APMA 0160 (A. Yew) Spring 2011

Homework #11

Deadline: May 15 (Sunday) 4:00am EST

Problem 1

(a) Consider the second-order linear differential equation

$$\frac{d^2y}{dx^2} + 3\,\frac{dy}{dx} + \frac{x^2}{4}y = -e^{-x^2} \qquad \text{for } -1 \le x \le 1$$

subject to the boundary conditions

$$y(-1) = 2.5, \qquad y(1) = 3$$

Compute the solution of this boundary value problem with the method of finite differences. Take $\Delta x = 0.04$, and use $O(\Delta x^2)$ centered difference formulas to approximate the derivatives y'' and y'. Save the y values of the computed solution at $x = -1 : 0.04 : 1$ as a column vector in `A11.dat`.
Find the maximum point on the computed solution curve, and save its x and y coordinates as a row vector in `A12.dat`.

(b) Consider the second-order linear differential equation

$$\frac{d^2y}{dx^2} + \frac{x^2}{4}y = -e^{-x^2} \qquad \text{for } -1 \le x \le 1$$

subject to the boundary conditions

$$y'(-1) = 1.5, \qquad y(1) = 3$$

(note that the condition at the left endpoint involves the *derivative* of y).

Compute the solution of this boundary value problem with the method of finite differences. Take $\Delta x = 0.04$, and use the $O(\Delta x^2)$ centered difference formula to approximate y''. To approximate the boundary condition at the left end, use an $O(\Delta x^2)$ formula also.
Save the y values of the computed solution at $x = -1 : 0.04 : 1$ as a column vector in `A13.dat`.
Find the maximum point on the computed solution curve, and save its x and y coordinates as a row vector in `A14.dat`.

(PLEASE TURN OVER)

Problem 2

The partial differential equation

$$(\star) \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (\kappa - 1)\frac{\partial u}{\partial x} - \kappa\, u$$

is a scaled version of the Black–Scholes equation from financial mathematics.

(a) With $\kappa = 1$, compute an approximation of the function $u(t,x)$ which satisfies the PDE $(\star)$ together with the initial condition

$$u(0,x) = 1 + \frac{x}{5} \qquad \text{for } 0 \le x \le 10$$

and the Dirichlet boundary conditions

$$u(t,0) = 1, \qquad u(t,10) = 3 \qquad \text{for all } t > 0$$

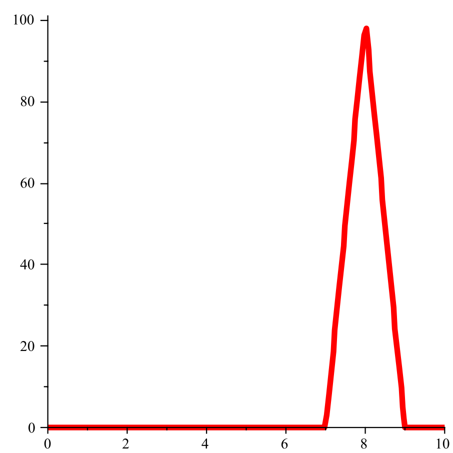
Choose the spatial grid points x_j so that $x_0 = 0$ and $x_{51} = 10$.
Use $O(\Delta x^2)$ centered difference formulas to approximate the spatial derivatives $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial u}{\partial x}$.
Then use `ode45` with default tolerances to do the time-stepping, asking Matlab to record the approximate solution values at $t = 0 : 0.2 : 5$.

Save the computed approximate solution, $u(t,x)$ for $t = 0 : 0.2 : 5$ and $x = x_0, x_1, x_2, \ldots, x_{51}$, as a 26×52 matrix in `A21.dat` (where each row represents a time slice and each column represents a particular spatial position).
Find the minimum value of u at $t = 5$, as well as the x-value at which this minimum occurs; save these x and u values (in that order) as a row vector in `A22.dat`.

(b) Do the same as in part (a), but this time with $\kappa = 0.2$. Save the computed solution in `A23.dat`, and save the x and u values of the minimum point at $t = 5$ in `A24.dat`.

(c) With $\kappa = 0.2$, compute an approximation of the function $u(t,x)$ which satisfies the PDE $(\star)$ together with the initial condition

$$u(0,x) = \begin{cases} 100(x-7) & \text{for } 7 \le x \le 8 \\ 100 - 100(x-8) & \text{for } 8 \le x \le 9 \\ 0 & \text{otherwise} \end{cases}$$



and the Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t,0) = 0, \qquad \frac{\partial u}{\partial x}(t,10) = 0 \qquad \text{for all } t > 0$$

Choose the spatial grid points x_j so that $x_0 = 0$ and $x_{51} = 10$.
Use $O(\Delta x^2)$ centered difference formulas to approximate the spatial derivatives $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial u}{\partial x}$.
Also use $O(\Delta x^2)$ formulas to approximate the boundary conditions at both ends.
Then use `ode45` with default tolerances to do the time-stepping, asking Matlab to record the approximate solution values at $t = 0 : 0.2 : 5$. Save the computed solution as a 26×52 matrix in `A25.dat`.
Find the maximum value of u at $t = 5$, as well as the x-value at which this maximum occurs; save these x and u values as a row vector in `A26.dat`.