Fixed point iterations

In the previous class we started to look at sequences generated by **iterated maps**:

$$x_{k+1} = \phi(x_k), \qquad \text{where } x_0 \text{ is given.}$$

A **fixed point** of a map ϕ is a number p for which $\phi(p) = p$.

If a sequence generated by $x_{k+1} = \phi(x_k)$ converges, then its limit must be a fixed point of ϕ.

Conversely, under what conditions is it possible to find a fixed point of a map ϕ by generating a convergent sequence from iterating the map?

<u>Theorem</u> (sufficient conditions)
(a) Suppose ϕ is continuous on $[a, b]$ and that $a \le \phi(x) \le b$ for all $x \in [a, b]$ (i.e. the range of ϕ is contained in its domain). Then ϕ has a fixed point in $[a, b]$.
(b) If, in addition, ϕ is differentiable on (a, b) and there is a positive constant $L < 1$ such that

$$|\phi'(x)| \le L \quad \text{for all } x \in (a, b),$$

then the fixed point p is unique and the sequence generated by $x_{k+1} = \phi(x_k)$ converges to p for any choice of initial point x_0 in $[a, b]$.

Root finding

One of the most basic problems in scientific computing is to find the solution of a scalar equation in one (real) variable:

$$f(x) = 0$$

In other words, we want to compute a "root" (also called a "zero") of the function f.

Note that any root-finding problem can be reformulated as a fixed-point problem, i.e. we can always rewrite $f(x) = 0$ in the form $x = \phi(x)$ for some function ϕ, so that a root of the original function f is a fixed point of the map ϕ. We can then try to generate a sequence by iterating ϕ. If this sequence converges, then its limit will be the root of f.

There are always multiple ways of rewriting $f(x) = 0$ as $x = \phi(x)$, i.e. there are multiple choices for the map ϕ. We can use the sufficient conditions given in the theorem above to help us select a ϕ that will give rise to a convergent sequence.

Example

By plotting its graph, it can be seen that $f(x) = x^3 + 4x^2 - 10$ has a unique root in the interval $[1, 2]$. Check that $x^3 + 4x^2 - 10 = 0$ can be rewritten as a fixed-point equation $x = \phi(x)$ where ϕ can take the following forms (among others):

(i) $\phi_1(x) = x - (x^3 + 4x^2 - 10)$

(ii) $\phi_2(x) = \dfrac{1}{2}\sqrt{10 - x^3}$

(iii) $\phi_3(x) = \sqrt{\dfrac{10}{x} - 4x}$

(iv) $\phi_4(x) = \sqrt{\dfrac{10}{4 + x}}$

(v) $\phi_5(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

Starting with $x_1 = 1.5$, iterate each of these maps and observe (a) whether the sequence converges; and (b) if so, how many iterations are needed for it to converge to within a given tolerance.

Stopping criteria for root-finding algorithms

$$\text{absolute error} = |\text{true value} - \text{computed approximation}|$$

$$\text{relative error} = \frac{|\text{true value} - \text{computed approximation}|}{|\text{true value}|}$$

These error formulas are not of much direct use in constructing stopping conditions for algorithms, as in most practical situations the "true value" is not known (which is why we are using a computational procedure to approximate it). However, if the algorithm naturally provides a **bound** on the error (of either kind), then we can use this bound to make a stopping criterion:

$$\text{error bound} \leq \text{tolerance}$$

(e.g. in the ellipse perimeter example, a bound on the absolute error was $\dfrac{\mathcal{P}_{\text{outer}} - \mathcal{P}_{\text{inner}}}{2}$).

Suppose we are generating a sequence $\{x_n\}$ to approximate the solution of a problem.
If sequence converges, then its terms must get closer and closer together as n increases.
So one commonly used stopping criterion is:

$$|x_n - x_{n-1}| \leq \text{tolerance}$$

Another, related, criterion is:

$$\left|1 - \frac{x_n}{x_{n-1}}\right| \leq \text{tolerance} \qquad \text{or} \qquad \left|1 - \frac{x_{n-1}}{x_n}\right| \leq \text{tolerance}$$

This criterion is often used because the left-hand side expressions can be viewed as estimates of the relative error.

For a root x^* of f, we have $f(x^*) = 0$ by definition, so if we use a sequence $\{x_n\}$ to approximate x^*, we should expect $f(x_n)$ to approach zero as n increases. Another stopping criterion is therefore:

$$|f(x_n)| \leq \text{tolerance}$$