
Iterated matrices and eigenvectors

We have looked at a couple of examples of sequences generated by iterated transition matrices:

$$\mathbf{x}_{k+1} = M\mathbf{x}_k, \qquad \text{where the initial state vector } \mathbf{x}_0 \text{ is given.}$$

Instead of writing a loop to perform the repeated matrix multiplications, we could use matrix powers to calculate the iterates:

$$\begin{aligned}
\mathbf{x}_1 &= M\mathbf{x}_0, \\
\mathbf{x}_2 &= M\mathbf{x}_1 = MM\mathbf{x}_0 = M^2\mathbf{x}_0, \\
\mathbf{x}_3 &= M\mathbf{x}_2 = MM^2\mathbf{x}_0 = M^3\mathbf{x}_0,
\end{aligned}$$

and so on. Thus, $\mathbf{x}_n = M^n\mathbf{x}_0$.

If the sequence of vectors $\{\mathbf{x}_k\}$ converges (i.e. all of the entries settle down), then the limit must be a vector $\mathbf{x}^*$ that satisfies

$$\mathbf{x}^* = M\mathbf{x}^*$$

Eigenvalues and eigenvectors

Suppose that A is a square $(n \times n)$ matrix. If there is a non-zero vector $\mathbf{v}$ of length n such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some scalar (constant) λ, then we say that $\mathbf{v}$ is an *eigenvector* of A corresponding to the *eigenvalue* λ. An eigenvalue can be a real or complex number, or zero. An eigenvector may also have real, complex or zero entries, but it cannot have all its entries equal to zero.

Geometrically, the relation $A\mathbf{v} = \lambda\mathbf{v}$ means that multiplying $\mathbf{v}$ by the matrix A results in a vector that is in the same direction as $\mathbf{v}$, with magnitude scaled by λ.

Note that if $\mathbf{v}$ is an eigenvector of A corresponding to eigenvalue λ, then so is any non-zero constant multiple of $\mathbf{v}$, because $A(c\mathbf{v}) = cA\mathbf{v} = c\lambda\mathbf{v} = \lambda(c\mathbf{v})$.

Eigenvalues and eigenvectors come up in a huge number of application-oriented problems.
For instance, the behavior of solutions of systems of linear differential equations

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}$$

or sequences generated by iterated matrices

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

is determined by eigenvalues and eigenvectors of the matrix A.

In particular, if a sequence $\{\mathbf{x}_k\}$ generated by iterating a transition matrix M converges, then the limit must be an eigenvector of M corresponding to the eigenvalue 1.

Example 1
Recall the Fibonacci numbers $f_0, f_1, f_2, \ldots$ defined via the recursion relation

$$f_{n+1} = f_n + f_{n-1} \quad \text{for } n = 1, 2, 3, \ldots$$

with $f_0 = 1$ and $f_1 = 1$. As $n \to \infty$, the ratios $r_n = \dfrac{f_n}{f_{n-1}}$ converge to a limit, which is the so-called "golden ratio" $\dfrac{1+\sqrt{5}}{2}$. This limiting ratio can be found from an eigenvalue/eigenvector if we formulate the problem as a matrix iteration:
Define the nth "state vector" to be

$$\mathbf{x}_n = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$$

Then

$$\mathbf{x}_{n+1} = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$

and these "current" and "next" state vectors are related via matrix multiplication:

$$\mathbf{x}_{n+1} = A\mathbf{x}_n, \qquad \text{i.e.} \quad \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$$

The nth ratio $r_n = \dfrac{f_n}{f_{n-1}}$ is obtained by dividing the first component of $\mathbf{x}_n$ by the second component.

For almost all initial conditions, the long-term behavior of $\{\mathbf{x}_n\}$ follows the eigenvector corresponding to the eigenvalue of greatest magnitude.

Example 2
The following table shows population data for a certain country in the year 1990.

	Juveniles (aged under 15 years)	Adults (aged 15 years and over)
Subpopulation size (millions)	10.92	46.49
Proportionate birth rate	0.0000	0.0172
Proportionate death rate	0.0008	0.0136

Let J_n and A_n denote the sizes (in millions) of the subpopulations of juveniles and adults, respectively, n years after 1990. Assume that in any year $\frac{1}{15}$ of surviving juveniles become adults. The population growth model can be written as a matrix equation

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \mathbf{M} \begin{pmatrix} J_n \\ A_n \end{pmatrix}$$

(a) Fill in the entries of the matrix $\mathbf{M}$ to 4 decimal places.
(b) Find the sizes of the subpopulations of juveniles and adults predicted by the model for the years 2010 and 2030.
(c) Predict the ratio of adults to juveniles in the long term.