Sums

(a) Compute the sum $\displaystyle\sum_{k=1}^{20} \sqrt{k}$

(b) Compute the sum $\displaystyle\sum_{k=17}^{35} k^3$

(c) Compute the sum $\dfrac{1}{4} + \dfrac{1}{7} + \dfrac{1}{10} + \dfrac{1}{13} + \cdots + \dfrac{1}{94} + \dfrac{1}{97}$

(d) The series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges to $\dfrac{\pi^2}{6}$; that is, the sequence of partial sums $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$ tends to $\dfrac{\pi^2}{6}$ as $n \to \infty$. Write a script to approximate the value of this limit, and find the smallest n such that the absolute error of the approximation is no more than 5×10^{-6}.

Sequences defined by recursion relations: Fibonacci numbers and the golden ratio

The Fibonacci numbers $f_0, f_1, f_2, \ldots$ are defined via the recursion relation

$$f_{n+1} = f_n + f_{n-1} \quad \text{for } n = 1, 2, 3, \ldots$$

with $f_0 = 1$ and $f_1 = 1$. In other words, the "next" Fibonacci number f_{n+1} is given by the "current" Fibonacci number f_n plus the "previous" Fibonacci number f_{n-1}.

It can be shown that, as $n \to \infty$, the ratios $r_n = \dfrac{f_n}{f_{n-1}}$ converge to a limit, which is the so-called "golden ratio" $\dfrac{1+\sqrt{5}}{2}$.

Write a script that displays the values of n, f_n and r_n for $n = 1, \ldots, N$, where N is the smallest n such that $|r_{n+1} - r_n| \le 10^{-15}$.

Perimeter of an ellipse

Consider the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, which can also be expressed parametrically as

$$\begin{cases} x(t) = a\cos t \\ y(t) = b\sin t \end{cases} \qquad 0 \le t \le 2\pi$$

There is no simple formula for the perimeter of the ellipse, so we shall approximate it by computing the perimeters of inscribed and circumscribed polygons.
For each integer $n \ge 3$, we can take n points

$$p_k = \left(a\cos\left(k \cdot \frac{2\pi}{n}\right),\ b\sin\left(k \cdot \frac{2\pi}{n}\right)\right), \quad k = 1, \dots n$$

on the ellipse, join up successive pairs of these points, and obtain a *inscribed* (inner) polygon with n sides. If, instead, we connect the tangent lines to the ellipse at the points P_k, then we obtain a *circumscribed* (outer) polygon with n sides. The vertices of the circumscribed polygon are given by

$$q_k = \left(a\left\{\cos\left(k \cdot \frac{2\pi}{n}\right) - \frac{1-\cos\frac{2\pi}{n}}{\sin\frac{2\pi}{n}} \cdot \sin\left(k \cdot \frac{2\pi}{n}\right)\right\},\ b\left\{\sin\left(k \cdot \frac{2\pi}{n}\right) + \frac{1-\cos\frac{2\pi}{n}}{\sin\frac{2\pi}{n}} \cdot \cos\left(k \cdot \frac{2\pi}{n}\right)\right\}\right)$$

Let $\mathcal{P}_{\text{ellipse}}$ denote the perimeter of the ellipse, and let $\mathcal{P}_{\text{inner}}$ and $\mathcal{P}_{\text{outer}}$ denote the perimeters of the inner and outer polygons, respectively. Then

$$\mathcal{P}_{\text{inner}} < \mathcal{P}_{\text{ellipse}} < \mathcal{P}_{\text{outer}}$$

If we take

$$\mathcal{P}_{\text{ave}} = \frac{\mathcal{P}_{\text{inner}} + \mathcal{P}_{\text{outer}}}{2}$$

as an approximation to $\mathcal{P}_{\text{ellipse}}$, then

$$\text{absolute error of the approximation} = |\mathcal{P}_{\text{ave}} - \mathcal{P}_{\text{ellipse}}| \le \frac{\mathcal{P}_{\text{outer}} - \mathcal{P}_{\text{inner}}}{2}$$

(a) Write a function `P_inner` whose:

- input arguments are the semimajor and semiminor axes a and b of the ellipse followed by n, the number of sides of the polygon;
- output argument is the perimeter of the inscribed polygon.

(b) Write a function `P_outer` whose:

- input arguments are the semimajor and semiminor axes a and b of the ellipse followed by n, the number of sides of the polygon;
- output argument is the perimeter of the circumscribed polygon.

(c) Write a script that calls the functions you wrote in parts (a) and (b) to compute the approximation $\mathcal{P}_{\text{ave}}$ to the perimeter of an ellipse with semimajor axis 5 and semiminor axis 3. Output the values of n and $\mathcal{P}_{\text{ave}}$ for $n = 3, \dots, N$, where N is the smallest n such that the absolute error of the approximation does not exceed 10^{-4}.