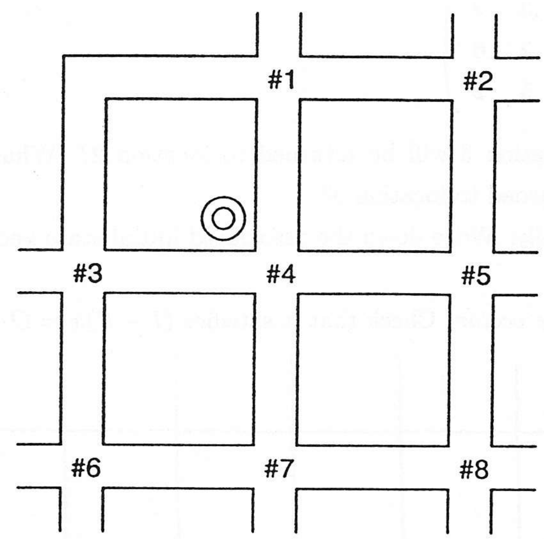
Homework #2

Deadline: March 2 (Wednesday) 4:00am EST
Scorelator will begin accepting submissions from February 24 (Thursday)

Problem 1

A police officer is assigned to control traffic at the eight intersections shown on the right. He spends one hour at each intersection and then moves to an adjacent intersection (NOTE: intersections #1 and #3 are considered to be adjacent). Each time he goes to intersection #4, he cannot resist eating a couple of donuts from the bakery there, but he figures that the amount of exercise he gets daily would be enough to counteract the calories from the donuts if he is at intersection #4 only once a day on average. Therefore, when each hour is up he selects at random the adjacent intersection he will go to next, so that all possible choices are equally likely; he thought that by doing this, he would be spending $\frac{1}{8}$ of his time at each intersection, i.e. on average he would be visiting each intersection once a day. However, as the weeks go by, the officer finds that his waistband is getting tighter and tighter and tighter...



(a) Construct a transition matrix containing the probabilities of the police officer moving from intersection j to intersection i (where j and i range from 1 to 8) after each hour of work. Save the matrix in the file `A11.dat`.

(b) Suppose that the officer is currently at intersection #6. Calculate the probability that he will be at each of the eight intersections 40 working hours later, and save this set of probabilities as a length-8 column vector in `A12.dat`.

(c) Using eigenvalues and eigenvectors, determine the steady state proportions of time that the officer will spend at the eight intersections. Save your answer as a length-8 column vector in `A13.dat` (you should find that, on average, he spends more than $\frac{1}{8}$ of his time at intersection #4).

Problem 2

The following table shows population data for the same country in 1990 that we considered in the class example, except here the population has been divided into three age groups—juveniles, workers and elderly.

	Juveniles (age < 15)	Workers (age 15–64 years)	Elderly (age ≥ 65)
Subpopulation size (millions)	10.92	37.50	8.99
Proportionate birth rate	0.0000	0.0213	0.0000
Proportionate death rate	0.0008	0.0031	0.0575

Let J_n, W_n and E_n denote the sizes (in millions) of the subpopulations of juveniles, workers and elderly, respectively, n years after 1990. Assume that in any year, $\frac{1}{15}$ of surviving juveniles enter the workforce and $\frac{1}{50}$ of surviving workers reach the retirement age of 65. The population growth model can be written as a matrix equation

$$\begin{pmatrix} J_{n+1} \\ W_{n+1} \\ E_{n+1} \end{pmatrix} = \mathbf{M} \begin{pmatrix} J_n \\ W_n \\ E_n \end{pmatrix}$$

(a) Fill in the entries of the matrix $\mathbf{M}$, and save the matrix in the file `A21.dat`.

(b) Find the sizes (in millions) of the subpopulations of juveniles, workers and adults predicted by the model for the year 2020. Save the results as a column vector in `A22.dat`.

(c) Using eigenvalues and eigenvectors, predict the ratio of elderly to workers and the ratio of juveniles to workers in the long term. Save your answers with at least 4 decimal places in `A23.dat` as a vertical list (ratio of elderly to workers followed by ratio of juveniles to workers).

Problem 3

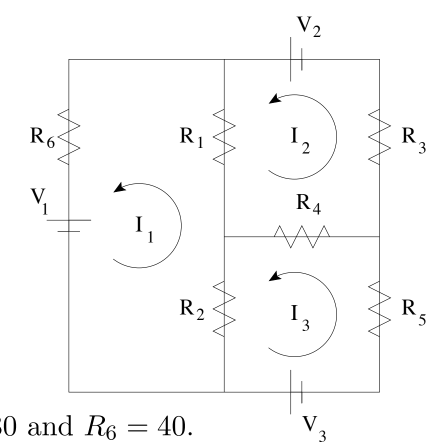
(a) Rewrite the linear system of equations

$$\begin{aligned} 3x_1 + x_2 - 4x_3 + 5x_5 &= 6 \\ -2x_2 - x_3 + x_4 - x_5 &= -5 \\ x_1 + 2x_3 - x_4 + x_5 &= 2 \\ 2x_1 + x_2 - x_3 + x_4 - 3x_5 &= 7 \\ x_1 - x_2 - x_3 - x_4 + x_5 &= 3 \end{aligned}$$

in matrix form $A\mathbf{x} = \mathbf{b}$. Find the LU factorization of A, so that the equation becomes $LU\mathbf{x} = \mathbf{b}$. Then let $U\mathbf{x} = \mathbf{y}$ and

(i) solve $L\mathbf{y} = \mathbf{b}$ by forward substitution (you will need to write a function to perform forward substitution, given a lower triangular matrix and a right-hand side vector), saving the $\mathbf{y}$ you found in `A31.dat`;

(ii) solve $U\mathbf{x} = \mathbf{y}$ by backward substitution (you can use the function developed in class, or write your own code), saving the $\mathbf{x}$ you found in `A32.dat`.



(b) From the electrical circuit depicted on the right, one can derive a system of equations for the currents I_1, I_2 and I_3:

$$\begin{aligned} R_6 I_1 + R_1(I_1 - I_2) + R_2(I_1 - I_3) &= V_1 \\ R_3 I_2 + R_4(I_2 - I_3) + R_1(I_2 - I_1) &= V_2 \\ R_5 I_3 + R_4(I_3 - I_2) + R_2(I_3 - I_1) &= V_3 \end{aligned}$$

The resistances are $R_1 = 20$, $R_2 = 10$, $R_3 = 25$, $R_4 = 10$, $R_5 = 30$ and $R_6 = 40$. For the voltages we have $V_2 = 0$ and $V_3 = 200$, while V_1 is variable.

Increase V_1 from 0 to 100 in steps of 10 (i.e. take $V_1 = 0, 10, 20, \ldots, 100$) and for each V_1 value calculate I_1, I_2 and I_3 by solving the system using LU factorization together with forward and backward substitution (as in part (a)). Save your results in `A33.dat` as a 3×11 matrix, where the first column contains the I_1, I_2, I_3 values corresponding to $V_1 = 0$, the second column contains the I_1, I_2, I_3 values corresponding to $V_1 = 10$, the third column contains the I_1, I_2, I_3 values corresponding to $V_1 = 20$, and so on.