Domino Tilings Beyond 2D

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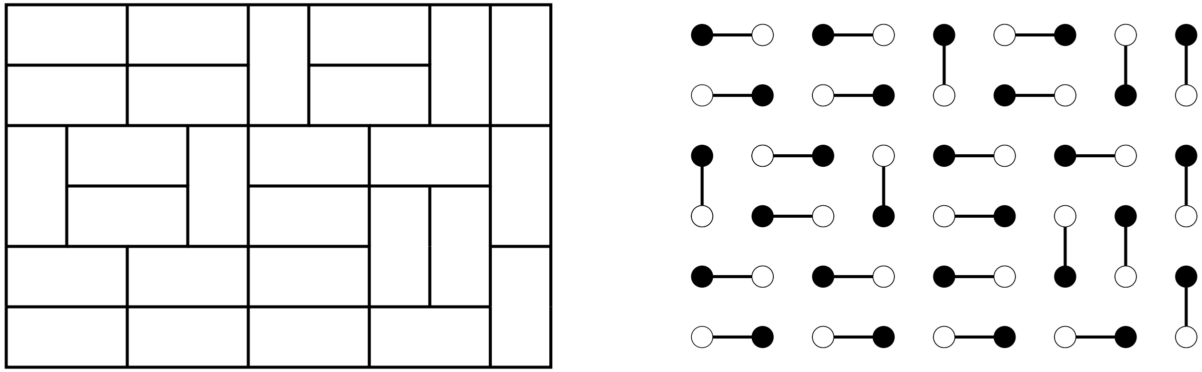
More Work by: Saldanha, de Marreiros, Khesin
Chandgotia, Sheffield, Wolfram

Two-Dimensional Tilings



A two dimensional tiling

Domino tilings of cubical Regions R

$$\updownarrow$$

Dimer covers of the dual graph R^*

Theory of Two-Dimensional Tilings

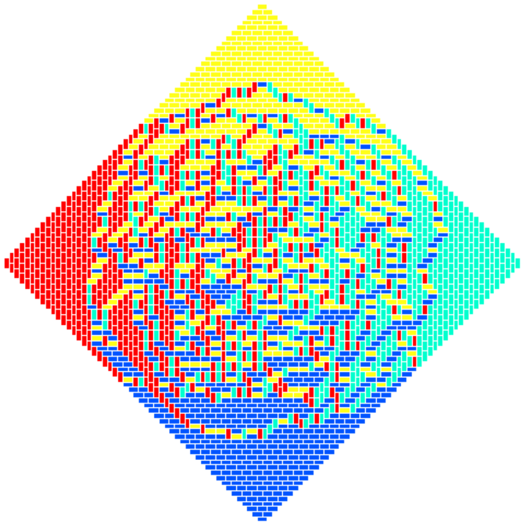
Combinatorics, Probability, Statistical Physics

Q: How many tilings are there of a given region?

Kasteleyn '61, Temperley and Fisher '61 Pfaffians, determinants, transfer matrix-method

Q: What does a random tiling look like?

Jockush, Propp, Shor '95 Arctic circle theorem



Q: The space of all tilings? Connectivity?

Thurston '90 Flip moves and height functions

Two-Dimensional Tilings

Q: How many tilings are there of a given region?

Kasteleyn '61, Temperley and Fisher '61 Pfaffians, determinants, transfer matrix-method

The number of domino tilings of the $m \times n$ rectangle is:

$$\prod_{j=1}^{\lceil \frac{m}{2} \rceil} \prod_{i=1}^{\lceil \frac{m}{2} \rceil} (4\cos^2 \frac{\pi j}{(m+1)} + 4\cos^2 \frac{\pi i}{n+1})$$

Determinant of a signed adjancency matrix:

$$\sqrt{\det(A(G))}$$

Two-Dimensional Tilings

Q: What does a random tiling look like?
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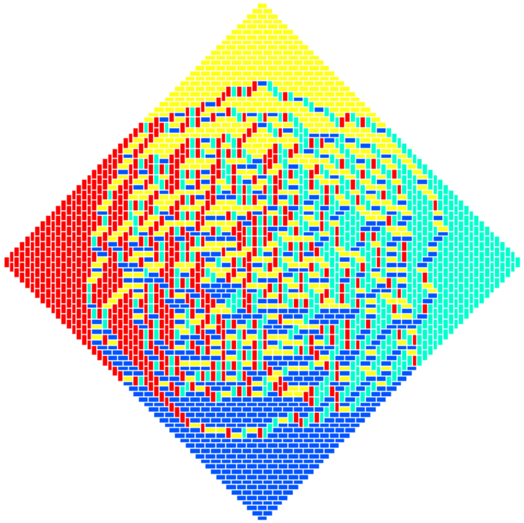


Dimers and amoebae Kenyon, Okounkov, Sheffield '06

Random tilings of the Aztec diamond.
Boundary of frozen regions described by algebraic curves.

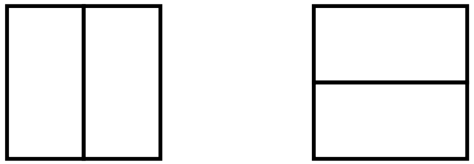
Large scale discrete vector fields and flows, statistical mechanics.

Two-Dimensional Tilings

Q: Can we move from one tiling to another?

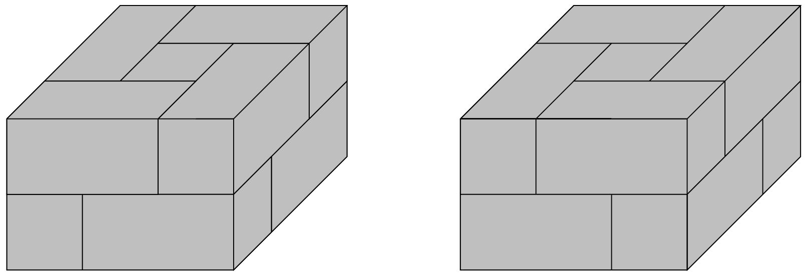
Is the space of tilings connected?

Flip: Remove two adjacent parallel dominoes and place them back rotated within a 2×2 block:



Theorem: (Thurston '90) In two dimensions, any two tilings of a simply connected region are flip connected.

Three-Dimensional Tilings



Dominoes: $2 \times 1 \times 1$ bricks of two adjacent cubes

Goal: All the questions from above:

Q: Can we move from one tiling to another?

Q: What does a typical 3-dimensional tiling look like?

Q: How many tilings are there?

Q: Four dimensions and beyond?

"The Third Rail of Tiling!"

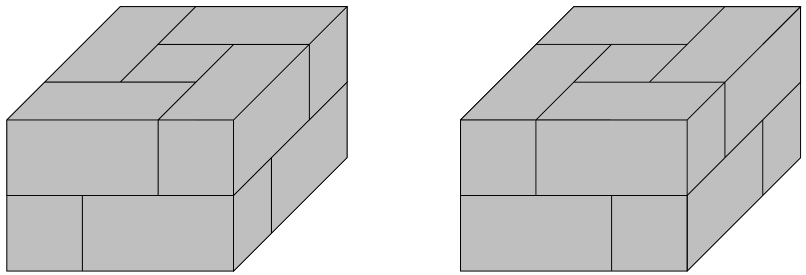
Connectivity by Local Moves

How and when can we move from one tiling to another?

Flip: Remove two adjacent parallel dominoes and place them back rotated within a $2 \times 2 \times 1$ block.

> **Theorem:** (Thurston) In two dimensions, any two tilings of a simply connected region are flip connected.

Not the case in 3D:



Two tilings of the $3 \times 3 \times 2$ box with no flips.

Connectivity by Flip Moves

$3 \times 3 \times 2$ box:

Number of tilings: 229

Connected components: 3

Sizes: $227, 1, 1$

$4 \times 4 \times 4$ box:

Number of tilings: $5,051,532,105$

Connected components: 93

Sizes: $4,412,646,453$

$2 \times 310,185,960$

$2 \times 8,237,514$

$2 \times 718,308$

$2 \times 283,044$

$6 \times 2,576$

24×618

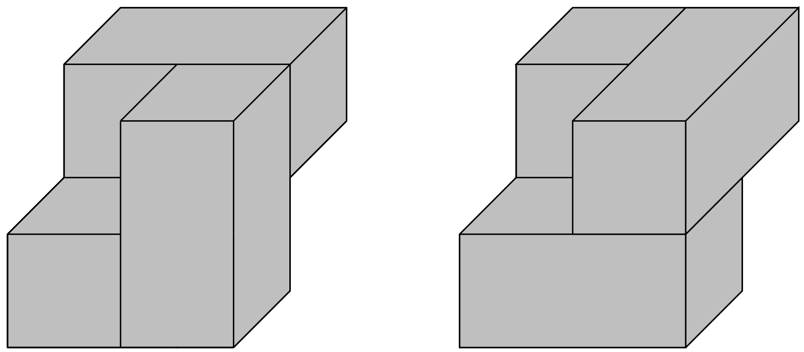
24×236

6×4

24×1

A Three Dimensional Move

Trit: Remove and replace 3 dominoes, one parallel to each axis inside a $2 \times 2 \times 2$ box.



A positive trit

Question: Are tilings of three-dimensional regions connected by flips and trits?

Connectivity by Flips and Trits

Question: Are all tilings of three-dimensional regions connected by flips and trits?

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Open:

- Boxes: $[0, L] \times [0, M] \times [0, N]$

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Simple and Natural Generalization of a well established theory

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Simple and Natural Open Problem

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Simple and Natural Generalization of a well established theory

Simple and Natural Open Problem

Novel Techniques in the pursuit of answers...

A Topological Approach

Homology, knot theory, cubical approximation of surfaces

Need two topological invariants:

Flux: "Flow across surfaces"
Twist: "Knottedness by trits"

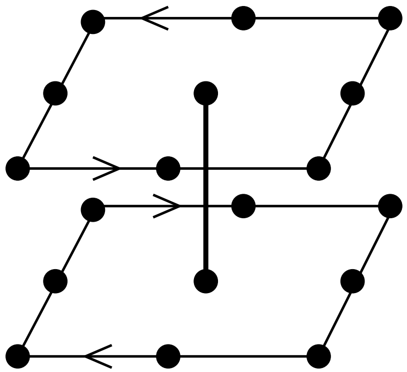
And need the notion of **Refinement**:

- Decompose each cube into $5 \times 5 \times 5$ smaller cubes
- Decompose each domino into $5 \times 5 \times 5$ smaller dominoes, each parallel to the original.

Difference of Tilings

For two tilings t_0, t_1:

- $t_1 - t_0 :=$ union of tiles (with orientation of t_0 reversed).



Yields a system of cycles.
(Ignore trivial 2-cycle.)

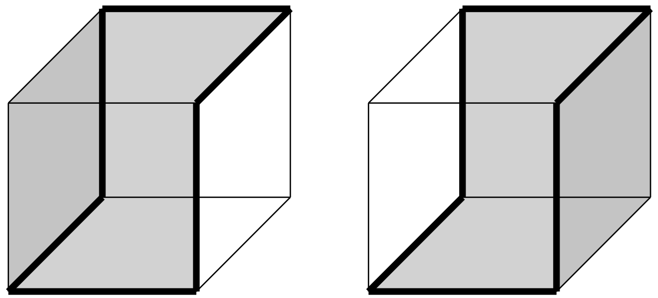
Homologically:
$t_1 - t_0 \in Z_1(R^*; \mathbb{Z})$

Fix a base tiling $t_\oplus$

$$\mathrm{Flux}(t) := [t - t_\oplus] \in H_1(R^*; \mathbb{Z})$$

Seifert Surfaces

A (discrete) Seifert surface for a pair t_0, t_1 is a connected embedded oriented topological surface S with boundary $t_0 - t_1$.



Lemma

If $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$ then for sufficiently large k there exists a discrete Seifert surface for the pair after k refinements.

The Twist of a Tiling

$$\phi(t;S) = \sum_v \varphi(v;t;S) \qquad \varphi(v;t;S) = \mathrm{c}(v) \cdot \begin{cases} +1, & \text{end above } S \\ 0, & \text{end on } S \\ -1, & \text{end below } S \end{cases}$$

Fix a base tiling $t_\oplus$:

$$\mathrm{Twist}(t) := \phi(t; t - t_\oplus)$$

> **Proposition**
>
> If $t_0 \rightsquigarrow$ trit $\rightsquigarrow t_1$ then
>
> $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$ and $\mathrm{Twist}(t_0) = \mathrm{Twist}(t_1) \pm 1$

- Intuitively, the twist records how "twisted" a tiling is by trits.

Connectivity - Local Moves

Main Theorem (FKMS '16)

For two tilings t_0 and t_1 of R:

- There exists a sequence of flips and trits connecting refinements of t_0 and t_1 *if and only if* $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$.

- There exists a sequence of flips connecting refinements of t_0 and t_1 *if and only if* $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$ and $\mathrm{Twist}(t_0) = \mathrm{Twist}(t_1)$.

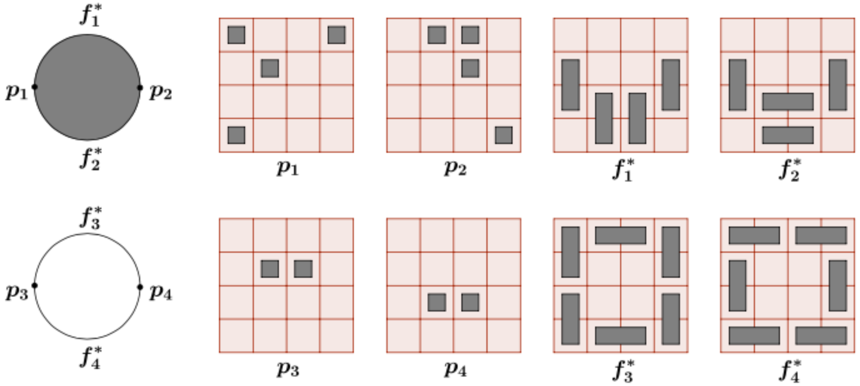
A second Topological Approach: The Domino Complex

Consider: Adding vertical space to disks.

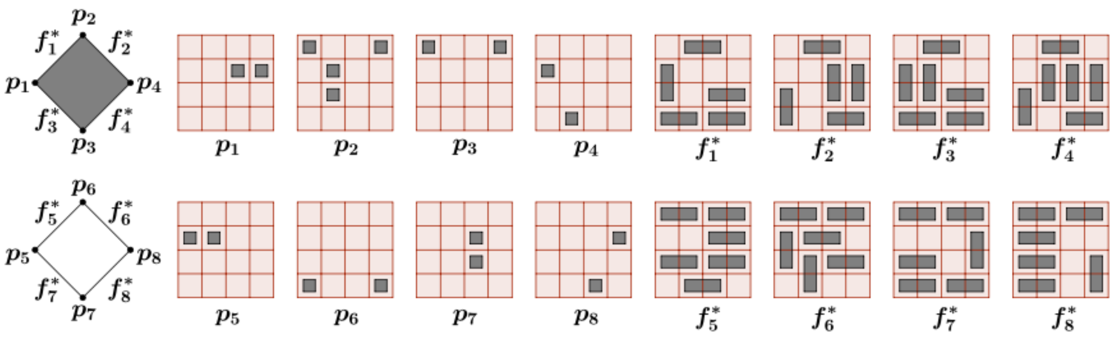
$$\text{Cylinders } = \mathcal{D} \times [0, N]$$

Domino Complex: $\mathcal{C}_{\mathcal{D}}$ = 2D CW-complex

- Vertices: Levels of the tiling
- Edges: Tilings within levels
- 2-cells: Bigons for horizontal moves, quads for vertical moves



The Domino Complex



Two tilings with vertical space added are flip connected if and only if their paths in $\mathcal{C}_{\mathcal{D}}$ are homotopic.

Domino Group $\mathcal{G}_{\mathcal{D}} = \pi_1(\mathcal{C}_{\mathcal{D}}, p_0)$

Twist: $\mathcal{G}_{\mathcal{D}} \to \mathbb{Z}$

Q: When is Twist an isomorphism?

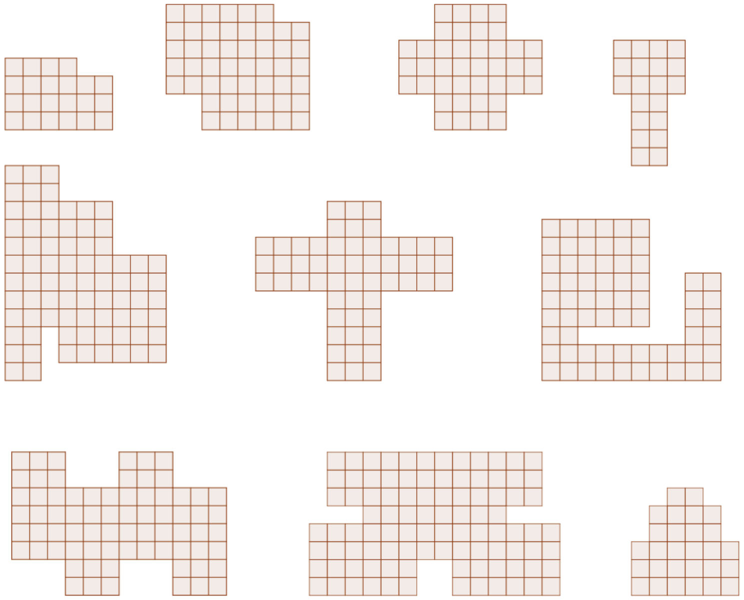
Precisely when two tilings of the same twist are flip connected (with possible vertical space)

The Domino Complex

Theorem (Saldanha '19, Saldanha, K. '22, de Marreiros '24)

Twist is an isomorphism for:

- Two dimensional boxes
- Arbitrary dimensional boxes
- Hamiltonian disks without bottlenecks*



A Commutative Algebra Approach

Binomial ideals of domino tilings Chin '19, Gross, Yazmon '21

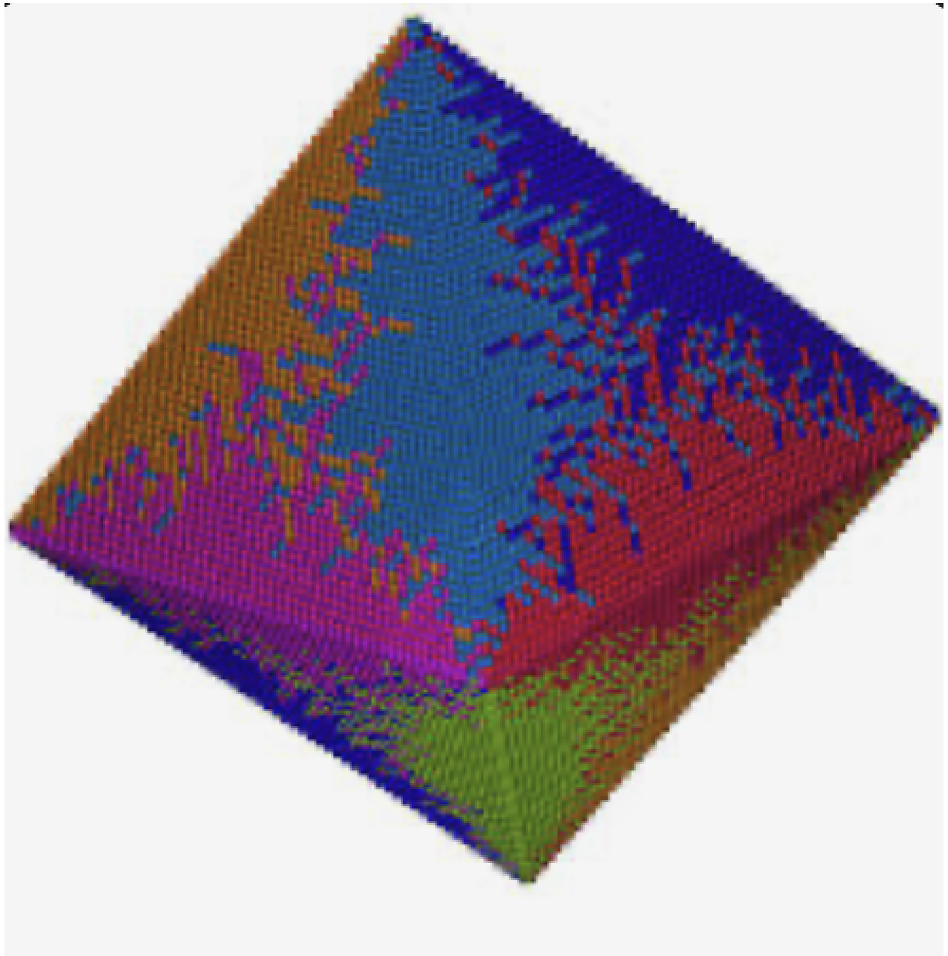
$$\prod_{x_e \in T_1} x_e - \prod_{x'_e \in T_2} x'_e = x^{T_1} - x^{T_2}$$

Tiling Ideal: $\quad I_{\text{Tiling}} := \langle x^{T_1} - x^{T_2} \rangle$

Flip Ideal: $\quad I_{\text{Flip}} := \langle x_a x_b - x_c x_d \,|\, (ab, cd) \text{ is a flip move } \rangle$

- Connectivity by relating ideals.
- Primary decompositions of Flip Ideals.

A Random Tiling



‘Large deviations for the 3D dimer model’ Chandgotia, Sheffield, Wolfram, ’23

(2D large deviations Cohn, Kenyon, Propp ’01)

Helicity as Twist

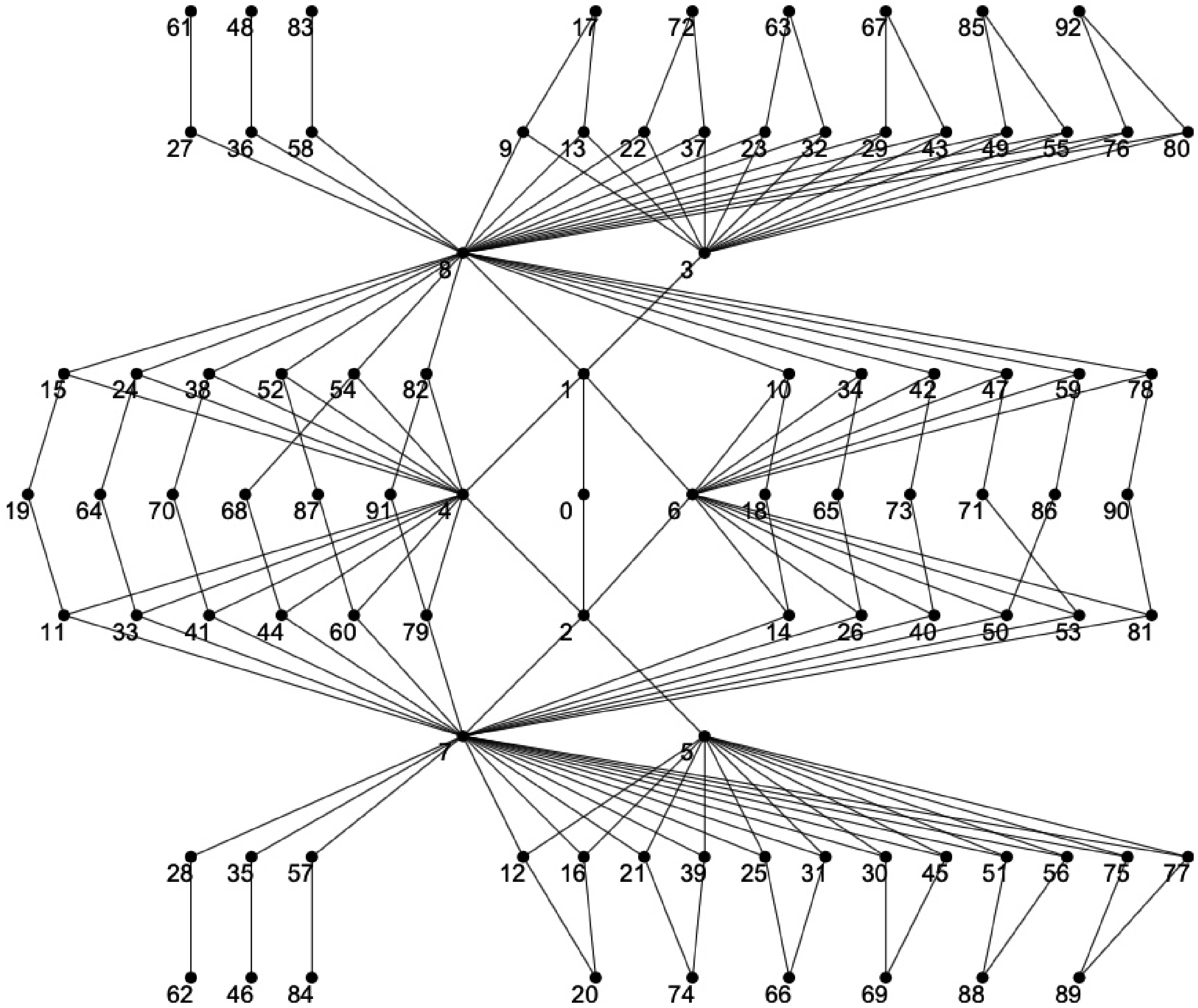
"The twist can be interpested as a discrete analogue of *helicity* arising in fluid mechanics and topological hydrodynamics... The helicity of a vector field on a domain in $\mathbb{R}^3$ is a measure of the self linkage of field lines. An importabnt recent result shows that helicity is the only integral invariant of volume-preserving transformations. " FKMS

'Relative Helicity and Tiling Twist' Khesin, Saldanha, 2024

" ... construct a smooth divergence-free vector field associated to an arbitrary tiling so that the twist invariant becomes, up to a factor, the relative helicity of that vector field."

" ... [relating] the flux invariant of a tiling to the rotation class of the associated vector field."

Poset of Connected Components by Twist



How Many Tilings are There?

Open: How many tilings are there of the d-dimensional box?

Theorem Valiant '79 Pak, Yang '13

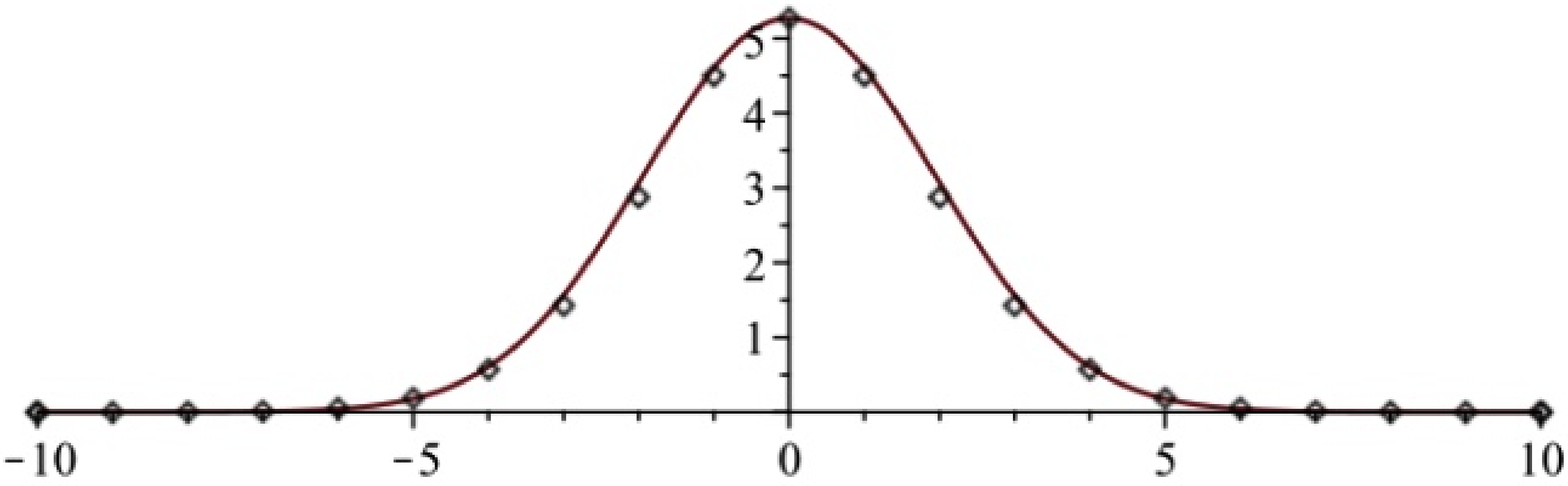
Counting 3D tilings is #P.

Theorem KS

$$\mathrm{pf}(K(R)) = \sum_T (-1)^{\mathrm{Twist}(T)}$$

$K(R)$ is a linear algebraic construction.

Distribution of Twist



#Tilings per value of Twist of the $4 \times 4 \times 60$ box.

Vertical axis $\times 10^{156}$

Red curve is true Gaussian

Open: Is the Twist normally distributed?

Questions

Q: How often are refinements necessary?

Open: Are they necessary at all for boxes?

Q: How many tilings are there?

Open: Is the Twist normally distributed?

Q: Topology beyond connectedness?

Q: More complicated domains?

Thank You