Curve fitting: polynomial interpolation

In least squares fitting, we try to fit a curve $y = f(x)$ to a set of data points

$$(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3),\ \ldots, (x_n, y_n)$$

by minimizing the 2-norm of the residual vector $\mathbf{r}$, whose components are $r_i = y_i - f(x_i),\ i = 1, 2, \ldots n$.

We have seen that, quite often, by choosing a function form $f(x)$ that contains more parameters ("degrees of freedom"), it is possible to obtain a closer fit. So one might ask: is it possible to choose $f(x)$ so that the curve goes through all of the data points exactly, giving a fitting error of zero?

The answer should be "yes" if $f(x)$ has the form of a polynomial: a polynomial of degree $n-1$ has n coefficients, and it depends linearly on each of these coefficients; so if we require $f(x_i) = y_i$ for $i = 1, 2, \ldots, n$, we get n linear equations for the n unknown coefficients, and in general such a linear system can be solved. In fact, for a polynomial

$$f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$

the linear system is

$$\begin{aligned}
a_{n-1}x_1^{n-1} + a_{n-2}x_1^{n-2} + \cdots + a_2x_1^2 + a_1x_1 + a_0 &= y_1 \\
a_{n-1}x_2^{n-1} + a_{n-2}x_2^{n-2} + \cdots + a_2x_2^2 + a_1x_2 + a_0 &= y_2 \\
&\vdots \\
a_{n-1}x_n^{n-1} + a_{n-2}x_n^{n-2} + \cdots + a_2x_n^2 + a_1x_n + a_0 &= y_n
\end{aligned}$$

or, in matrix form, $V\boldsymbol{\alpha} = \boldsymbol{\beta}$ where

$$V = \begin{pmatrix} \quad & \quad & \quad \\ \\ \\ \\ \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} \quad \\ \\ \\ \\ \end{pmatrix}, \qquad \boldsymbol{\beta} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

V is called a Vandermonde matrix.

It is also possible to write down an explicit formula for the polynomial that goes through all the data points. The idea is to express $f(x)$ as

$$f(x) = y_1L_1(x) + y_2L_2(x) + \cdots + y_{n-1}L_{n-1}(x) + y_nL_n(x) = \sum_{i=1}^{n} y_iL_i(x)$$

where each $L_i(x)$ is a polynomial of degree $n-1$ with the property that

$$L_i(x_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

Lagrange's formula for $L_i(x)$ is:

and then this $f(x)$ is called the "Lagrange interpolating polynomial".