

CHIP-FIRING AND ENERGY MINIMIZATION ON M-MATRICES

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ABSTRACT. We consider chip-firing dynamics defined by arbitrary M-matrices. M-matrices generalize graph Laplacians and were shown by Gabrielov to yield avalanche finite systems. Building on the work of Baker and Shokrieh, we extend the concept of energy minimizing chip configurations. Given an M-matrix, we show that there exists a unique energy minimizing configuration in each equivalence class defined by the matrix.

We consider the class of z -superstable configurations. We prove that for any M-matrix, the z -superstable configurations coincide with the energy minimizing configurations. Moreover, we prove that the z -superstable configurations are in simple duality with critical configurations. Thus for all avalanche-finite systems (including all directed graphs with a global sink) there exist unique critical, energy minimizing and z -superstable configurations. The critical configurations are in simple duality with energy minimizers which coincide with z -superstable configurations.

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1. INTRODUCTION

There is a large literature on the dynamics and combinatorics of chip-firing games. They were originally studied in the context of self-organized criticality and sandpile models [3, 13, 14], as balancing games on graphs [2, 23], and for their algebraic structure [8, 12]. More recently, chip-firing has appeared in a surprising variety of new connections. For example, chip firing plays a central role in a Riemann-Roch theorem for graphs [5] and linear systems in tropical geometry [15]. Our starting point will be the recent work of Baker and Shokrieh on chip-firing, potential theory and energy minimization on graphs [4]. Building on this new connection to energy minimization, we are able to return to some of the first questions concerning the long-term stability of chip-firing dynamics.

There are many variants to the chip-firing game. Typically one considers a finite graph with integer values associated to the vertices. A single vertex is distinguished as the sink (or bank). The value of the sink may be arbitrary but all other vertices have non-negative values, which we think of as the number of chips associated to the vertex. A chip firing rule is given as follows: if any non-sink vertex has at least as many chips as it has neighbors, then it “fires” by sending one chip to each of its neighbors. The value of each neighbor is increased by one and the value at the vertex that fired is decreased by its degree. In particular, if we consider the number of chips at each vertex as an integer vector, called a chip configuration, then “firing” a vertex subtracts the corresponding row of the graph Laplacian from the configuration. Two chip configurations are considered equivalent

Date: November 21, 2014.

Key words and phrases. chip-firing, M-matrices, energy minimization, combinatorial Laplacians.

if their difference is in the image of the graph Laplacian. Informally, two chip configurations are equivalent if one chip configuration can transform to the other via fires and reverse-fires.

Of great interest is the long term behavior of such systems. If the system has a sink, as above, then every configuration does stabilize in the sense that eventually no non-sink vertex will be able to fire. Imposing further stability requirements leads to important classes of chip configurations. Briefly, *superstable* configurations (also known as G -parking functions or reduced divisors) are stable configurations such that no subset of vertices can simultaneously fire and result in a non-negative configuration. *Critical* configurations (also known as recurrent configurations) are stable configurations that can be reached from sufficiently large starting configurations. It is well known that superstable configurations and critical configurations exist and are unique per equivalence class of chip configurations. Furthermore, critical and superstable configurations are in simple duality with each other. They are also in bijection with the spanning trees of the graph; see e.g. [11] or [6].

Baker and Shokrieh [4] introduced a norm on chip configurations in terms of the graph Laplacian for undirected graphs. The norm is thought of as an energy function and they investigated energy-minimizing chip configurations. In particular, they prove that energy-minimizers are precisely the superstable configurations and hence unique per equivalence class and in duality with critical configurations.

Following the work of Dhar [13], Gabrielov [14] considered more general chip-firing dynamics in terms of a class of dissipation matrices which is broader than the graph Laplacians above. He worked with *avalanche-finite matrices*, which precisely guarantee that all configurations eventually stabilize using legal firing moves (see Section 2 for specifics). Gabrielov proved that critical configurations exist and are unique per equivalence class for all avalanche-finite matrices.

More recently, much attention has focused on the intermediate case of chip-firing on directed graphs. In this case, the existence and uniqueness of critical configurations is guaranteed by Gabrielov's earlier work, because the associated graph Laplacians are a special case of avalanche-finite matrices. In this setting, the term superstable is used in at least two different ways. We will use the notation χ - and z -superstable to distinguish the classes of configurations (see Section 4.1). The uniqueness of χ -superstable configurations and the duality with critical configurations appears in [17, 22] for the special case of Eulerian directed graphs. For all directed graphs with a global sink, a stronger form of stability is required for an analogous result. The uniqueness of z -superstable configurations and the duality with critical configurations appears originally in [21] and later in [20] and [1].

We unify and generalize these results as follows. First, building on Baker and Shokrieh's work, we define a class of norms and energy-minimizing configurations for all avalanche-finite matrices. We prove the existence and uniqueness of energy-minimizers per equivalence class of these matrices (Theorem 3.4). We show that the z -superstable configurations are precisely the energy-minimizers (Theorem 4.6) and are in simple duality with critical configurations (Theorem 4.14).

Namely, for all avalanche-finite matrices, there exist unique critical, energy-minimizing, and z -superstable configurations. The first are in simple duality with the latter two which coincide. The number of such configurations is given by the determinant of the matrix.

2. M-MATRICES

The dynamics of chip-firing on graphs is dictated by the reduced graph Laplacian. Let G be a directed (multi)-graph with $n + 1$ vertices. The graph Laplacian $\tilde{\Delta}(G)$ is given by

$$\tilde{\Delta}_{ij} = \begin{cases} -a_{ij} & i \neq j \text{ and } (i, j) \in E \\ \text{outdeg}(i) & i = j \\ 0 & \text{otherwise,} \end{cases}$$

where a_{ij} is the number of edges from i to j . A reduced graph Laplacian is any matrix resulting from deleting a single row and column from a graph Laplacian. All graphs we will consider will have a global sink. A graph G has a global sink, s , if for every vertex $v \neq s$ there is a directed path from v to s . When referring to the reduced Laplacian $\Delta(G)$ for a graph with a global sink, we will always assume the row and column corresponding to the sink has been deleted.

Given a graph on $n + 1$ vertices with the last vertex a global sink, a chip configuration $c = (c_1, c_2, \dots, c_n)$ is a non-negative integer vector, $c \in \mathbb{Z}_{\geq 0}^n$. The value c_i is thought of as the number of chips at vertex i . Starting with a configuration c , firing vertex i results in subtracting the i th row of the reduced Laplacian from c ; $c - Le_i$ where $L = \Delta^T$ and e_i is the i th standard basis vector in \mathbb{R}^n . In this notation, the $n + 1$ st vertex is the sink vertex and we will not be concerned with its “chip value”.

Gabrielov considered more general chip-firing systems by replacing the reduced graph Laplacian with a broader class of matrices [14]. For an arbitrary $n \times n$ integer matrix N , we consider a system with n states. A chip configuration is any integer vector $c \in \mathbb{Z}^n$. Firing a state i is defined to be the process which replaces the configuration c with $c - N^T e_i$, namely subtracting the i th row of N . Two configurations c and d are considered equivalent if their difference $c - d$ is in the \mathbb{Z} -image of N . In this more general setup, a state i is allowed to fire if $c_i \geq N_{ii}$. Following the physicality of the original model, Gabrielov restricted to matrices with a positive diagonal and non-positive off-diagonal. Therefore, a state must have a certain positive amount of chips in order to fire and firing a state increases the number of chips on neighboring states. These are referred to as redistribution matrices in [14]. A configuration is stable if $c_i < N_{ii}$ for all states i . A natural question arises: for which such matrices does the chip-firing process eventually stabilize versus producing an infinite process. An *avalanche-finite* matrix is one for which every non-negative chip configuration stabilizes. We will look closely at such matrices. We start with some definitions.

Definition 2.1. An $n \times n$ matrix L such that $L_{ij} \leq 0$ for all $i \neq j$ is called a Z-matrix.

Definition 2.2. Let L be a $n \times n$ Z-matrix. If any of the following equivalent conditions hold then L is called a non-singular M-matrix:

- (1) L is avalanche finite.
- (2) The real part of the eigenvalues are positive.
- (3) L^{-1} exists and all the entries of L^{-1} are non-negative.
- (4) There exists a vector $x \in \mathbb{R}^n$ with $x \geq 0$ such that Lx has all positive entries.

The equivalence of the last three conditions can be found, for example, in Plemmons [24]. The equivalence of first condition is due to Gabrielov [14]. M-matrices appear in many different fields including economics, operations research, finite difference and finite element analysis ; see for example [7, 9, 10, 18, 16, 25]. In particular, if the stiffness matrix (e.g. the discrete Laplacian) of the finite element method is an M-matrix then the solution satisfies a discrete maximum principle [9, 10, 25]. The discrete maximum principle was an important property used by Baker and Shokrieh [4] in their work on the graph Laplacian for undirected graphs.

We note that these conditions do not necessitate that M has either positive row or column sums. The desired properties of chip-firing such as the existence of unique critical, superstable and energy minimizing configurations will all hold in this more general setting.

3. ENERGY MINIMIZATION

In this section, building on work of Baker and Shokrieh [4], we will prove that for any M-matrix, energy-minimizing configurations exist and are unique per equivalence class.

Given an M-matrix L and an integer vector q define the following energy,

$$(3.1) \quad E(q) = \|L^{-1}q\|_2^2,$$

where $\|v\|_2^2 = v \cdot v$.

This energy is different from the form used in [4]. They defined energy minimizers in terms of a norm using any pseudo-inverse of the (undirected) graph Laplacian. One of the reasons to choose the energy form here is that it allows us to consider non-symmetric matrices L . This will be particularly important in the directed graph case.

The energy form can be extended to any energy from the class $E(q) = \|L^{-1}q\|_G^2$ where $\|v\|_G^2 = v^t G v$ and G is a symmetric positive definite matrix with $L^{-1}G \geq 0$. All of our results below hold for this more general setting. Moreover, if L is symmetric then setting $G = L$ recovers the energy used in [4]. For ease of exposition, we use the simplest form where $G = I$.

Furthermore, there are more energies one can consider. For instance, instead of basing the energy on the 2-norm, as in (3.1), one can base the energy on a p -norm:

$$(3.2) \quad E(q) = \|L^{-1}q\|_p^p,$$

where $\|D\|_p^p = \sum_{i=1}^n |D_i|^p$. The case $p = 1$ was considered by Baker and Shokrieh [4] where they called this quantity the potential. In fact, all of our results will hold for energies defined in the following way

$$(3.3) \quad E(q) = \sum_{i=1}^n \phi_i((L^{-1}q)_i),$$

where the functions $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$ are non-negative and strictly increasing. For example, $\phi_i(x) = |x|^p$ for all $1 \leq i \leq n$ as in the case of (3.2). Another example, could be $\phi_i(x) = \log(1 + |x|)$ for all i . In the appendix we prove that the main results of the paper hold for these more general energies. However, for simplicity, up until the appendix we will restrict the discussion to the energy (3.1).

Baker and Shokrieh work in the undirected graph case and show that energy-minimizing configurations are precisely the superstable configurations (which they refer to as reduced divisors) of the graph. Hence energy-minimizers exist and are unique per equivalence class. In the current section we work directly with the energy minimization problem for arbitrary M-matrices. We make the connection to superstable configurations in Section 4.

The energy minimization problem is posed on equivalence classes induced by L . In order to have cleaner notation, we will adopt the following convention throughout: Let Δ be an M-matrix and $L = \Delta^T$. In this way, we will not have to continually write the transpose for row operations. Also note that the transpose of an M-matrix is always an M-matrix.

Definition 3.1. Two configurations $f, g \in \mathbb{Z}^n$ are equivalent, denoted $f \sim g$, if $g - f = Lz$ for some $z \in \mathbb{Z}^n$. The equivalence class of f is denoted by $[f]$.

Given $f \in \mathbb{Z}^n$ with $f \geq 0$ consider the following problem:

$$(3.4) \quad \min_{g \sim f, g \geq 0} E(g).$$

A solution to the minimization problem is a non-negative configuration equivalent to f with smallest energy. We call such a configuration an energy-minimizer. Since we are working in a discrete space it is not difficult to see that minimizers always exist. We will prove that for any M-matrix there is a *unique* energy-minimizer per equivalence class. To do this, we first prove two preliminary lemmas. We need the following notation. Given $z \in \mathbb{Z}^n$ define $z^+ \in \mathbb{Z}_{\geq 0}^n$ by

$$z_i^+ = \begin{cases} z_i & \text{if } z_i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, define $z^- \in \mathbb{Z}_{\leq 0}^n$ by replacing all positive entries of z with 0.

Lemma 3.2. *Let L be a Z -matrix. If $f, g \geq 0$ and $g = f - Lz$ then $h = f - Lz^+ \geq 0$.*

Proof. Suppose that $z_i^+ = 0$. Then it is clear that $-(Lz^+)_i \geq 0$. Hence, $h_i \geq f_i \geq 0$. On the other hand suppose that $z_i^+ > 0$, then $z^- = z - z^+$ satisfies $z_i^- = 0$ and so $(Lz^-)_i \geq 0$, or equivalently $(Lz)_i \geq (Lz^+)_i$ and so $f_i - (Lz^+)_i \geq f_i - (Lz)_i \geq 0$. \square

The next Lemma expresses the difference in energy of two equivalent configurations.

Lemma 3.3. *Let L be an M -matrix and suppose that $g = f - Lz$, then*

$$E(g) = E(f) + z^t z - 2z^t L^{-1} f = E(f) - z^t z - 2z^t L^{-1} g.$$

Proof.

$$\begin{aligned} E(g) &= \|L^{-1}(f - Lz)\|_2^2 \\ &= \|L^{-1}f - z\|_2^2 \\ &= \|L^{-1}f\|_2^2 + z^t z - 2z^t L^{-1} f \\ &= E(f) + z^t z - 2z^t L^{-1} f \\ &= E(f) - z^t z - 2z^t L^{-1} g. \quad \square \end{aligned}$$

We now state our main theorem for this section.

Theorem 3.4. *Let L be an M -matrix. For every configuration f , there exists a unique energy minimizer equivalent to f . Namely, for every configuration f , there exists a unique solution to problem (3.4).*

Proof. Suppose that $g \sim f$ and $w \sim f$ with $g, w \geq 0$ both minimizers to problem (3.4). We will show that $g = w$. Because g is equivalent to w , there exists z such that $g = w - Lz$ for some $z \in \mathbb{Z}^n$. By Lemma 3.2 we know that $h = w - Lz^+ \geq 0$ and of course $h \sim w \sim f$. By Lemma 3.3 we have

$$E(h) = E(w) - (z^+)^t z^+ - 2(z^+)^t L^{-1} h.$$

Using that L^{-1} is a non-negative matrix and $h \geq 0$, $L^{-1}h \geq 0$. This implies that $-2(z^+)^t L^{-1}h \leq 0$, and so

$$E(h) \leq E(w) - (z^+)^t z^+.$$

Since w is a minimizer it must be that $z^+ = 0$ or that $z \leq 0$.

On the other hand, we similarly have

$$E(w) = E(g) + z^t z - 2z^t L^{-1} w.$$

Since $z \leq 0$ this shows that $E(g) < E(w)$ unless $z = 0$. \square

4. CHIP-FIRING ON M-MATRICES

In Section 2 we defined chip-firing on M -matrices. For an $n \times n$ M -matrix Δ , we consider a system with n states. A configuration is any integer vector $c \in \mathbb{Z}^n$, with c_i considered the number of chips at state i . For a configuration c , state i is allowed to fire if $c_i \geq \Delta_{ii}$ (recall that M -matrices have non-negative diagonal entries). The resulting configuration is $c' = c - Le_i$ where $L = \Delta^T$.

In Section 4.1 we exam three important types of chip configurations - stable, χ -superstable, and z -superstable. In Section 4.2, we prove that energy-minimizers coincide with z -superstable configurations. In Section 4.3, we prove that z -superstable configurations are in duality with critical configurations.

4.1. Stability. We consider three notions of ‘stable’ configurations, each strictly stronger than the previous. The definitions could be made with respect to any matrix, again we have in mind that L is the transpose of an M-matrix.

Definition 4.1. A vector $f \in \mathbb{Z}^n$ is stable if for all i , $f_i < L_{ii}$.

Definition 4.2. A vector $f \in \mathbb{Z}^n$ with $f \geq 0$ is χ -superstable if for every $\chi \in \{0, 1\}^n$ with $\chi \neq 0$ there exists $1 \leq i \leq n$ such that

$$f_i - (L\chi)_i < 0.$$

Definition 4.3. A vector $f \in \mathbb{Z}^n$ with $f \geq 0$ is z -superstable if for every $z \in \mathbb{Z}^n$ with $z \geq 0$ and $z \neq 0$ there exists $1 \leq i \leq n$ such that

$$f_i - (Lz)_i < 0.$$

The above notion of stable configuration is standard in the literature. A stable configuration is one in which no individual state can fire. A χ -superstable configuration is one in which no subset of states can simultaneously fire and result in a non-negative configuration. A z -superstable configuration is one in which no multiset of states can simultaneously fire and result in a non-negative configuration.

In [17], the term superstable is used for χ -superstables. In [20], the term superstable is used for z -superstables. For undirected graphs and Eulerian directed graphs, the notions coincide, i.e. a configuration is χ -superstable if and only if it is z -superstable. Moving to non-Eulerian directed graphs and more generally to M-matrices, the z -superstable condition is strictly stronger.

It is immediately clear that if f is z -superstable then it is χ -superstable. The following result gives sufficient conditions on a matrix for the converse to hold.

Theorem 4.4. *Suppose L is a matrix with non-positive off diagonal entries and **non-negative row sums**¹. Then, if $f \geq 0$ is χ -superstable it is z -superstable.*

Proof. Let $z \in \mathbb{Z}^n$ with $z \geq 0$ and $z \neq 0$. We will show that $f - Lz$ must have a negative entry. To this end, let $\kappa = \max_i z_i$ and so $\kappa > 0$. Define $\chi \in \{0, 1\}^n$ such that $\chi_i = 1$ if $z_i > 0$ and $\chi_i = 0$ if $z_i \leq 0$. Let $\tilde{z} = z - \chi$ and let $\tilde{\kappa} = \max_i \tilde{z}_i$. Then $\tilde{z}_j = \tilde{\kappa}$ for every j such that $\chi_j = 1$. Using that L has non-positive off diagonal entries we have that $(L\chi)_i < 0$ for every i such that $\chi_i = 0$. Therefore, $f_i - (L\chi)_i \geq f_i \geq 0$ for i such that $\chi = 0$. Since f is χ -superstable this means that $f_j - (L\chi)_j < 0$ for some j where $\chi_j = 1$. Consider such a j then we argued that $\tilde{z}_j = \tilde{\kappa}$ and so

$$(L\tilde{z})_j = L_{jj}\tilde{z}_j + \sum_{i \neq j} L_{ji}\tilde{z}_i \geq \tilde{\kappa} \sum_i L_{ji} \geq 0,$$

since we are assuming the row sums are non-negative.

Hence,

$$f_j - (Lz)_j = f_j - (L\chi)_j - (L\tilde{z})_j < f_j - (L\chi)_j < 0. \quad \square$$

The next example shows that if the non-negative row sum condition is dropped, then χ -superstable configurations may not be z -superstable and not unique per equivalence class.

Example 4.5. Consider the following M-matrix which does not have positive row or column sums:

$$L = \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}.$$

¹In the graphical case, L is the transpose of the reduced graph Laplacian. Hence this result applies to graphs whose Laplacians have non-negative column sums.

An explicit calculation shows the image of the three non-zero characteristic vectors:

$$L \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad L \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \quad L \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence, the χ -superstable configurations are: $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Of these three configurations, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is not z -superstable since it is in the image of L , $L \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Note this shows that $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are equivalent under L and so the χ -superstable configurations are not unique per equivalence class.

A z -superstable configuration is one in which no subset of states can fire *with multiplicity* and result in a non-negative configuration. Again we note that z -superstable configurations are the same as the well-known χ -superstable configurations for undirected graphs and Eulerian directed graphs [17, 22]. At the level of M-matrices (which include graph Laplacians from non-Eulerian directed graphs with global sink), z -superstability is the the natural notion to consider.

4.2. z -superstables and energy minimizers. The next results show that for an arbitrary M-matrix, z -superstable configurations coincide with energy minimizers (compare to Theorem 4.14 of [4]).

Theorem 4.6. *Let L be an M-matrix. A vector $f \in \mathbb{Z}^n$ with $f \geq 0$ is z -superstable if and only if it is the minimizer of*

$$\min_{g \sim f, g \geq 0} E(g).$$

Proof. First, suppose that f is z -superstable and let $g \sim f$ with $g \geq 0$. Then we know that there exists $z \in \mathbb{Z}^n$ such that $g = f - Lz$. By Lemma 3.2 $h = f - Lz^+ \geq 0$, but since f is z -superstable then it must be that $z^+ = 0$, or in other words $z \leq 0$. Since by Lemma 3.3

$$E(g) = E(f) + z^t z - 2z^t L^{-1} f$$

we have that $E(g) \geq E(f)$.

On the other hand suppose that f is the minimizer. Assume for the moment f is not z -superstable. Then this implies there exists $z \in \mathbb{Z}^n$ with $z \geq 0$ and z not identically zero such that $g = f - Lz \geq 0$. Since

$$E(g) = E(f) - z^t z - 2z^t L^{-1} f,$$

this implies that

$$E(g) \leq E(f) - z^t z < E(f).$$

However, this contradicts that f is the minimizer. Hence, it must be that f is z -superstable. \square

We have now shown that energy-minimizers are unique up to equivalence class (see Theorem 3.4) and that they coincide with z -superstable configurations. Of course, this implies that the z -superstable configurations are unique up to equivalence class.

Corollary 4.7. *Let L be an M-matrix. For every equivalence class defined by L , there exists a unique z -superstable configuration.*

In the special case of non-negative row sums, another immediate corollary of the Theorems 4.4 and 4.6 follows.

Corollary 4.8. *Let L be an M -matrix with non-negative row sums. For every equivalence class, there exists a unique energy-minimizer, a unique z -superstable configuration, a unique χ -superstable configuration all of which coincide.*

Before making the connection between z -superstable configurations and critical configurations, we note a few properties about z -superstable configurations. The proposition below shows that if f is a z -superstable configuration and any entry of f is reduced but remains non-negative, then the result is also a z -superstable configuration. In the (undirected) graphical case, χ -superstable configurations satisfy a stronger condition that for all *maximal* superstable configurations, the sum of the coordinates is the same. The example below shows that this does not extend to z -superstable configurations of arbitrary M -matrices.

Proposition 4.9. *Suppose f is a z -superstable configuration with respect to L and $g \leq f$, i.e. g is coordinate-wise less than or equal to f . Then g is z -superstable. Namely, z -superstable configurations are component-wise downward closed.*

Proof. Suppose f and g are as above, and g is not z -superstable. Then there exist z such that $g - Lz \geq 0$. Since $(Lz)_i \leq g_i \leq f_i$ for all i we see that $f - Lz \geq 0$, contradicting the fact that f is z -superstable. \square

On the other hand, z -superstable configurations do not form a *pure* order ideal of \mathbb{N}^n . Consider the following M -matrix,

$$L = \begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix}.$$

It is easily checked that L has seven z -superstable configurations. The two maximal configurations under the component-wise partial order are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Hence the two maximal configurations do not have equal sums, a situation that can not occur in the graphical case, see e.g. [19, 6].

4.3. z -superstable and critical configurations. In this section we prove the duality pairing between z -superstable configurations and critical configurations.

Given a matrix $L \in \mathbb{Z}^{n \times n}$ with positive diagonal entries, define $D^L \in \mathbb{Z}^n$ by $D_i^L = L_{ii} - 1$ for all i . Namely, D^L is the vector formed by taking the diagonal entries of L and subtracting 1 from each. Recall that for a given matrix L , a configuration $f \in \mathbb{Z}^n$ is said to be stable if $f \leq D^L$. A configuration $f \in \mathbb{Z}^n$ is said to be unstable if it is not stable.

Definition 4.10. A configuration $c \in \mathbb{Z}^n$ is a critical configuration if it is stable and if there exists a configuration $g \in \mathbb{Z}^n$ with $g_i \geq L_{ii}$ for all i with

$$c = g - \sum_{j=1}^k L e_{i_j},$$

and the requirement

$$g - \sum_{j=1}^{\ell} L e_{i_j} \geq L_{i_{\ell+1}, i_{\ell+1}}$$

for all $\ell < k$.

Interpreting the notation above, the definition states that a configuration c is critical if there exists a configuration g whose entries are at least as large as the diagonal of L and such that g can legally fire a single vertex at a time and result in the configuration c . Critical configurations are also referred to as recurrent configurations. In the case of chip-firing on graphs, critical configurations are often

defined using the idea of firing the sink vertex. The definition given here is more appropriate for our setting of M-matrices where the model does not have a site designated as the sink.

Gabrielov established the existence of critical configurations for any avalanche-finite system.

Proposition 4.11 ([14]). *For any M-matrix, critical configurations exist and are unique per equivalence class.*

In order to prove the connection between z -superstable configurations and critical configurations for M-matrices we need two simple lemmas.

Lemma 4.12. *Suppose that $L \in \mathbb{Z}^{n \times n}$ is an M-matrix. Given any vector $y \in \mathbb{Z}^n$ with $y \geq 0$ there exists a vector $z \in \mathbb{Z}^n$ with $z \geq 0$ such that $Lz \geq y$.*

Proof. Let $g \in \mathbb{Q}^n$ be given by $g = L^{-1}y$. Since L is an M-matrix and y is non-negative, $g \geq 0$. Let $g_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathbb{N}$. Let $\lambda = b_1 b_2 \cdots b_n$ then $z = \lambda g \in \mathbb{Z}^n$ with $z \geq 0$ and $Lz = \lambda y \geq y$. \square

Lemma 4.13. *Suppose that L is a Z-matrix. If $c = g - \sum_{j=1}^k L e_{i_j}$ with c stable and g not stable then for every ℓ with $g_\ell \geq L_{\ell\ell}$ there exists $1 \leq j \leq k$ so that $i_j = \ell$.*

Proof. Suppose that $g_\ell \geq L_{\ell\ell}$ and suppose that $i_j \neq \ell$ for every $j = 1, \dots, k$. Then since L is a Z matrix we have

$$\sum_{j=1}^k (L e_{i_j})_\ell \leq 0,$$

and so

$$c_\ell \geq g_\ell \geq L_{\ell\ell},$$

contradicting the fact that c is stable. \square

Theorem 4.14. *Let L be an M-matrix. If $f \in \mathbb{Z}^n$ is z -superstable then $D^L - f$ is a critical configuration.*

Proof. Let f be z -superstable. It is not difficult to show that $D^L - f$ is stable. By Lemma 4.12 there exists a vector $z \geq 0$ such that $(D^L - f + Lz)_i \geq L_{ii}$ for all i . Set $g = D^L - f + Lz$ so that $g_i \geq L_{ii}$ for all i . Note that since $z \geq 0$ we can write $z = \sum_{j=1}^k e_{i_j}$. We know that

$$D^L - f = g - \sum_{j=1}^k L e_{i_j}.$$

The proof will be complete if we can show there exists a permutation σ of $\{1, \dots, k\}$ so that

$$g^\ell := g - \sum_{j=1}^{\ell} L e_{i_{\sigma(j)}},$$

is such that

$$(4.1) \quad g_{i_{\sigma(\ell+1)}}^\ell \geq L_{i_{\sigma(\ell+1)} i_{\sigma(\ell+1)}},$$

for $\ell = 1, \dots, k-1$.

We proceed to define the permutation σ inductively. Suppose that we have chosen $\sigma(1), \sigma(2), \dots, \sigma(r-1)$ with $r \leq k$ so that (4.1) holds for $1 \leq \ell \leq r-2$. We know that

$$g^{r-1} = D^L - f + L\tilde{z}$$

or equivalently

$$f - L\tilde{z} = D^L - g^{r-1},$$

where

$$\tilde{z} = \sum_{j=1}^k Le_{i_j} - \sum_{j=1}^{r-1} Le_{i_{\sigma(j)}} \geq 0$$

and $\tilde{z} \neq 0$. Since f is z -superstable we know that there exists a q such that $(D^L - g^{r-1})_q < 0$ or equivalently $g_q^{r-1} \geq L_{qq}$. Also, since $c = g^{r-1} - (\sum_{j=1}^k Le_{i_j} - \sum_{j=1}^{r-1} Le_{i_{\sigma(j)}})$, by Lemma 4.13 there exists $1 \leq \sigma(r) \leq k$ such that $\sigma(r) \neq \sigma(j)$ for $j = 1, 2, \dots, r-1$ such that $i_{\sigma(r)} = q$. This completes the proof. \square

Gabrielov [14] showed that critical configurations are unique up to equivalence class for M-matrices. We have shown that z -superstable configurations are unique up to equivalence class and their duals are critical configurations, this is enough to show the following converse of Theorem 4.14.

Theorem 4.15. *Let $L \in \mathbb{Z}^{n \times n}$ be an M-matrix. If c is a critical configuration then $D^L - c$ is z -superstable.*

For completeness we will give an alternative proof of Theorem 4.15. To do so, we need the following known lemma which appears for example in [14]. We also give a proof of this lemma for completeness.

Lemma 4.16. *Let $L \in \mathbb{Z}^{n \times n}$ be a Z-matrix with positive diagonal entries. Let c be a stable configuration and $c = g - \sum_{j=1}^k Le_{i_j}$ with $g_{i_{\ell+1}}^\ell \geq L_{i_{\ell+1}i_{\ell+1}}$ for $\ell = 1, \dots, k-1$ where $g^\ell = g - \sum_{j=1}^\ell Le_{i_j}$. If $g - Lw$ is stable where $w \geq 0$, then $w \geq z$ where $z = \sum_{j=1}^k e_{i_j}$.*

Proof. Suppose that w is not greater than z . Then, this implies there exists $1 \leq \ell \leq k$ such that $w = \sum_{j=1}^{\ell-1} e_{i_j} + \tilde{w}$ with $\tilde{w} \geq 0$ and $\tilde{w}_{i_\ell} = 0$. However, by our hypothesis $g_{i_\ell}^{\ell-1} \geq L_{i_\ell i_\ell}$. Moreover, $g - Lw = g^{\ell-1} - L\tilde{w}$, so by Lemma 4.13 it must be $\tilde{w}_{i_\ell} > 0$. Hence, we reached a contradiction. \square

Now we turn to the proof of Theorem 4.15.

Proof of Theorem 4.15. Let c be a critical configuration and let $f = D^L - c$. Suppose that f is not z -superstable. Then, this implies there exists a vector $z \geq 0$ with $z \neq 0$ such that $f - Lz \geq 0$. Since c is critical, there exists a vector $g > D^L$ such that

$$c = g - \sum_{j=1}^k Le_{i_j},$$

with $g_{i_{\ell+1}}^\ell \geq L_{i_{\ell+1}i_{\ell+1}}$ for $\ell = 1, \dots, k-1$, where $g^\ell = g - \sum_{j=1}^\ell Le_{i_j}$. Setting $w = \sum_{j=1}^k e_{i_j}$, we see that

$$f - Lz = D^L - g + L(w - z).$$

Which gives

$$L(w - z) \geq g - D^L > 0.$$

Since L is an M-matrix this implies that $w - z \geq 0$. Also, we have

$$D^L \geq g - L(w - z),$$

which implies $g - L(w - z)$ is stable. However, by Lemma 4.16 $w - z \geq w$. Which implies that $z \leq 0$. Hence, we have reached a contradiction. \square

4.4. Graph Laplacians and G -parking functions. As mentioned before, the fact that χ -superstable configurations are unique (up to equivalence class) and are in simple duality with critical configurations is well known for graph Laplacians of undirected graphs.

In the directed graph case, this result was first extended to Eulerian graphs; see for example [17, 22]. The Eulerian condition ensures the reduced Laplacian has non-negative column sums. In this case, χ -superstable configurations continue to coincide with z -superstable configurations.

The duality was further extended to all directed graphs with a global sink. The result first appears in [21] and later in [20] and [1]. In this case, χ -superstable configurations are not the same as z -superstable configurations. Critical configurations are in duality with the z -superstable configurations, which are called superstable configurations in [20, 21] and reduced divisors in [1]. If Δ is the reduced Laplacian resulting from any directed graph with a global sink then Δ is an M-matrix, this was shown explicitly for example in [22]. Hence we also recover this result as a special case of our duality pairing for any system defined by an M-matrix.

As a final remark, in an attempt to clarify the literature, we relate these notions to G -parking functions. For a directed graph G , a parking function is a non-negative integer sequence (a_1, a_2, \dots, a_n) such that for every subset $I \subseteq [n]$ there exists $i \in I$ such that

$$a_i < d_I(i),$$

where $d_I(i)$ is the number of edges from i to vertices not in I . In the undirected (and directed Eulerian) graph case, G -parking functions, χ -superstable configurations, z -superstable configurations, and reduced divisors all coincide.

However, in the non-Eulerian directed graph case,

$$\chi\text{-superstables} \neq z\text{-superstables} = \text{reduced divisors} \neq G\text{-parking functions.}$$

This distinction is implicit in [22]. We end with an explicit example illustrating the difference.

Example 4.17. Consider the graph on 3 vertices with directed graph Laplacian equal to:

$$\begin{array}{c} 1 \quad 2 \quad s \\ 1 \left(\begin{array}{ccc} 3 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{array} \right). \\ 2 \\ s \end{array}$$

Vertex 1 has three edges directed to vertex 2. Vertex 2 has a single edge to vertex 1 and a single edge to the sink. The sink has no outgoing edges. The transpose of the reduced graph Laplacian is:

$$L = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}.$$

It is not hard to check that all four 0/1-vectors of length two are χ -superstable for this graph. On the other hand, the all ones vector is not z -superstable as it is equal to $L \cdot (1, 2)^T$. In particular, the all ones configuration is equivalent to the all zeros configuration. For this graph, $D^L = (2, 1)$ and hence the critical configurations are $(2, 1)$, $(1, 1)$, and $(2, 0)$, the z -superstables are $(0, 0)$, $(1, 0)$, $(0, 1)$. It is easily checked that the G -parking functions are $(0, 0)$, $(1, 0)$, $(2, 0)$.

Acknowledgements The authors thank David Perkinson for many helpful discussions and an anonymous reviewer for a thoughtful question concerning energy forms.

5. APPENDIX

Here we show that z -superstable configurations are also the minimizers of a more general class of energies. We consider any energy of the form

$$(5.1) \quad E(q) = \sum_{i=1}^n \phi_i((L^{-1}q)_i),$$

where the functions $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$ are non-negative and strictly increasing.

We first prove that there is a unique $E(q)$ -minimizer per equivalence class. The minimization problem remains the same.

Given $f \in \mathbb{Z}^n$ with $f \geq 0$ consider the following problem:

$$(5.2) \quad \min_{g \sim f, g \geq 0} E(g).$$

The following result is the generalization of Theorem 3.4. In fact, the wording of the statement is exactly the same except that we now consider the energy (5.1). Also, the reader will notice that the beginning of the proof is identical to that of Theorem 3.4.

Theorem 5.1. *Let L be an M -matrix. For every configuration f , there exists a unique energy minimizer equivalent to f . Namely, for every configuration f , there exists a unique solution to problem (5.2).*

Proof. Suppose that $g \sim f$ and $w \sim f$ with $g, w \geq 0$ are both minimizers to problem (3.4). We will show that $g = w$. Because g is equivalent to w , there exists z such that $g = w - Lz$ for some $z \in \mathbb{Z}^n$. By Lemma 3.2 we know that $h = w - Lz^+ \geq 0$ and of course $h \sim w \sim f$. We have that $H = W - z^+$ where $H = L^{-1}h$ and $W = L^{-1}w$. Since L is an M -matrix the entries of L^{-1} are non-negative and so $H, W > 0$. Note that $0 \leq H_i \leq W_i$ and since ϕ_i is strictly increasing we have $\phi_i(|H_i|) \leq \phi_i(|W_i|)$ for all $1 \leq i \leq n$ with strict inequality whenever $z_i^+ > 0$.

Therefore,

$$E(h) = \sum_{i=1}^n \phi_i(|H_i|) \leq \sum_{i=1}^n \phi_i(|W_i|) = E(w),$$

with strict inequality if at least one z_i^+ is greater than zero. Since w is a minimizer it must be that z^+ is identically zero. In other words, $z \leq 0$. If we let $G = L^{-1}g$ then we have $G = W - z$ and therefore we have that $G_i \geq W_i \geq 0$ with strict inequality whenever $z_i < 0$. Hence,

$$E(g) = \sum_{i=1}^n \phi_i(|G_i|) \geq \sum_{i=1}^n \phi_i(|W_i|) = E(w),$$

with strict inequality if at least one z_i is less than zero. Since g is a minimizer it must be that z is identically zero. That is, $g = w$. \square

The following result is the generalization to Theorem 4.6.

Theorem 5.2. *Let L be an M -matrix. A vector $f \in \mathbb{Z}^n$ with $f \geq 0$ is z -superstable if and only if it is the minimizer of*

$$\min_{g \sim f, g \geq 0} E(g).$$

Proof. First, suppose that f is z -superstable and let $g \sim f$ with $g \geq 0$. Then we know that there exists $z \in \mathbb{Z}^n$ such that $g = f - Lz$. By Lemma 3.2 $h = f - Lz^+ \geq 0$, but since f is z -superstable then it must be that $z^+ = 0$, or in other words $z \leq 0$. We have $G = F - z$ where $G = L^{-1}g$ and $F = L^{-1}f$ and so $G_i \geq F_i \geq 0$. Therefore, we see that $\phi_i(|G_i|) \geq \phi_i(|F_i|)$ for every $1 \leq i \leq n$. Hence, we have $E(g) \geq E(f)$, and so f is a minimizer.

On the other hand, suppose that f is the minimizer. Assume for the moment f is not z -superstable. Then this implies there exists $z \in \mathbb{Z}^n$ with $z \geq 0$ and z not identically zero such that $g = f - Lz \geq 0$. Since $G = F - z$ it must be that $0 \leq G_i \leq F_i$ for every i and $G_i < F_i$ for at least one i . Hence, $\phi_i(|G_i|) \leq \phi_i(|F_i|)$ for every i and $\phi_i(|G_i|) < \phi_i(|F_i|)$ for at least one i . Therefore, $E(g) < E(f)$, but this contradicts that f is a minimizer. Hence, it must be that f is z -superstable. \square

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