
Iterated maps

Consider a sequence defined by a recursion relation of the form

$$x_{k+1} = \phi(x_k), \qquad \text{where } x_0 \text{ is given.}$$

In this context, the function ϕ is usually referred to as a "map". We have

$$\begin{aligned} x_1 &= \phi(x_0), \\ x_2 &= \phi(x_1) = \phi(\phi(x_0)), \\ x_3 &= \phi(x_2) = \phi(\phi(x_1)) = \phi(\phi(\phi(x_0))), \end{aligned}$$

and so on. So we can think of x_n as the result of applying the map ϕ repeatedly for n times.
In other words, the sequence is generated from an "iterated map".

Example

The logistic equation

$$x_{k+1} = rx_k(1 - x_k) \quad \text{with } r > 0$$

was originally proposed as a model of population growth. Here the map is $\phi(x) = rx(1 - x)$.
Starting with $x_1 = 0.5$, take r to be each of the values

$$0.8,\ 1.5,\ 2.8,\ 3.2,\ 3.5,\ 3.7$$

and iterate the map to generate the sequence $x_1, x_2, x_3, \ldots, x_{50}$; then plot the sequence.
Put the six sequences (one for each r-value) into a 6×50 matrix, and save it as an ASCII data file.

Fixed points

A fixed point of a map ϕ is a number p for which $\phi(p) = p$.
Geometrically, a fixed point is an intersection of the graph of $y = \phi(x)$ with the line $y = x$.

If a sequence generated by the relation $x_{k+1} = \phi(x_k)$ converges, then its limit must be a fixed point of ϕ.

<u>THEOREM</u>

(a) Suppose ϕ is continuous on $[a, b]$ and that $a \le \phi(x) \le b$ for all $x \in [a, b]$ (i.e. the range of ϕ is contained in its domain). Then ϕ has a fixed point in $[a, b]$.

(b) If, in addition, ϕ is differentiable on (a, b) and there exists a positive constant $L < 1$ such that

$$|\phi'(x)| \le L \quad \text{for all } x \in (a, b),$$

then the fixed point p is unique and the sequence generated by $x_{k+1} = \phi(x_k)$ converges to p for any choice of the initial point x_0 in $[a, b]$.