\rangle \\
&= \sum_{j=1}^{r} (-1)^j \det L^{\mathrm{du}}_{A,A\cup\sigma\setminus\sigma_j} \langle \partial\sigma_j, \partial\rho\rangle \\
&= \sum_{j=1}^{r} (-1)^j \det L^{\mathrm{du}}_{A,A\cup\sigma\setminus\sigma_j} L^{\mathrm{du}}_{\rho,\sigma_j} \\
&= \det L^{\mathrm{du}}_{A\cup\rho,A\cup\sigma},
\end{aligned}$$

where the last equality comes from expanding the row corresponding to ρ. Note that $\det L^{\mathrm{du}}_{A\cup\rho,A\cup\sigma} \neq 0$ for $\rho = \sigma$, so $\partial^* v \neq 0$. On the other hand, by Cramer's rule, v is orthogonal to $\partial\sigma_1, \ldots, \partial\sigma_{r-1}$, so in fact $\langle \partial^* v, \rho\rangle = 0$ for all $\rho \in \Sigma_d \setminus B$. This establishes the desired formula for $\partial^* v$, and then $\mathrm{supp}(\partial^* v) = B$ by Lemma 4.4. □

Equation (9) does not provide a canonical cut-vector associated to a given bond B, because $\partial^* v$ depends on the choice of A and σ. On the other hand, the bond B can always be expressed as a fundamental bond $\mathsf{bo}(\Upsilon, \sigma)$ (equivalently, fundamental cocircuit; see equation (4) in Section 2.4) by taking σ to be an arbitrary facet of B and taking $\Upsilon = A \cup \sigma$, where A is a maximal acyclic subset of $\Sigma \setminus B$. This observation suggests that the underlying combinatorial data that gives rise to a cut-vector is really the pair (Υ, σ).

Definition 4.6. Let $\Upsilon = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ be a cellular spanning forest of Σ, and let $\sigma = \sigma_i \in \Upsilon$. The *(uncalibrated) characteristic vector* of the bond $\mathsf{bo}(\Upsilon, \sigma)$ is:

$$\bar{\chi}(\Upsilon, \sigma) = \sum_{j=1}^{r} (-1)^j (\det L^{\mathrm{du}}_{\Upsilon\setminus\sigma,\Upsilon\setminus\sigma_j}) L^{\mathrm{du}}\sigma_j$$

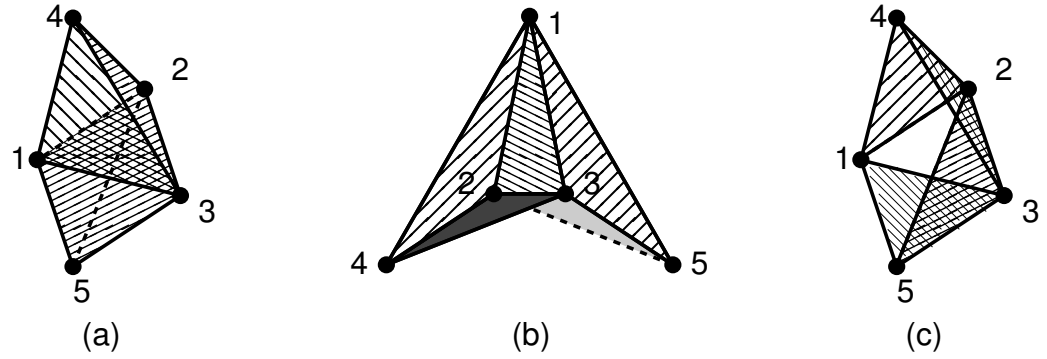

FIGURE 1. (a) The bipyramid Θ. (b) A simplicial spanning tree (unfolded). (c) Deleting the bond $\{123, 134, 125\}$.

By Lemma 4.5, taking $A = \Upsilon \setminus \sigma$, we have

$$\bar{\chi}(\Upsilon, \sigma) = \sum_{\rho \in \mathsf{bo}(\Upsilon,\sigma)} (\det L^{\mathrm{du}}_{\Upsilon\setminus\sigma\cup\rho,\Upsilon})\rho,$$

a cut-vector supported on $\mathsf{bo}(\Upsilon, \sigma)$.

The next result is the cellular analogue of [18, Lemma 14.1.3].

Theorem 4.7. *The family* $\{\bar{\chi}(\Upsilon, \sigma) \colon \sigma \in \Upsilon\}$ *is a* $\mathbb{R}$*-vector space basis for the cut space of* Σ*.*

Proof. Let $\sigma \in \Upsilon$. Then $\operatorname{supp} \bar{\chi}(\Upsilon, \sigma) = \mathsf{bo}(\Upsilon, \sigma)$ contains σ, but no other facet of Υ. Therefore, the set of characteristic vectors is linearly independent, and its cardinality is $|\Upsilon_d| = r = \dim \mathrm{Cut}_d(\Sigma)$. □

Example 4.8. The *equatorial bipyramid* is the two-dimensional simplicial complex Θ with facet set $\{123, 124, 125, 134, 135, 234, 235\}$ (Figure 1(a)). Let Υ be the simplicial spanning tree with facets $\{123, 124, 234, 135, 235\}$ (unfolded in Figure 1(b)). Then

$$\begin{aligned} &\mathsf{bo}(\Upsilon, 123) = \{123, 125, 134\}, && \mathsf{bo}(\Upsilon, 124) = \{124, 134\}, \\ &\mathsf{bo}(\Upsilon, 135) = \{135, 125\}, && \mathsf{bo}(\Upsilon, 234) = \{234, 134\}, \\ &\mathsf{bo}(\Upsilon, 235) = \{235, 125\}. \end{aligned}$$

In each case, the removal of the bond leaves a 1-dimensional hole (as shown for the bond $\{123, 125, 134\}$ in Figure 1(c)). By Theorem 4.7, we have

$$\begin{aligned} &\bar{\chi}(\Upsilon, 123) = 75([123] + [125] - [134]), && \bar{\chi}(\Upsilon, 124) = 75([124] + [134]), \\ &\bar{\chi}(\Upsilon, 135) = 75([125] + [135]), && \bar{\chi}(\Upsilon, 234) = 75([134] + [234]), \\ &\bar{\chi}(\Upsilon, 235) = 75([235] - [125]), \end{aligned}$$

which indeed form a basis for $\mathrm{Cut}_2(\Theta)$.

4.2. **Calibrating the characteristic vector of a bond.** The term "characteristic vector" suggests that the coefficients of $\bar{\chi}(\Upsilon, \sigma)$ should all be 0 or 1, but this is not necessarily possible, even by scaling, as Example 4.12 below will show. We would like to define the characteristic vector of a bond so that it carries combinatorial or topological information, avoiding extra factors such as the 75 in Example 4.8. We will show that the number

$$\mu_\Upsilon := \mathbf{t}_{d-1}(\Upsilon) \sum_\Gamma \frac{\mathbf{t}_{d-1}(\Sigma, \Gamma)^2}{\mathbf{t}_{d-1}(\Sigma)^2}, \tag{10}$$

where the sum runs over all relatively acyclic $(d-1)$-subcomplexes $\Gamma \subseteq \Sigma$, divides every coefficient of the characteristic vector. Moreover, we will show that the entries of $\frac{1}{\mu_\Upsilon}\bar{\chi}(\Upsilon, \sigma)$ are (up to sign) the torsion coefficients of the cellular forests $\{\Upsilon \setminus \sigma \cup \rho\}$ for $\rho \in \mathsf{bo}(\Upsilon, \sigma)$.

Let $\varepsilon^A_{\sigma,\sigma'}$ be the relative sign of $\partial\sigma, \partial\sigma'$ with respect to ∂A; that is, it is $+1$ or -1 according to whether $\partial\sigma$ and $\partial\sigma'$ lie on the same or the opposite sides of the hyperplane in $\operatorname{im}\partial$ spanned by ∂A. In the language of oriented matroids, this sign is simply a product of the entries corresponding to $\partial\sigma$ and $\partial\sigma'$ in one of the cocircuits corresponding to the hyperplane and determines the relative signs of a basis orientation on $A \cup \sigma$ and $A \cup \sigma'$ [6, Section 3.5].

Proposition 4.9. *Let Σ be a cell complex of rank r. Let $\Upsilon = A \cup \sigma$ and $\Upsilon' = A \cup \sigma'$ be d-dimensional cellular spanning forests of Σ. Let $L^{\mathrm{du}}_{\Upsilon,\Upsilon'}$ be the restriction of the down-up Laplacian $L^{\mathrm{du}} = \partial^*\partial$ to the rows indexed by Υ and the columns indexed by Υ'. Then*

$$\det L^{\mathrm{du}}_{\Upsilon,\Upsilon'} = \det \partial^*_\Upsilon \partial_{\Upsilon'} = \varepsilon \mu_\Upsilon \mathbf{t}_{d-1}(\Upsilon')$$

where $\varepsilon = \varepsilon^A_{\sigma,\sigma'}$.

Proof. By the Binet-Cauchy formula, we have

$$\det L^{\mathrm{du}}_{\Upsilon,\Upsilon'} = \sum_S (\det \partial^*_{\Upsilon,S})(\det \partial_{S,\Upsilon'}) = \sum_S (\det \partial_{S,\Upsilon})(\det \partial_{S,\Upsilon'}), \tag{11}$$

where the sum runs over all sets $S \subseteq \Sigma_{d-1}$ with $|S| = r$, and $\partial_{S,\Upsilon}$ is the corresponding $r \times r$ submatrix of ∂. By Proposition 3.2, $\det \partial_{S,\Upsilon}$ is nonzero if and only if $\Gamma_S = \Sigma_{d-1} \setminus S$ is relatively acyclic, and for those summands we have by Proposition 3.4 $|\det \partial_{S,\Upsilon}| = \mathbf{t}_{d-1}(\Upsilon, \Gamma_S) = \mathbf{t}_{d-1}(\Upsilon)\mathbf{t}_{d-1}(\Sigma, \Gamma_S)/\mathbf{t}_{d-1}(\Sigma)$. The sign of each summand in equation (11) is just ε, so

$$\begin{aligned}
\det L^{\mathrm{du}}_{\Upsilon,\Upsilon'} &= \sum_S (\det \partial_{S,\Upsilon})(\det \partial_{S,\Upsilon'}) \\
&= \sum_S \varepsilon \left(\frac{\mathbf{t}_{d-1}(\Upsilon)\mathbf{t}_{d-1}(\Sigma, \Gamma_S)}{\mathbf{t}_{d-1}(\Sigma)}\right)\left(\frac{\mathbf{t}_{d-1}(\Upsilon')\mathbf{t}_{d-1}(\Sigma, \Gamma_S)}{\mathbf{t}_{d-1}(\Sigma)}\right) \\
&= \varepsilon \frac{\mathbf{t}_{d-1}(\Upsilon)\mathbf{t}_{d-1}(\Upsilon')}{\mathbf{t}_{d-1}(\Sigma)^2} \sum_\Gamma \mathbf{t}_{d-1}(\Sigma, \Gamma)^2 \\
&= \varepsilon \mu_\Upsilon \mathbf{t}_{d-1}(\Upsilon').
\end{aligned}$$

□

Theorem 4.10. *Let B be a bond. Fix a facet $\sigma \in B$ and a cellular forest $A \subseteq \Sigma_d \setminus B$, so that in fact $B = \mathsf{bo}(A \cup \sigma, \sigma)$. Define the* characteristic vector of B with respect to A *as*

$$\chi_A(B) := \frac{1}{\mu_\Upsilon} \bar{\chi}(A \cup \sigma, \sigma) = \sum_{\rho \in B} \varepsilon^A_{\sigma,\rho} \mathbf{t}_{d-1}(A \cup \rho)\ \rho.$$

Then $\chi_A(B)$ is in the cut space of Σ, and has integer coefficients. Moreover, it depends on the choice of σ only up to sign.

Proof. Apply the formula of Proposition 4.9 to the formula of Definition 4.6 for the characteristic vector, and factor the integer μ_Υ out of every coefficient. Meanwhile, replacing σ with a different facet $\sigma' \in B$ merely multiplies all coefficients by $\varepsilon^A_{\sigma',\rho}/\varepsilon^A_{\sigma,\rho} = \varepsilon^A_{\sigma,\sigma'} \in \{\pm 1\}$. □

If Υ is a cellular spanning forest of Σ and $\sigma \in \Upsilon_d$, then we define the *characteristic vector* of the pair (Υ, σ) by

$$\chi(\Upsilon, \sigma) := \chi_{\Upsilon \setminus \sigma}(\mathsf{bo}(\Upsilon, \sigma)) = \frac{1}{\mu_\Upsilon} \bar{\chi}(\Upsilon, \sigma). \tag{12}$$

Example 4.11. Let Θ be the bipyramid of Example 4.8. Every cellular spanning forest $\Upsilon \subseteq \Theta$ is torsion-free. Moreover, the relatively acyclic subcomplexes Γ that appear in equation (10) are the spanning trees of the 1-skeleton $\Theta_{(1)}$ (see Section 3), which is the graph K_5 with one edge removed; Accordingly, we have $\mu_\Upsilon = \tau(\Theta_{(1)}) = 75$, so the calibrated characteristic vectors are as in Example 4.8, with all factors of 75 removed.

On the other hand, μ_Υ is not necessarily the *greatest* common factor of the entries of each uncalibrated characteristic vector, as the following example illustrates.

Example 4.12. Consider the cell complex Σ with a single vertex v, two 1-cells e_1 and e_2 attached at v, and four 2-cells attached via the boundary matrix

$$\partial = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cccc} \sigma_2 & \sigma_3 & \sigma_5 & \sigma_7 \end{array} \\ \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 5 & 7 \end{pmatrix} \end{array}.$$

Let B be the bond $\{\sigma_2, \sigma_3\}$, so that the obvious candidate for a cut-vector supported on B is the row vector $\begin{bmatrix} 2 & 3 & 0 & 0 \end{bmatrix}$. On the other hand, taking $A = \{\sigma_5\}$ (a cellular spanning forest of $\Sigma \setminus B$), the calibrated characteristic vector of Theorem 4.10 is

$$\chi_A(B) = \begin{bmatrix} 10 & 15 & 0 & 0 \end{bmatrix}.$$

For $\Upsilon = A \cup \{\sigma_2\}$, the uncalibrated characteristic vector of Definition 4.6 is

$$\bar{\chi}(\Upsilon, \sigma_2) = \begin{bmatrix} 100 & 150 & 0 & 0 \end{bmatrix}.$$

On the other hand, the calibration factor μ_Υ is not $\gcd(100, 150) = 50$, but rather 10, since $\mathbf{t}_1(\Upsilon) = 10$ and the summation of equation (10) has only

one term, namely $\Gamma = \Sigma_{(0)}$. Similarly, for $A' = \{\sigma_7\}$ and $\Upsilon' = A' \cup \{\sigma_2\}$, we have

$$\chi_{A'}(B) = \begin{bmatrix}14 & 21 & 0 & 0\end{bmatrix}, \quad \bar{\chi}(\Upsilon', \sigma_2) = \begin{bmatrix}196 & 294 & 0 & 0\end{bmatrix}, \quad \mu_{\Upsilon'} = 14.$$

Remark 4.13. Torsion plays a role in the characteristic vectors of bonds, even when Σ is a simplicial complex. For example, let Σ be the the complete 2-dimensional simplicial complex on 6 vertices and let Υ be the triangulation of $\mathbb{RP}^2$ obtained by identifying opposite faces in an icosahedron. Then Υ is a cellular spanning forest of Σ (and in fact Σ has twelve spanning forests of this kind). For any facet $\sigma \in \Upsilon$, we have $\mathsf{bo}(\Upsilon, \sigma) = \Sigma_2 \setminus \Upsilon_2 \cup \{\sigma\}$, and the entries of the calibrated cut-vector include both ± 2 (in position σ) as well as ± 1's (in positions $\Sigma \setminus \Upsilon$).

Remark 4.14. When Σ is a graph and Υ is a spanning forest, μ_Υ is just the number of vertices of Σ. Then, for any edge σ in Υ, the vector $\chi_\Upsilon(\sigma)$ is the usual characteristic vector of the fundamental bond $\mathsf{bo}(\Upsilon, \sigma)$.

Remark 4.15. Taking $\Upsilon = \Upsilon'$ in the calculation of Proposition 4.9 gives the equality

$$\sum_\Gamma \mathbf{t}_{d-1}(\Sigma, \Gamma)^2 = \frac{\mathbf{t}_{d-1}(\Sigma)^2}{\mathbf{t}_{d-1}(\Upsilon)^2} \det L_\Upsilon^{\mathrm{du}}$$

which can be viewed as a dual form of the cellular matrix-forest theorem, Theorem 3.5 (enumerating relatively acyclic $(d-1)$-subcomplexes, rather than cellular spanning forests).

5. The Flow Space

In this section we describe the flow space of a cell complex. We begin by observing that the cut and flow spaces are orthogonal to each other.

Proposition 5.1. *The cut and flow spaces are orthogonal complements under the standard inner product on $C_d(\Sigma; \mathbb{R})$.*

Proof. First, we show that the cut and flow spaces are orthogonal. Let $\alpha \in \mathrm{Cut}_d = \operatorname{im} \partial_d^*$ and $\beta \in \mathrm{Flow}_d = \ker \partial_d$. Then $\alpha = \partial^* \gamma$ for some $(d-1)$-chain γ, and $\langle \alpha, \beta \rangle = \langle \partial^* \gamma, \beta \rangle = \langle \gamma, \partial \beta \rangle = 0$ by equation (8).

It remains to show that Cut_d and Flow_d have complementary dimensions. Indeed, let $n = \dim C_d(\Sigma; \mathbb{R})$; then $\dim \mathrm{Flow}_d = \dim \ker \partial_d = n - \dim \operatorname{im} \partial_d = n - \dim \operatorname{im} \partial_d^* = n - \dim \mathrm{Cut}_d$. □

Next we construct a basis of the flow space whose elements correspond to fundamental circuits of a given cellular spanning forest. Although cuts and flows are in some sense dual constructions, it is easier in this case to work with kernels than images, essentially because of Proposition 2.1. As a consequence, we can much more directly obtain a characteristic flow vector whose coefficients carry topological meaning.

We need one preliminary result from linear algebra.

Proposition 5.2. *Let N be a matrix with c columns and r rows such that* $\operatorname{rank} N = c - 1$. *In particular, $r \geq c - 1$ and* $\dim \ker N = 1$. *Then* $\ker N$ *has a spanning vector $v = (v_1, \ldots, v_c)$ such that*

$$v_i = \pm|\mathbf{T}(\operatorname{coker} N_{\bar{\imath}})|$$

where $N_{\bar{\imath}}$ denotes the submatrix of N obtained by deleting the i^{th} column. In particular, $v_i \neq 0$ for all i.

Proof. Let Q be a $r \times r$ matrix whose first $r-(c-1)$ rows form a $\mathbb{Z}$-module basis for $\ker(N^T)$, and whose remaining $c-1$ rows extend it to a basis of $\mathbb{Z}^r$ (see Proposition 2.1). Then Q is invertible over $\mathbb{Z}$, and the matrix $P = QN = (N^T Q^T)^T$ has the form

$$P = \left[\begin{array}{c} 0 \\ \hline M \end{array}\right]$$

where M is a $(c-1) \times c$ matrix whose column matroid is the same as that of N. Then $\ker N = \ker P = \ker M$. Meanwhile, by Cramer's rule, $\ker M$ is the one-dimensional space spanned by $v = (v_1, \ldots, v_c)$, where $v_i = (-1)^i \det M_{\bar{\imath}} = \pm|\operatorname{coker} M_{\bar{\imath}}| = \pm|\mathbf{T}(\operatorname{coker} P_{\bar{\imath}})|$. For each i, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im} N_{\bar{\imath}} & \longrightarrow & \widehat{\operatorname{im} N_{\bar{\imath}}} & \longrightarrow & \mathbf{T}(\operatorname{coker} N_{\bar{\imath}}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle Q} & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{im} P_{\bar{\imath}} & \longrightarrow & \widehat{\operatorname{im} P_{\bar{\imath}}} & \longrightarrow & \mathbf{T}(\operatorname{coker} P_{\bar{\imath}}) & \longrightarrow & 0 \end{array}$$

where the hat denotes saturation (see Section 2.3). Since Q is invertible, it induces isomorphisms as labeled, so by the snake lemma, the third vertical arrow is an isomorphism as well. Hence $|\mathbf{T}(\operatorname{coker} P_{\bar{\imath}})| = |\mathbf{T}(\operatorname{coker} N_{\bar{\imath}})|$, completing the proof. □

Recall that a set of facets $C \subseteq \Sigma_d$ is a circuit of the cellular matroid $\mathcal{M}(\Sigma)$ if and only if it corresponds to a minimal linearly dependent set of columns of ∂_d. Applying Proposition 5.2 with $N = \partial_C$ (i.e., the restriction of ∂ to the columns indexed by C), we obtain a flow vector whose support is exactly C. We call this the *characteristic vector* $\varphi(C)$.

Theorem 5.3. *Let C be a circuit of the cellular matroid $\mathcal{M}(\Sigma)$, and let $\Delta \subseteq \Sigma$ be the subcomplex $\Sigma_{(d-1)} \cup C$. Then*

$$\varphi(C) = \sum_{\sigma \in C} \pm \mathbf{t}_{d-1}(\Delta \setminus \sigma)\, \sigma.$$

Proof. For $\sigma \in C$, let $N_{\bar{\sigma}}$ denote N with the column σ removed. By Proposition 5.2, it suffices to show that the two groups

$$\tilde{H}_{d-1}(\Delta \setminus \sigma; \mathbb{Z}) = \frac{\ker \partial_{d-1}}{\operatorname{im} N_{\bar{\sigma}}}, \qquad \operatorname{coker} N_{\bar{\sigma}} = \frac{C_{d-1}(\Sigma; \mathbb{Z})}{\operatorname{im} N_{\bar{\sigma}}}$$

have the same torsion summands. But this is immediate because $\ker \partial_{d-1}$ is a summand of $C_{d-1}(\Sigma;\mathbb{Z})$ as a free $\mathbb{Z}$-module. □

Example 5.4. Consider the cell complex Σ with two vertices v_1 and v_2, three one-cells e_1, e_2, and e_3, each one with endpoints v_1 and v_2, and three two-cells σ_1, σ_2, and σ_3 attached to the 2-cells so that the 2-dimensional boundary matrix is

$$\partial = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc} \sigma_1 & \sigma_2 & \sigma_3 \end{array} \\ \begin{pmatrix} 2 & 2 & 0 \\ -2 & 0 & 1 \\ 0 & -2 & -1 \end{pmatrix} \end{array}.$$

The only circuit in Σ is the set C of all three 2-cells; then $\varphi(C) = 2\sigma_1 + 2\sigma_2 + 4\sigma_3$, because $\tilde{H}_2(\Delta \setminus \sigma_1;\mathbb{Z}) = \tilde{H}_2(\Delta \setminus \sigma_2;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$, but $\tilde{H}_2(\Delta \setminus \sigma_3;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

For a cellular spanning forest Υ and facet $\sigma \notin \Upsilon$, let $\mathsf{ci}(\Upsilon, \sigma)$ denote the *fundamental circuit of σ with respect to Υ*, that is, the unique circuit in $\Upsilon \cup \sigma$.

Theorem 5.5. *Let Σ be a cell complex and $\Upsilon \subseteq \Sigma$ a cellular spanning forest. Then the set*

$$\{\varphi(\mathsf{ci}(\Upsilon, \sigma)) \colon \sigma \notin \Upsilon\}$$

forms a $\mathbb{R}$-vector space basis for the flow space of Σ.

Proof. The flow space is the kernel of a matrix with $|\Sigma_d|$ columns and rank $|\Upsilon_d|$, so its dimension is $|\Sigma_d| - |\Upsilon_d|$. Therefore, it is enough to show that the $\varphi(\mathsf{ci}(\Upsilon, \sigma))$ are linearly independent. Indeed, consider the matrix W whose rows are the vectors $\varphi(\mathsf{ci}(\Upsilon, \sigma))$; its maximal square submatrix W' whose columns correspond to $\Sigma \setminus \Upsilon$ has nonzero entries on the diagonal but zeroes elsewhere. □

Example 5.6. Recall the bipyramid of Example 4.8, and its spanning tree Υ. Then $\mathsf{ci}(\Upsilon, 125) = \{125, 123, 135, 235\}$, and $\mathsf{ci}(\Upsilon, 134) = \{134, 123, 124, 234\}$. If we instead consider the spanning tree $\Upsilon' = \{124, 125, 134, 135, 235\}$, then $\mathsf{ci}(\Upsilon', 123) = \{123, 125, 135, 235\}$, and $\mathsf{ci}(\Upsilon', 234) = \{234, 124, 125, 134, 135, 235\}$. Each of these circuits is homeomorphic to a 2-sphere, and the corresponding flow vectors are the homology classes they determine. Furthermore, each of $\{\mathsf{ci}(\Upsilon, 125), \mathsf{ci}(\Upsilon, 134)\}$ and $\{\mathsf{ci}(\Upsilon', 123), \mathsf{ci}(\Upsilon', 234)\}$ is a basis of the flow space.

6. Integral Bases for the Cut and Flow Lattices

Recall that the *cut lattice* and *flow lattice* of Σ are defined as

$$\mathcal{C} = \mathcal{C}(\Sigma) = \mathrm{im}_{\mathbb{Z}}\, \partial_d^* \subseteq \mathbb{Z}^n, \qquad \mathcal{F} = \mathcal{F}(\Sigma) = \ker_{\mathbb{Z}} \partial_d \subseteq \mathbb{Z}^n.$$

In this section, we study the conditions under which the vector space bases of Theorems 4.10 and 5.5 are integral bases for the cut and flow lattices respectively.

Theorem 6.1. *Suppose that Σ has a cellular spanning forest Υ such that $\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})$ is torsion-free. Then*

$$\{\chi(\Upsilon,\sigma)\colon \sigma \in \Upsilon\}$$

is an integral basis for the cut lattice $\mathcal{C}(\Sigma)$, where $\chi(\Upsilon,\sigma)$ is defined as in equation (12)*.*

Proof. Consider the $n \times r$ matrix with columns $\chi(\Upsilon,\sigma)$ for $\sigma \in \Upsilon_d$. Its restriction to the rows Υ_d is diagonal, and by Theorem 4.7 and the hypothesis on $\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})$, its entries are all ± 1. Therefore, the $\chi(\Upsilon,\sigma)$ form an integral basis for the lattice $\mathrm{Cut}_d(\Sigma) \cap \mathbb{Z}^n$. Meanwhile,

$$(\mathrm{Cut}_d(\Sigma) \cap \mathbb{Z}^n)/\mathcal{C}_d(\Sigma) = \mathbf{T}(\tilde{H}^d(\Sigma;\mathbb{Z})) \cong \mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z}))$$

where the first equality is because $\mathrm{Cut}_d(\Sigma) \cap \mathbb{Z}^n$ is a summand of $\mathbb{Z}^n$, and the second one is equation (1). On the other hand, $\tilde{H}_{d-1}(\Sigma;\mathbb{Z})$ is a quotient of $\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})$ of equal rank; in particular, $\mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z})) = 0$ and in fact $\mathrm{Cut}_d(\Sigma) \cap \mathbb{Z}^n = \mathcal{C}_d(\Sigma)$. □

Next we consider integral bases of the flow lattice. For a circuit C, define

$$\hat{\varphi}(C) = \frac{1}{g}\varphi(C)$$

where $\varphi(C)$ is the characteristic vector defined in Section 5 and g is the gcd of its coefficients. Thus $\hat{\varphi}(C)$ generates the rank-1 free $\mathbb{Z}$-module of flow vectors supported on C.

Theorem 6.2. *Suppose that Σ has a cellular spanning forest Υ such that $\tilde{H}_{d-1}(\Upsilon;\mathbb{Z}) = \tilde{H}_{d-1}(\Sigma;\mathbb{Z})$. Then $\{\hat{\varphi}(\mathsf{ci}(\Upsilon,\sigma))\colon \sigma \notin \Upsilon\}$ is an integral basis for the flow lattice $\mathcal{F}(\Sigma)$.*

Proof. By the hypothesis on Υ, the columns of ∂ indexed by the facets in Υ form a $\mathbb{Z}$-basis for the column space. That is, for every $\sigma \notin \Upsilon$, the column ∂_σ is a $\mathbb{Z}$-linear combination of the columns of Υ; equivalently, there is an element w_σ of the flow lattice, with support $\mathsf{ci}(\Upsilon,\sigma)$, whose coefficient in the σ position is ± 1. But then w_σ and $\hat{\varphi}(\mathsf{ci}(\Upsilon,\sigma))$ are integer vectors with the same linear span, both of which have the gcd of their entries equal to 1; therefore, they must be equal up to sign. Therefore, retaining the notation of Theorem 5.5, the matrix W' is in fact the identity matrix, and it follows that the lattice spanned by the $\hat{\varphi}(\mathsf{ci}(\Upsilon,\sigma))$ is saturated, so it must equal the flow lattice of Σ. □

If Σ is a graph, then all its subcomplexes and relative complexes are torsion-free (equivalently, its incidence matrix is totally unimodular). Therefore, Theorems 6.1 and 6.2 give integral bases for the cut and flow lattices respectively. These are, up to sign, the integral bases constructed combinatorially in, e.g., [18, Chapter 14].

7. Groups and Lattices

In this section, we define the critical, cocritical, and cutflow groups of a cell complex. We identify the relationships between these groups and to the discriminant groups of the cut and flow lattices. The case of a graph was studied in detail by Bacher, de la Harpe and Nagnibeda [2] and Biggs [4], and is presented concisely in [18, Chapter 14].

Throughout this section, let Σ be a cell complex of dimension d with n facets, and identify both $C_d(\Sigma;\mathbb{Z})$ and $C^d(\Sigma;\mathbb{Z})$ with $\mathbb{Z}^n$.

Definition 7.1. The *critical group* of Σ is

$$K(\Sigma) := \mathbf{T}(\ker \partial_{d-1}/\operatorname{im} \partial_d \partial_d^*) = \mathbf{T}(\operatorname{coker}(\operatorname{im} \partial_d \partial_d^*)).$$

Here and henceforth, all kernels and images are taken over $\mathbb{Z}$.

Note that the second and third terms in the definition are equivalent because $\ker \partial_{d-1}$ is a summand of $C_{d-1}(\Sigma;\mathbb{Z})$ as a free $\mathbb{Z}$-module. This definition coincides with the usual definition of the critical group of a graph in the case $d = 1$, and with the authors' previous definition in [15] in the case that Σ is $\mathbb{Q}$-acyclic in codimension one (when $\ker \partial_{d-1}/\operatorname{im} \partial_d \partial_d^*$ is its own torsion summand).

Definition 7.2. The *cutflow group* of Σ is $\mathbb{Z}^n/(\mathcal{C}(\Sigma) \oplus \mathcal{F}(\Sigma))$.

Note that the cutflow group is finite because the cut and flow spaces are orthogonal complements in $\mathbb{R}^n$ (Proposition 5.1), so in particular $\mathcal{C} \oplus \mathcal{F}$ spans $\mathbb{R}^n$ as a vector space. Observe also that the cutflow group does not decompose into separate cut and flow pieces; that is, it is not isomorphic to the group $G = ((\mathrm{Flow}_d \cap \mathbb{Z}^n)/\ker \partial_d) \oplus ((\mathrm{Cut}_d \cap \mathbb{Z}^n)/\operatorname{im} \partial_d^*)$, even when Σ is a graph. For example, if Σ is the complete graph on three vertices, whose boundary map can be written as

$$\partial = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

then $\ker \partial = \operatorname{span}\{(1,1,1)^T\}$ and $\operatorname{im} \partial = \operatorname{span}\{(1,-1,0)^T, (1,0,-1)^T\}$. So G is the trivial group, while $\mathbb{Z}^n/(\ker \partial_d \oplus \operatorname{im} \partial_d^*) = K_1(\Sigma) \cong \mathbb{Z}_3$.

In order to define the cocritical group of a cell complex, we first need to introduce the notion of acyclization.

Definition 7.3. An *acyclization* of Σ is a $(d+1)$-dimensional complex Ω such that $\Omega_{(d)} = \Sigma$ and $\tilde{H}_{d+1}(\Omega;\mathbb{Z}) = \tilde{H}_d(\Omega;\mathbb{Z}) = 0$.

Algebraically, this construction corresponds to finding an integral basis for $\ker \partial_d(\Sigma)$ and declaring its elements to be the columns of $\partial_{d+1}(\Omega)$ (so in particular $|\Omega_{(d+1)}| = \tilde{\beta}_d(\Sigma)$). The definition of acyclization and equation (1) together imply that $\tilde{H}^{d+1}(\Omega;\mathbb{Z}) = 0$; that is, $\partial_{d+1}^*(\Omega)$ is surjective.

Definition 7.4. The *cocritical group* $K^*(\Sigma)$ is

$$K^*(\Sigma) := C_{d+1}(\Omega;\mathbb{Z})/\operatorname{im}\partial_{d+1}^*\partial_{d+1} = \operatorname{coker} L_{d+1}^{\mathrm{du}}.$$

It is not immediate that the group $K^*(\Sigma)$ is independent of the choice of Ω; we will prove this independence as part of Theorem 7.7. For the moment, it is at least clear that $K^*(\Sigma)$ is finite, since $\operatorname{rank}\partial_{d+1}^* = \operatorname{rank} L_{d+1}^{\mathrm{du}} = \operatorname{rank} C_{d+1}(\Omega;\mathbb{Z})$. In the special case of a graph, the cocritical group coincides with the discriminant group of the lattice generated by the columns of the "intersection matrix" defined by Kotani and Sunada [22]. (See also [5, Sections 2, 3].)

Remark 7.5. As in [15], one can define critical and cocritical groups in every dimension by $K_i(\Sigma) = \mathbf{T}(C_i(\Sigma;\mathbb{Z})/\operatorname{im}\partial_{i+1}\partial_{i+1}^*)$ and $K_i^*(\Sigma) = \mathbf{T}(C_i(\Sigma;\mathbb{Z})/\operatorname{im}\partial_i^*\partial_i)$. If the cellular chain complexes of Σ and Ψ are algebraically dual (for example, if Σ and Ψ are Poincaré dual cell structures on a compact orientable d-manifold), then $K_i(\Psi) = K_{d-i}^*(\Sigma)$ for all i.

We now come to the main results of the second half of the paper: the critical and cocritical groups are isomorphic to the discriminant groups of the cut and flow lattices respectively, and the cutflow group mediates between the critical and cocritical groups, with an "error term" given by homology.

Theorem 7.6. *Let Σ be a cell complex of dimension d with n facets. Then there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F}) & \xrightarrow{\psi} & \mathcal{C}^\sharp/\mathcal{C} & \longrightarrow & \mathbf{T}(\tilde{H}^d(\Sigma;\mathbb{Z})) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \operatorname{im}\partial_d/\operatorname{im}\partial_d\partial_d^* & \longrightarrow & K(\Sigma) & \longrightarrow & \mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z})) & \longrightarrow & 0 \end{array} \tag{13}$$

in which all vertical maps are isomorphisms. In particular, $K(\Sigma) \cong \mathcal{C}^\sharp/\mathcal{C}$.

Proof. Step 1: Construct the bottom row of (13). The inclusions $\operatorname{im}\partial_d\partial_d^* \subseteq \operatorname{im}\partial_d \subseteq \ker\partial_{d-1}$ give rise to the short exact sequence

$$0 \to \operatorname{im}\partial_d/\operatorname{im}\partial_d\partial_d^* \to \ker\partial_{d-1}/\operatorname{im}\partial_d\partial_d^* \to \ker\partial_{d-1}/\operatorname{im}\partial_d \to 0.$$

The first term is finite (because $\operatorname{rank}\partial_d = \operatorname{rank}\partial_d\partial_d^*$), so taking torsion summands yields the desired short exact sequence.

Step 2: Construct the top row of (13). Let $r = \operatorname{rank}\partial_d$, let $\{v_1,\dots,v_r\}$ be an integral basis of $\mathcal{C}$, and let V be the matrix with columns $v_1,\dots,v_r$. By Proposition 2.2, the dual basis $\{v_1^*,\dots,v_r^*\}$ for $\mathcal{C}^\sharp$ consists of the columns of the matrix $W = V(V^TV)^{-1}$. Let ψ be the orthogonal projection $\mathbb{R}^n \to \mathrm{Cut}(\Sigma)$, which is given by the matrix $P = WV^T = V(V^TV)^{-1}V^T$ (see Proposition 2.2). Then

$$\operatorname{im}\psi = \operatorname{colspace}\left(\underbrace{\begin{bmatrix} v_1^* & \cdots & v_r^* \end{bmatrix}}_{W}\ \underbrace{\begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix}^T}_{V^T}\right).$$

The i^{th} column of P equals W times the i^{th} column of V^T. If we identify $\mathcal{C}^\sharp$ with $\mathbb{Z}^r$ via the basis $\{v_1^*, \dots, v_r^*\}$, then $\operatorname{im}(\psi)$ is just the column space of V^T. So $\mathcal{C}^\sharp / \operatorname{im}\psi \cong \mathbb{Z}^r / \operatorname{colspace}(V^T)$, which is a finite group because $\operatorname{rank} V = r$. Since the matrices V and V^T have the same invariant factors, we have

$$\mathbb{Z}^r / \operatorname{colspace}(V^T) \cong \mathbf{T}(\mathbb{Z}^n / \operatorname{colspace}(V)) = \mathbf{T}(C^d(\Sigma;\mathbb{Z}) / \operatorname{im}\partial_d^*) = \mathbf{T}(\tilde{H}^d(\Sigma;\mathbb{Z})).$$

Meanwhile, $\operatorname{im}\psi \supseteq \mathcal{C}$ because $PV = V$. Since $\ker\psi = \mathcal{F}$, we have $(\operatorname{im}\psi)/\mathcal{C} = (\mathbb{Z}^n/\mathcal{F})/\mathcal{C} = \mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F})$. Therefore, the inclusions $\mathcal{C} \subseteq \operatorname{im}\psi \subseteq \mathcal{C}^\sharp$ give rise to the short exact sequence in the top row of (13).

Step 3: Describe the vertical maps in (13). The maps α and β are each induced by ∂_d in the following ways. First, the image of the cutflow group under ∂_d is

$$\partial_d\left(\mathbb{Z}^n/(\mathcal{F}\oplus\mathcal{C})\right) = \partial_d\left(\mathbb{Z}^n/(\ker\partial_d \oplus \operatorname{im}\partial_d^*)\right) = \operatorname{im}\partial_d / \operatorname{im}\partial_d\partial_d^*.$$

On the other hand, ∂_d acts injectively on the cutflow group (since the latter is a subquotient of $\mathbb{Z}^n/\ker\partial_d$). So the map labeled α is an isomorphism.

The cellular boundary map ∂_d also gives rise to the map $\beta\colon \mathcal{C}^\sharp/\mathcal{C} \to K(\Sigma)$, as we now explain. First, note that $\partial_d\mathcal{C}^\sharp \subseteq \operatorname{im}_{\mathbb{R}}\partial_d \subseteq \ker_{\mathbb{R}}\partial_{d-1}$. Second, observe that for every $w \in \mathcal{C}^\sharp$ and $\rho \in C_{d-1}(\Sigma;\mathbb{Z})$, we have $\langle\partial w, \rho\rangle = \langle w, \partial^*\rho\rangle \in \mathbb{Z}$, by equation (8) and the definition of dual lattice. Therefore, $\partial_d\mathcal{C}^\sharp \subseteq C_{d-1}(\Sigma;\mathbb{Z})$. It follows that ∂_d maps $\mathcal{C}^\sharp$ to $(\ker_{\mathbb{R}}\partial_{d-1}) \cap C_{d-1}(\Sigma;\mathbb{Z}) = \ker_{\mathbb{Z}}\partial_{d-1}$, hence defines a map $\beta\colon \mathcal{C}^\sharp/\mathcal{C} \to \ker_{\mathbb{Z}}\partial_{d-1}/\operatorname{im}_{\mathbb{Z}}\partial_d\partial_d^*$. Since $\mathcal{C}^\sharp/\mathcal{C}$ is finite, the image of β is purely torsion, hence contained in $K(\Sigma)$. Moreover, β is injective because $(\ker\partial_d)\cap\mathcal{C}^\sharp = \mathcal{F}\cap\mathcal{C}^\sharp = 0$ by Proposition 5.1.

Every element of $\mathbb{R}^n$ can be written uniquely as $c + f$ with $c \in \operatorname{Cut}(\Sigma)$ and $f \in \operatorname{Flow}(\Sigma)$. The map ψ is orthogonal projection onto $\operatorname{Cut}(\Sigma)$, so $\partial_d(c+f) = \partial_d c = \partial_d(\psi(c+f))$. Hence the left-hand square commutes. The map γ is then uniquely defined by diagram-chasing.

The snake lemma now implies that $\ker\gamma = 0$. Since the groups $\mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z}))$ and $\mathbf{T}(\tilde{H}^d(\Sigma;\mathbb{Z}))$ are abstractly isomorphic by equation (1), in fact γ must be an isomorphism and $\operatorname{coker}\gamma = 0$ as well. Applying the snake lemma again, we see that all the vertical maps in (13) are isomorphisms. □

Theorem 7.7. *Let Σ be a cell complex of dimension d with n facets. Then there is a short exact sequence*

$$0 \to \mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z})) \to \mathbb{Z}^n/(\mathcal{C}\oplus\mathcal{F}) \to \mathcal{F}^\sharp/\mathcal{F} \to 0. \tag{14}$$

Moreover, $K^*(\Sigma) \cong \mathcal{F}^\sharp/\mathcal{F}$.

Proof. Let Ω be an acyclization of Σ. By construction, the columns of the matrix A representing $\partial_{d+1}(\Omega)$ form an integral basis for $\mathcal{F} = \ker\partial_d$. Again, the matrix $Q = A(A^TA)^{-1}A^T$ represents orthogonal projection $\mathbb{R}^n \to \operatorname{Flow}(\Sigma)$. The maximal minors of A have gcd 1 (because $\mathcal{F}$ is a summand of $\mathbb{Z}^n$, so the columns of A are part of an integral basis), so by Proposition 2.2, the columns of Q generate the lattice $\mathcal{F}^\sharp$. Therefore, if we regard Q as a map

of $\mathbb{Z}$-modules, it defines a surjective homomorphism $\mathbb{Z}^n \to \mathcal{F}^\sharp$. This map fixes $\mathcal{F}$ pointwise and its kernel is the saturation $\hat{\mathcal{C}} := (\mathcal{C} \otimes \mathbb{R}) \cap \mathbb{Z}^n$. So we have short exact sequences $0 \to \hat{\mathcal{C}} \to \mathbb{Z}^n/\mathcal{F} \to \mathcal{F}^\sharp/\mathcal{F} \to 0$ and

$$0 \to \hat{\mathcal{C}}/\mathcal{C} \to \mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F}) \to \mathcal{F}^\sharp/\mathcal{F} \to 0.$$

Since $\hat{\mathcal{C}}$ is a summand of $\mathbb{Z}^n$ by Proposition 2.1, we can identify $\hat{\mathcal{C}}/\mathcal{C}$ with $\mathbf{T}(\tilde{H}^d(\Sigma;\mathbb{Z})) \cong \mathbf{T}(\tilde{H}_{d-1}(\Sigma;\mathbb{Z}))$, which gives the short exact sequence (14).

We will now show that $\mathcal{F}^\sharp/\mathcal{F} \cong K^*(\Sigma)$. To see this, observe that $\partial^*_{d+1}(\mathcal{F}^\sharp) = \partial^*_{d+1}(\mathrm{colspace}(Q)) = \mathrm{colspace}(A^T Q) = \mathrm{colspace}(A^T) = \operatorname{im} \partial^*_{d+1} = C_{d+1}(\Omega)$ (by the construction of an acyclization). In addition, $\ker \partial^*_{d+1}$ is orthogonal to $\mathcal{F}^\sharp$, hence their intersection is zero. Therefore, ∂^*_{d+1} defines an isomorphism $\mathcal{F}^\sharp \to C_{d+1}(\Omega)$. Moreover, the same map ∂^*_{d+1} maps $\mathcal{F} = \ker \partial_d = \operatorname{im} \partial_{d+1}$ surjectively onto $\operatorname{im} \partial^*_{d+1}\partial_{d+1}$. □

Corollary 7.8. *If $\tilde{H}_{d-1}(\Sigma;\mathbb{Z})$ is torsion-free, then the groups $K(\Sigma)$, $K^*(\Sigma)$, $\mathcal{C}^\sharp/\mathcal{C}$, $\mathcal{F}^\sharp/\mathcal{F}$, and $\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})$ are all isomorphic to each other.*

Corollary 7.8 includes the case that Σ is a graph, as studied by Bacher, de la Harpe and Nagnibeda [2] and Biggs [4]. It also includes the combinatorially important family of Cohen-Macaulay simplicial complexes, as well as cellulations of compact orientable manifolds.

Example 7.9. Suppose that $\tilde{H}_d(\Sigma;\mathbb{Z}) = \mathbb{Z}$ and that $\tilde{H}_{d-1}(\Sigma;\mathbb{Z})$ is torsion-free. Then the flow lattice is generated by a single element, and it follows from Corollary 7.8 that $K(\Sigma) \cong K^*(\Sigma) \cong \mathcal{F}^\sharp/\mathcal{F}$ is a cyclic group. For instance, if Σ is homeomorphic to a cellular sphere or torus, then the critical group is cyclic of order equal to the number of facets. (The authors had previously proved this fact for simplicial spheres [15, Theorem 3.7], but this approach using the cocritical group makes the statement more general and the proof transparent.)

Example 7.10. Let Σ be the standard cellulation $e^0 \cup e^1 \cup e^2$ of the real projective plane, whose cellular chain complex is

$$\mathbb{Z} \xrightarrow{\partial_2 = 2} \mathbb{Z} \xrightarrow{\partial_1 = 0} \mathbb{Z}.$$

Then $\mathcal{C} = \operatorname{im} \partial_2^* = 2\mathbb{Z}$, $\mathcal{C}^\sharp = \frac{1}{2}\mathbb{Z}$, and $K(\Sigma) = \mathcal{C}^\sharp/\mathcal{C} = \mathbb{Z}_4$. Meanwhile, $\mathcal{F} = \mathcal{F}^\sharp = \mathcal{F}^\sharp/\mathcal{F} = K^*(\Sigma) = 0$. The cutflow group is $\mathbb{Z}_2$. Note that the rows of Theorem 7.6 are not split in this case.

Example 7.11. Let $a, b \in \mathbb{Z} \setminus \{0\}$. Let Σ be the cell complex whose cellular chain complex is

$$\mathbb{Z} \xrightarrow{\partial_2 = [a\ b]} \mathbb{Z} \xrightarrow{\partial_1 = 0} \mathbb{Z}.$$

Topologically, Σ consists of a vertex e^0, a loop e^1, and two facets of dimension 2 attached along e^1 by maps of degrees a and b. Then

$$\mathcal{C}^\sharp/\mathcal{C} = \mathbb{Z}_\tau, \qquad \mathbb{Z}^2/(\mathcal{C} \oplus \mathcal{F}) = \mathbb{Z}_{\tau/g}, \qquad \mathcal{F}^\sharp/\mathcal{F} = \mathbb{Z}_{\tau/g^2},$$

where $\tau = a^2 + b^2$ and $g = \gcd(a, b)$. Note that $\tau = \tau_2(\Sigma)$ is the complexity of Σ (see equation (3)) and that $g = |\tilde{H}_1(\Sigma; \mathbb{Z})|$. The short exact sequence of Theorem 7.7 is in general not split (for example, if $a = 6$ and $b = 2$).

8. Enumeration

For a connected graph, the cardinality of the critical group equals the number of spanning trees. In this section, we calculate the cardinalities of the various group invariants of Σ.

Examples 7.10 and 7.11 both indicate that $K(\Sigma) \cong \mathcal{C}^\sharp/\mathcal{C}$ should have cardinality equal to the complexity $\tau(\Sigma)$. Indeed, in Theorem 4.2 of [15], the authors proved that $|K(\Sigma)| = \tau(\Sigma)$ whenever Σ has a cellular spanning tree Υ such that $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z}) = \tilde{H}_{d-1}(\Sigma; \mathbb{Z}) = 0$ (in particular, Σ must be not merely $\mathbb{Q}$-acyclic, but actually $\mathbb{Z}$-acyclic, in codimension one). Here we prove that this condition is actually not necessary: for any cell complex, the order of the critical group $K(\Sigma)$ equals the torsion-weighted complexity $\tau(\Sigma)$. Our approach is to determine the size of the discriminant group $\mathcal{C}^\sharp/\mathcal{C}$ directly, then use the short exact sequences of Theorems 7.6 and 7.7 to calculate the sizes of the other groups.

Theorem 8.1. *Let Σ be a d-dimensional cell complex and let $\mathbf{t} = \mathbf{t}_{d-1}(\Sigma) = |\mathbf{T}(\tilde{H}_{d-1}(\Sigma; \mathbb{Z}))|$. Then*

$$\begin{aligned} |\mathcal{C}^\sharp/\mathcal{C}| = |K(\Sigma)| &= \tau_d(\Sigma), \\ |\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})| &= \tau_d(\Sigma)/\mathbf{t}, \quad \textit{and} \\ |\mathcal{F}^\sharp/\mathcal{F}| = |K^*(\Sigma)| &= \tau_d(\Sigma)/\mathbf{t}^2. \end{aligned}$$

Proof. By Theorems 7.6 and 7.7, it is enough to prove that $|\mathcal{C}^\sharp/\mathcal{C}| = \tau_d(\Sigma)$.

Let R be a set of $(d-1)$-cells corresponding to a row basis for ∂ (hence a vector space basis for $\mathrm{Cut}(\Sigma)$); let $\mathcal{R}$ be the lattice spanned by those rows (which is a full-rank integral sublattice of $\mathcal{C}$); and let $\Gamma = (\Sigma_{d-1} \setminus R) \cup \Sigma_{(d-2)}$. The inclusions $\mathcal{R} \subseteq \mathcal{C} \subseteq C^d(\Sigma)$ give rise to a short exact sequence $0 \to \mathcal{C}/\mathcal{R} \to \tilde{H}^d(\Sigma, \Gamma; \mathbb{Z}) \to \tilde{H}^d(\Sigma; \mathbb{Z}) \to 0$. Since $\mathcal{C}/\mathcal{R}$ is finite, the torsion summands form a short exact sequence (see Section 3). Taking cardinalities and using equation (1), we get

$$|\mathcal{C}/\mathcal{R}| = \frac{|\mathbf{T}(\tilde{H}^d(\Sigma, \Gamma; \mathbb{Z}))|}{|\mathbf{T}(\tilde{H}^d(\Sigma; \mathbb{Z}))|} = \frac{\mathbf{t}_{d-1}(\Sigma, \Gamma)}{\mathbf{t}_{d-1}(\Sigma)}. \tag{15}$$

The inclusions $\mathcal{R} \subseteq \mathcal{C} \subseteq \mathcal{C}^\sharp \subseteq \mathcal{R}^\sharp$ give $|\mathcal{R}^\sharp/\mathcal{R}| = |\mathcal{R}^\sharp/\mathcal{C}^\sharp| \cdot |\mathcal{C}^\sharp/\mathcal{C}| \cdot |\mathcal{C}/\mathcal{R}|$. Moreover, $\mathcal{R}^\sharp/\mathcal{C}^\sharp \cong \mathcal{C}/\mathcal{R}$. By equation (15) and Binet-Cauchy, we have

$$\begin{aligned}|\mathcal{C}^\sharp/\mathcal{C}| &= \frac{|\mathcal{R}^\sharp/\mathcal{R}|}{|\mathcal{R}^\sharp/\mathcal{C}^\sharp| \cdot |\mathcal{C}/\mathcal{R}|} = \frac{\mathbf{t}_{d-1}(\Sigma)^2}{\mathbf{t}_{d-1}(\Sigma,\Gamma)^2}\ |\mathcal{R}^\sharp/\mathcal{R}| \\ &= \frac{\mathbf{t}_{d-1}(\Sigma)^2}{\mathbf{t}_{d-1}(\Sigma,\Gamma)^2}\ \det(\partial_R \partial_R^*) \\ &= \sum_{\Upsilon \subseteq \Sigma_d\colon |\Upsilon|=r} \frac{\mathbf{t}_{d-1}(\Sigma)^2}{\mathbf{t}_{d-1}(\Sigma,\Gamma)^2}\ \det(\partial_{R,\Upsilon})^2.\end{aligned}$$

By Proposition 3.2, the summand is nonzero if and only if Υ is a cellular spanning forest. In that case, the matrix $\partial_{R,\Upsilon}$ is the cellular boundary matrix of the relative complex (Υ, Γ), and its determinant is (up to sign) $\mathbf{t}_{d-1}(\Upsilon,\Gamma)$, so by Proposition 3.4 we have

$$|\mathcal{C}^\sharp/\mathcal{C}|\ = \sum_\Upsilon \frac{\mathbf{t}_{d-1}(\Sigma)^2}{\mathbf{t}_{d-1}(\Sigma,\Gamma)^2}\ \mathbf{t}_{d-1}(\Upsilon,\Gamma)^2\ = \sum_\Upsilon \mathbf{t}_{d-1}(\Upsilon)^2$$

with the sums over all cellular spanning forests $\Upsilon \subseteq \Sigma$. □

Dually, we can interpret the cardinality of the cocritical group as enumerating cellular spanning forests by relative torsion (co)homology, as follows.

Theorem 8.2. *Let Ω be an acyclization of Σ. Then*

$$|K^*(\Sigma)| = \sum_\Upsilon |\tilde{H}^{d+1}(\Omega, \Upsilon; \mathbb{Z})|^2 = \sum_\Upsilon |\tilde{H}_d(\Omega, \Upsilon; \mathbb{Z})|^2$$

with the sums over all cellular spanning forests $\Upsilon \subseteq \Sigma$.

Proof. Let $\partial_{d+1} = \partial_{d+1}(\Omega)$. Note that $\operatorname{rank} \partial_{d+1} = \tilde{\beta}_d(\Sigma)$; abbreviate this number as b. By Binet-Cauchy, we have

$$|K^*(\Sigma)| = |\det \partial_{d+1}^* \partial_{d+1}| = \sum_{B \subseteq \Omega_d\colon |B|=b} (\det \partial_B^*)^2$$

where ∂_B denotes the submatrix of ∂ with rows B. Letting $\Upsilon = \Sigma \backslash B$, we can regard ∂_B as the cellular boundary map of the relative complex (Ω, Υ), which consists of b cells in each of dimensions d and $d+1$. By Proposition 3.2, the summand is nonzero if and only if Υ is a cellular spanning tree of $\Omega_{(d)} = \Sigma$. (Note that the $d+1, B, \Upsilon, \Omega_{(d)}$ in the present context correspond respectively to the d, R, Γ, Υ of Proposition 3.2.) For these summands, $H^{d+1}(\Omega, \Upsilon; \mathbb{Z}) = H_d(\Omega, \Upsilon; \mathbb{Z})$ is a finite group of order $|\det \partial_B|$. □

Remark 8.3. Let $\tau^*(\Sigma) = \sum_\Upsilon |\tilde{H}_d(\Omega, \Upsilon; \mathbb{Z})|^2$, as in Theorem 8.2. Then combining Theorems 8.1 and Theorems 8.2 gives

$$\begin{aligned}|\mathcal{C}^\sharp/\mathcal{C}| = |K(\Sigma)| = \tau(\Sigma) &= \tau^*(\Sigma) \cdot \mathbf{t}^2, \\ |\mathcal{F}^\sharp/\mathcal{F}| = |K^*(\Sigma)| = \tau^*(\Sigma) &= \tau(\Sigma)/\mathbf{t}^2, \\ |\mathbb{Z}^n/(\mathcal{C} \oplus \mathcal{F})| = \tau(\Sigma)/\mathbf{t} &= \tau^*(\Sigma) \cdot \mathbf{t},\end{aligned}$$

highlighting the duality between the cut and flow lattices.

9. Bounds on combinatorial invariants from lattice geometry

Let $n \geq 1$ be an integer. The *Hermite constant* γ_n is defined as the maximum value of

$$\left(\min_{x \in \mathcal{L} \setminus \{0\}} \langle x, x \rangle \right) (|\mathcal{L}^\sharp / \mathcal{L}|)^{-1/n} \tag{16}$$

over all lattices $\mathcal{L} \subseteq \mathbb{R}^n$. The Hermite constant arises both in the study of quadratic forms and in sphere packing; see [23, Section 4]. It is known that γ_n is finite for every n, although the precise values are known only for $1 \leq n \leq 8$ and $n = 24$ [8].

As observed by Kotani and Sunada [22], if $\mathcal{L} = \mathcal{F}$ is the flow lattice of a connected graph, then the shortest vector in $\mathcal{F}$ is the characteristic vector of a cycle of minimum length; therefore, the numerator in equation (16) is the girth of G. Meanwhile, $|\mathcal{F}^\sharp / \mathcal{F}|$ is the number of spanning trees. We now generalize this theorem to cell complexes.

Definition 9.1. Let Σ be a cell complex. The *girth* and the *connectivity* are defined as the cardinalities of, respectively the smallest circuit and the smallest cocircuit of the cellular matroid of Σ.

Theorem 9.2. *Let Σ be a cell complex of dimension d with girth g and connectivity k, and top boundary map of rank r. Let $b = \operatorname{rank} \tilde{H}_{d-1}(\Sigma; \mathbb{Z})$. Then*

$$k\tau(\Sigma)^{-1/r} \leq \gamma_r \qquad \text{and} \qquad g\tau^*(\Sigma)^{-1/b} \leq \gamma_b.$$

Proof. Every nonzero vector of the cut lattice $\mathcal{C}$ contains a cocircuit in its support, so $\min_{x \in \mathcal{C} \setminus \{0\}} \langle x, x \rangle \geq k$. Likewise, every nonzero vector of the flow lattice $\mathcal{F}$ of Σ contains a circuit in its support, so $\min_{x \in \mathcal{F} \setminus \{0\}} (x \cdot x) \geq g$. Meanwhile, $|\mathcal{C}^\sharp / \mathcal{C}| = \tau$ and $|\mathcal{F}^\sharp / \mathcal{F}| = \tau^*$ by Theorem 8.1. The desired inequalities now follow from applying the definition of Hermite's constant to the cut and flow lattices respectively. □

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