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The H-Function

Theory and Applications

Springer

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Hans J. Haubold is the chief scientist at the outer space division of the United Nations, situated at Vienna, Austria. His research contribution is mainly in the area of theoretical physics. He has published over 300 papers, over 30 of them are jointly with A.M. Mathai on stellar and solar models, energy generation, neutrino problem, gravitational instability problem, etc. He has authored a number of papers jointly with A.M. Mathai and R.K. Saxena on applications of fractional calculus to reaction–diffusion problems. In the beginning of 2008 he has published the book *Special Functions for Applied Scientists*, jointly with A.M. Mathai (Springer, New York). His research collaboration with A.M. Mathai goes back to 1984.

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Preface

The H-function or popularly known in the literature as Fox's H-function has recently found applications in a large variety of problems connected with reaction, diffusion, reaction–diffusion, engineering and communication, fractional differential and integral equations, many areas of theoretical physics, statistical distribution theory, etc. One of the standard books and most cited book on the topic is the 1978 book of Mathai and Saxena. Since then, the subject has grown a lot, mainly in the fields of applications. Due to popular demand, the authors were requested to upgrade and bring out a revised edition of the 1978 book. It was decided to bring out a new book, mostly dealing with recent applications in statistical distributions, pathway models, nonextensive statistical mechanics, astrophysics problems, fractional calculus, etc. and to make use of the expertise of Hans J. Haubold in astrophysics area also.

It was decided to confine the discussion to H-function of one scalar variable only. Matrix variable cases and many variable cases are not discussed in detail, but an insight into these areas is given. When going from one variable to many variables, there is nothing called a unique bivariate or multivariate analogue of a given function. Whatever be the criteria used, there may be many different functions qualified to be bivariate or multivariate analogues of a given univariate function. Some of the bivariate and multivariate H-functions, currently in the literature, are also questioned by many authors. Hence, it was decided to concentrate on one variable case and to put some multivariable situations in an appendix; only the definitions and immediate properties are given here.

Chapter 1 gives the definitions, various contours, existence conditions, and some particular cases. Chapter 2 deals with various types of transforms such as Laplace, Fourier, Hankel, etc. on H-functions, their properties, and some relationships among them. Chapter 3 goes into fractional calculus and their connections to H-functions. All the popular fractional differential and fractional integral operators are examined in this chapter.

Chapter 4 is on the applications of H-function in various areas of statistical distribution theory, various structures of random variables, generalized distributions, Mathai's pathway models, a versatile integral which is connected to different fields, etc. Chapter 5 gives a glimpse into functions of matrix argument, mainly real-valued scalar functions of matrix argument when the matrices are real or Hermitian positive definite. H-function of matrix argument is defined only in the form of a class of functions satisfying a certain integral equation and hence a detailed discussion is not attempted here.

Chapter 6 examines applications of H-function into various problems in physics. The problems examined are the following: solar and stellar models, gravitational instability problem, energy generation, solar neutrino problem, generalized entropies, Tsallis statistics, superstatistics, Mathai's pathway analysis, input–output models, kinetic equations, reaction, diffusion, and reaction–diffusion problems where H-functions prop up in the analytic solutions to these problems.

The book is intended as a reference source for teachers and researchers, and it can also be used as a textbook in a one-semester graduate (post-graduate) course on H-function. In this context, a more or less exhaustive and up-to-date bibliography on H-function is included in the book.

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Contents

Chapter 1
On the H-Function With Applications

1.1 A Brief Historical Background

Mellin–Barnes integrals are discovered by Salvatore Pincherle, an Italian mathematician in the year 1888. These integrals are based on the duality principle between linear differential equations and linear difference equations with rational coefficients. The theory of these integrals has been developed by Mellin (1910) and has been used in the development of the theory of hypergeometric functions by Barnes (1908). Important contributions of Salvatore Pincherle are recently given in a paper by Mainardi and Pagnini (2003). In the year 1946, these integrals were used by Meijer to introduce the G-function into mathematical analysis. From 1956 to 1970 lot of work has been done on this function, which can be seen from the bibliography of the book by Mathai and Saxena (1973a).

In the year 1961, in an attempt to discover a most generalized symmetrical Fourier kernel, Charles Fox (1961) defined a new function involving Mellin–Barnes integrals, which is a generalization of the G-function of Meijer. This function is called Fox's H-function or the H-function. The importance of this function is realized by the scientists, engineers and statisticians due to its vast potential of its applications in diversified fields of science and engineering. This function includes, among others, the functions considered by Boersma (1962), Mittag-Leffler (1903), generalized Bessel function due to Wright (1934), the generalization of the hypergeometric functions studied by Fox (1928), and Wright (1935, 1940), Krätzel function (Krätzel 1979), generalized Mittag-Leffler function due to Dzherbashyan (1960), generalized Mittag-Leffler function due to Prabhakar (1971) and multi-index Mittag-Leffler function due to Kiryakova (2000), etc. Except the functions of Boersma (1962), the aforesaid functions cannot be obtained as special cases of the G-function of Meijer (1946), hence a study of the H-function will cover wider range than the G-function and gives general, deeper, and useful results directly applicable in various problems of physical, biological, engineering and earth sciences, such as fluid flow, rheology, diffusion in porous media, kinematics in viscoelastic media, relaxation and diffusion processes in complex systems, propagation of seismic waves, anomalous diffusion and turbulence, etc. see, Caputo (1969), Glöckle

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and Nonnenmacher (1993), Mainardi et al. (2001), Saichev and Zaslavsky (1997), Hilfer (2000), Metzler and Klafter (2000), Podlubny (1999), Schneider (1986) and Schneider and Wyss (1989) and others.

1.2 The H-Function

Notation 1.1.

$$H(x) = H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\text{: H-function.} \tag{1.1}$$

Definition 1.1. The H-function is defined by means of a Mellin–Barnes type integral in the following manner (Mathai and Saxena 1978)

$$\begin{aligned} H(x) = H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] &= H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right] \\ &= \frac{1}{2\pi i}\int_L \Theta(s) z^{-s} \mathrm{d}s, \end{aligned} \tag{1.2}$$

where $i = (-1)^{\frac{1}{2}}, z \neq 0$, and $z^{-s} = \exp[-s\{\ln|z| + i \arg z\}]$, where $\ln|z|$ represents the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value. Here

$$\Theta(s) = \frac{\{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\}\{\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)\}}{\{\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\}\{\prod_{j=n+1}^{p}\Gamma(a_j + A_j s)\}}. \tag{1.3}$$

An empty product is always interpreted as unity; $m, n, p, q \in N_0$ with $0 \le n \le p$, $1 \le m \le q, A_i, B_j \in R_+, a_i, b_j \in R$, or C, $i = 1, \ldots, p; j = 1, \ldots, q$. L is a suitable contour separating the poles

$$\zeta_{jv} = -\left(\frac{b_j + v}{B_j}\right), \; j = 1, \ldots, m; \quad v = 0, 1, 2, \ldots \tag{1.4}$$

of the gamma functions $\Gamma(b_j + sB_j)$ from the poles

$$\omega_{\lambda k} = \left(\frac{1 - a_\lambda + k}{A_\lambda}\right), \; \lambda = 1, \ldots, n; \quad k = 0, 1, 2, \ldots \tag{1.5}$$

of the gamma functions $\Gamma(1 - a_\lambda - sA_\lambda)$, that is

$$A_\lambda(b_j + v) \neq B_j(a_\lambda - k - 1), \; j = 1, \cdots, m; \lambda = 1, \ldots, n; \;\; v, k = 0, 1, 2, \ldots \tag{1.6}$$

The contour L exists on account of (1.6). These assumptions will be retained throughout. The contour L is either $L_{-\infty}, L_{+\infty}$ or $L_{i\gamma\infty}$. The following are the definitions of these contours.

(i) $L = L_{-\infty}$ is a loop beginning and ending at $-\infty$ and encircling all the poles of $\Gamma(b_j + B_j s), j = 1, \ldots, m$ once in the positive direction but none of the poles of $\Gamma(1 - a_\lambda - A_\lambda s), \lambda = 1, \ldots, n$. The integral converges for all z if $\mu > 0$ and $z \neq 0$; or $\mu = 0$ and $0 < |z| < \beta$. The integral also converges if

$$\mu = 0, |z| = \beta \quad \text{and} \quad \Re(\delta) < -1, \tag{1.7}$$

where

$$\beta = \left\{ \prod_{j=1}^{p} (A_j)^{-A_j} \right\} \left\{ \prod_{j=1}^{q} (B_j)^{B_j} \right\}, \tag{1.8}$$

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j, \text{ and} \tag{1.9}$$

$$\delta = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}. \tag{1.10}$$

(ii) $L = L_{+\infty}$ is a loop beginning and ending at $+\infty$ and encircling all the poles of $\Gamma(1 - a_\lambda - A_\lambda s), \lambda = 1, \ldots, n$ once in the negative direction but none of the poles of $\Gamma(b_j + B_j s), j = 1, \ldots, m$. The integral converges for all z if

$$\mu < 0 \text{ and } z \neq 0 \text{ or } \mu = 0 \text{ and } |z| > \beta. \tag{1.11}$$

The integral also converges if the conditions given in (1.7) are satisfied.

(iii) $L = L_{i\gamma\infty}$ is a contour starting at the point $\gamma - i\infty$ and going to $\gamma + i\infty$ where $\gamma \in R = (-\infty, +\infty)$ such that all the poles of $\Gamma(b_j + B_j s), j = 1, \ldots, m$ are separated from those of $\Gamma(1 - a_\lambda - A_\lambda s), \lambda = 1, \ldots, n$. The integral converges if

$$\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha, \ a \neq 0. \tag{1.12}$$

The integral also converges if $\alpha = 0$, $\gamma\mu + \Re(\delta) < -1$, $\arg z = 0$ and $z \neq 0$ where

$$\alpha = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \tag{1.13}$$

A detailed and comprehensive account of the H-function is available from the monographs Mathai and Saxena (1978), Prudnikov et al. (1990), Kilbas and Saigo (2004).

Existence conditions for the *H*-function. In many applied problems associated with fractional differential equations and fractional integral equations, the solutions of certain problems are obtained in terms of the H-function. The H-function naturally occurs as solutions of such equations. In order to find the existence conditions of the solution of the problem, we therefore need the existence conditions for the H-function. The existence conditions for the H-function are enumerated below. It is presumed that the condition (1.6) is satisfied throughout this book unless otherwise stated.

Theorem 1.1. *The H-function is an analytic function of z and exists in the following cases:*

$$\textit{Case 1}: q \geq 1, \mu > 0,\ H\textit{-function exists for all } z \neq 0, \tag{1.14}$$
$$\textit{Case 2}: q \geq 1, \mu = 0,\ H\textit{-function exists for } 0 < |z| < \beta, \tag{1.15}$$
$$\textit{Case 3}: q \geq 1, \mu = 0, \Re(\delta) < -1,\ H\textit{-function exists for } |z| = \beta, \tag{1.16}$$
$$\textit{Case 4}: p \geq 1, \mu < 0,\ H\textit{-function exists for all } z, z \neq 0, \tag{1.17}$$
$$\textit{Case 5}: p \geq 1, \mu = 0,\ H\textit{-function exists for } |z| > \beta, \tag{1.18}$$
$$\textit{Case 6}: p \geq 1, \mu = 0 \textit{ and } \Re(\delta) < -1,\ H\textit{-function exists for } |z| = \beta, \tag{1.19}$$
$$\textit{Case 7}: \alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha,\ H\textit{-function exists for all } z \neq 0, \tag{1.20}$$
$$\textit{Case 8}: \alpha = 0, \gamma\mu + \Re(\delta) < -1,\ H\textit{-function exists for } \arg z = 0 \textit{ and } z \neq 0. \tag{1.21}$$

In what follows

$$c^* = m + n - \frac{1}{2}p - \frac{1}{2}q. \tag{1.22}$$

Proof 1.1. The proof of the existence conditions can be obtained by finding the convergence of the integral (1.2), which depends on the asymptotic estimate of $\Theta(s)$ at infinity. Such a result can be found by using the following asymptotic relation for the gamma function $\Gamma(z), z = x + iy,\ x, y \in R$ at infinity on lines parallel to the coordinate axes given by Kilbas and Saigo (1999, p. 193):

$$|x + iy| \sim \sqrt{2\pi}|x|^{x-\frac{1}{2}} \exp[-x - x(1 - \mathrm{sign}(x))y/2], \quad |x| \to \infty, \tag{1.23}$$

and

$$|x + iy| \sim \sqrt{2\pi}|y|^{x-\frac{1}{2}} \mathrm{e}^{-x-\pi|y|/2}, \quad |y| \to \infty. \tag{1.24}$$

The proof of the above results (1.23) and (1.24) can be developed by making use of the Stirling formula (Erdélyi et al. 1953, p. 47, 1.18(2))

$$\Gamma(z) \sim \sqrt{2\pi}\ z^{z-\frac{1}{2}} \mathrm{e}^{-z} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - 0(z^{-4})\right], \quad |\arg z| < \pi. \tag{1.25}$$

For details of the proof, see Kilbas and Saigo (1999).

In order to prove Theorem 1.1, we first establish the following two lemmas. These lemmas will then be applied in finding the asymptotic relations along the lines τ_1, τ_2 and τ_γ, defined by

$$\tau_1 = \{t + i\varphi_1 : t \in R\}, \tau_2 = \{t + \varphi_2 : t \in R\}, \tau_\gamma = \{\gamma + it : t \in R\}, \quad (1.26)$$

where $\varphi_1, \varphi_2, \gamma \in R$. □

Lemma 1.1. *For $\sigma, t \in R$, there holds the asymptotic estimate*

$$\Theta(t + i\sigma)| \sim A\left(\frac{\mathrm{e}}{t}\right)^{\mu t} \beta^{-t} t^{\Re(\delta)}, t \to \infty, \quad (1.27)$$

where

$$A = (2\pi)^{c^*} \mathrm{e}^{q-m-n} \frac{\left\{\prod_{j=1}^{q} [B_j^{\Re(b_j)-\frac{1}{2}} \mathrm{e}^{-\Re(b_j)}]\right\} \left\{\prod_{j=1}^{n} \mathrm{e}^{\pi[\sigma A_j + \mathrm{Im}(a_j)]}\right\}}{\left\{\prod_{j=1}^{p} [A_j^{\Re(a_j)-\frac{1}{2}} \mathrm{e}^{-\Re(a_j)}]\right\} \left\{\prod_{j=1}^{n} \mathrm{e}^{\pi[\sigma B_j + \mathrm{Im}(b_j)]}\right\}}, \quad (1.28)$$

and

$$|\Theta(t + i\sigma)| \sim B\left(\frac{\mathrm{e}}{|t|}\right)^{\mu|t|} \beta^{-|t|} |t|^{\Re(\delta)},\ t \to -\infty, \quad (1.29)$$

where

$$B = (2\pi)^{c^*} \mathrm{e}^{q-m-n} \frac{\left\{\prod_{j=1}^{q} [B_j^{\Re(b_j)-\frac{1}{2}} \mathrm{e}^{-\Re(b_j)}]\right\} \left\{\prod_{j=n+1}^{p} \mathrm{e}^{\pi[\sigma A_j + \mathrm{Im}(a_j)]}\right\}}{\left\{\prod_{j=1}^{p} [A_j^{\Re(a_j)-\frac{1}{2}} \mathrm{e}^{-\Re(a_j)}]\right\} \left\{\prod_{j=1}^{m} \mathrm{e}^{\pi[\sigma B_j + \mathrm{Im}(b_j)]}\right\}}, \quad (1.30)$$

and β, μ and δ are defined in (1.8), (1.9), and (1.10) respectively.

Lemma 1.2. *For $\sigma, t \in R$ there holds the asymptotic relation*

$$|\Theta(\sigma + it) \sim C|t|^{\mu\sigma + \Re(\delta)} \exp[-\pi\{|t|\alpha + \mathrm{Im}(\nu) sign(t)\}/2], |t| \to \infty, \quad (1.31)$$

uniformly on σ on any bounded interval in R, where

$$C = (2\pi)^{c^*} \exp\{-c^* - \mu\sigma - \Re(\delta)\} \beta^{\sigma} \left\{\prod_{j=1}^{p} A_j^{\frac{1}{2}-a_j}\right\} \left\{\prod_{j=1}^{q} B_j^{b_j - \frac{1}{2}}\right\}, \quad (1.32)$$

where μ, δ, c^* are defined in (1.9), (1.10) and (1.22) respectively, and

$$\nu = \sum_{j=1}^{n} a_j - \sum_{j=n+1}^{p} a_j + \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j.$$

The Lemma 1.1 and Lemma 1.2 follow from (1.3), (1.19) and (1.20). By virtue of the above Lemmas 1.1 and 1.2. it is not difficult to derive the following asymptotic relations at infinity of the integrand of (1.2):

$$|\Theta(z)z^{-s}| \sim B_i \mathrm{e}^{\phi_j \arg z} \left(\frac{\mathrm{e}}{|t|}\right)^{\mu|t|} \left(\frac{|z|}{\beta}\right)^{|t|} |t|^{\Re(\delta)},\ s = t + i\phi_j \in \tau_j,\ j = 1, 2, \tag{1.33}$$

as $t \to -\infty$;

$$|\Theta(z)z^{-s}| \sim A_j \mathrm{e}^{\phi_j \arg z} \left(\frac{\mathrm{e}}{|t|}\right)^{-\mu|t|} \left(\frac{\beta}{|z|}\right)^{t} |t|^{\Re(\delta)},\ s = t + i\phi_j \in \tau_j,\ j = 1, 2, \tag{1.34}$$

as $t \to +\infty$;

$$|\Theta(z)z^{-s}| \sim C_1 \exp[-\gamma \log|z| + \pi \,\mathrm{Im}(\nu)\, sign(t)/2]|t|^{\gamma\mu + \Re(\delta)}, \tag{1.35}$$

$$\times \exp[-\pi|t|\frac{\alpha}{2} + t \arg z],\ s = \gamma + it \in \tau_\gamma, \tag{1.36}$$

as $|t| \to \infty$. Here A_1 and A_2, B_1 and B_2 are defined in (1.24) and (1.28) with σ replaced by ϕ_1 and ϕ_2 respectively, and C_1 by (1.28) with σ replaced by γ.

The conditions for the existence of the H-function then follow as a consequence of these asymptotic relations.

Remark 1.1. Existence conditions for the H-function are given by Braaksma (1964, p. 240), Mathai (1993c) and Kilbas and Saigo (2004). The conditions described here are based on the results given by Kilbas and Saigo (1998, p. 44), also see Kilbas and Saigo (2004); which provide slight improvement over the conditions given in the theorem initially given by Prudnikov, Brychkov, and Marichev (1990, Sect. 8.3.1, p. 627).

Note 1.1. Due to the presence of the factor z^{-s} in the integrand of (1.2), the H-function is, in general, multivalued but one-valued on the Riemann surface of $\ln z$ (Braaksma 1964).

Note 1.2. The convergence of a general Mellin–Barnes integral is already given in the book by Erdélyi et al. (1953, pp. 49–50). Asymptotic estimates for the function $\Theta(\sigma + it)$ and its derivative $\Theta'(\sigma + it)$ as $|t| \to \infty$ are given by Kilbas et al. (1993).

Remark 1.2. An extension of the definition of the H-function has been given by Skibinski (1970), Inayat-Hussain (1987), and Südland et al. (1998). Definition of some of these extensions will be presented in the Appendix.

1.3 Illustrative Examples

The simplest examples of the H-function involve the exponential function, Mittag-Leffler functions (Erdélyi et al. (1955, Sect. 18.1); Mittag-Leffler (1903)), and generalized Mittag-Leffler function (Prabhakar 1971), which are directly applicable in fractional reaction, fractional relaxation and fractional reaction–diffusion problems of science and engineering. These functions will be introduced with the help of the following examples:

Example 1.1. Evaluate

$$f(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)z^{-s}\mathrm{d}s, \left(|\arg z| < \frac{1}{2}\pi; z \neq 0\right), \tag{1.37}$$

where the path of integration is a straight line $\Re(s) = \gamma, \gamma > 0$, lying on the right of the poles of $\Gamma(s)$ given by $s = -v,\ v = 0, 1, 2, \ldots$ and express it in terms of the H-function.

Solution 1.1. Evaluating the integral as the sum of residues we have

$$\begin{aligned} f(z) &= \sum_{v=o}^{\infty}\lim_{s\to -v}(s+v)\Gamma(s)z^{-s} \\ &= \sum_{v=0}^{\infty}\lim_{s\to -v}\frac{(s+v)(s+v-1)\ldots s}{(s+v-1)\cdots s}\Gamma(s)z^{-s} \\ &= \sum_{v=0}^{\infty}\lim_{s\to -v}\frac{\Gamma(s+v+1)}{(s+v-1)\ldots s}z^{-s} = \sum_{v=0}^{\infty}\frac{(-1)^v}{v!}z^v = \mathrm{e}^{-z}. \end{aligned} \tag{1.38}$$

On comparing the equation (1.37) with the definition of the H-function (1.2), we obtain the relation

$$\mathrm{e}^{-z} = H_{0,1}^{1,0}\left[z\big|_{(0,1)}\right]. \tag{1.39}$$

Note 1.3. Equation (1.37) gives the Mellin–Barnes integral for the exponential function e^{-z}. This integral is called Cahen–Mellin integral and is very useful in evaluating integrals involving product of two exponential functions or one exponential function and one special function in a compact form. This integral is also useful in the study of statistical distributions.

Example 1.2. Prove that

$$(1-z)^{-a} = \frac{1}{2\pi i\ \Gamma(a)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(-s)\Gamma(s+a)(-z)^s\mathrm{d}s,\ |\arg(-z)| < \pi, \tag{1.40}$$

where $0 < \Re(\gamma) < \Re(a)$ and the contour is a straight line $\Re(s) = \gamma$, separating the poles of $\Gamma(-s)$ at the points $-s = -v,\ v = 0, 1, \ldots$ from those of $\Gamma(s+a)$ at the points $s = -a - v,\ v = 0, 1, \ldots$.

Solution 1.2. As in the preceding example, evaluating the integral as the sum of residues we have

$$\frac{1}{2\pi i\ \Gamma(a)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(-s)\Gamma(s+a)(-z)^s\mathrm{d}s$$
$$=\frac{1}{\Gamma(a)}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu\Gamma(a+\nu)(-z)^\nu}{\nu!}=\sum_{\nu=0}^{\infty}\frac{(a)_\nu}{\nu!}z^\nu$$
$$={}_1F_0(a;\ ;z)=(1-z)^{-a},|z|<1, \tag{1.41}$$

where $(a)_k,\ a\in C, k\in N_0$, is the Pochhammer symbol or shifted factorial, defined by

$$(a)_0=1,\ (a)_k=a(a+1)\dots(a+k-1), a\neq 0$$
$$=\frac{\Gamma(a+k)}{\Gamma(a)}, \tag{1.42}$$

when $\Gamma(a)$ is defined.

The result (1.42) can be expressed in terms of the H-function as

$$(1-z)^{-a}=\frac{1}{\Gamma(a)}H_{1,1}^{1,1}\left[-z\Big|_{(0,1)}^{(1-a,1)}\right]. \tag{1.43}$$

Notation 1.2. $E_\alpha(z)$: Mittag-Leffler function (Mittag-Leffler 1903).

Definition 1.2.

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\alpha\in C,\Re(\alpha)>0,z\in C. \tag{1.44}$$

Notation 1.3. $E_{\alpha,\beta}(z)$: Generalized Mittag-Leffler function (Erdélyi et al. (1955), Sect. 18.1, Wiman (1905)).

Definition 1.3.

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\alpha,\beta\in C,\Re(\alpha)>0,\Re(\beta)>0,z\in C. \tag{1.45}$$

Note 1.4. Both the functions defined by (1.44) and (1.45) are entire functions of order.

$$\rho=\frac{1}{\alpha}\quad\text{and}\quad\text{type }\sigma=1.$$

Notation 1.4. $E^{\gamma}_{\alpha,\beta}(z)$: Generalized Mittag-Leffler function.

Definition 1.4.

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta) k!}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, z \in C. \quad (1.46)$$

This function is also an entire function with $\rho = \frac{1}{\Re(\alpha)}$, see Prabhakar (1971).

Example 1.3. Evaluate the Mellin–Barnes integral

$$f(z) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)} (-z)^{-s} \mathrm{d}s, |\arg z| < \pi, \quad (1.47)$$

where $\alpha \in R^+$ and show that $f(z)$ is the Mittag-Leffler function $E_\alpha(z)$ defined by the series (1.44).

Solution 1.3. We have

$$\begin{aligned} f(z) &= \sum_{v=0}^{\infty} \lim_{s \to -v} \frac{(s+v)\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)} (-z)^{-s} = \sum_{v=0}^{\infty} \frac{z^v}{\Gamma(\alpha v + 1)} \quad (1.48) \\ &= E_\alpha(z) = H^{1,1}_{1,2}\left[-z\Big|^{(0,1)}_{(0,1),(0,\alpha)}\right], \end{aligned}$$

on comparing the results (1.2) and (1.48).

Example 1.4. Establish the Mellin–Barnes integral

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} \mathrm{d}s, \ |\arg z| < \pi, \quad (1.49)$$

where $\alpha \in R^+, \beta \in C, \Re(\beta) > 0$ and $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function defined by the series (1.45).

Solution 1.4. Evaluating the contour integral as a sum of residues, we find that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma - \infty}^{\gamma + i\infty} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} \mathrm{d}s &= \sum_{v=0}^{\infty} \lim_{s \to -v} \frac{(s+v)\Gamma(s)\Gamma(1-s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} \\ &= \sum_{v=0}^{\infty} \frac{z^v}{\Gamma(\alpha v + \beta)} = E_{\alpha,\beta}(z) \\ &= H^{1,1}_{1,2}\left[-z\Big|^{(0,1)}_{(0,1),(1-\beta,\alpha)}\right], \quad (1.50) \end{aligned}$$

where we have used the definition of the generalized Mittag-Leffler function (1.45) and the definition of the H-function (1.2).

In a similar manner, we can prove the next example.

Example 1.5. Prove that the generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ defined by (1.46) is represented as a Mellin–Barnes integral in the form

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{2\pi i\ \Gamma(\gamma)} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\mathrm{d}s,\ |\arg z| < \pi, \tag{1.51}$$

where $\alpha \in R^{+}, \beta, \gamma \in C, \Re(\beta) > 0, \gamma \neq 0, -1, -2, \ldots$

Solution 1.5. Proceed as in Solution 1.4 to establish the result.

Note 1.5. Applications of the generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ in finite-size scaling in anisotropic systems can be found in the papers by Tonchev (2005, 2007) and Chamati and Tonchev (2006). This function is studied by Prabhakar (1971), Kilbas et al. (2002, 2004) and Saxena and Saigo (2005).

Example 1.6. Evaluate the following reaction rate integral of physics in terms of the H-function.

$$I(a,b,c;\rho) = \int_0^{\infty} t^{a-1}\exp(-bt - ct^{-\rho})\mathrm{d}t, \tag{1.52}$$

where $a, b, c > 0$.

Solution 1.6. Expressing the right hand side of the above expression with the help of the convolution property of the Mellin transform and then taking the inverse Mellin transform one has

$$\begin{aligned}\int_0^{\infty} t^{a-1}\exp(-bt-ct^{-\rho})\mathrm{d}t &= \frac{1}{\rho b^{a}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\alpha+s)\Gamma\left(\frac{s}{\rho}\right)(bc^{\frac{1}{\rho}})^{-s}\mathrm{d}s \\ &= \frac{1}{\rho b^{a}} H^{2,0}_{0,2}\left[bc^{\frac{1}{\rho}}\Big|_{(0,1),(0,\frac{1}{\rho})}\right].\end{aligned} \tag{1.53}$$

Remark 1.3. The integral of this example defines the Krätzel function (Krätzel 1979). For a detailed account of this function, the reader may consult the book by Kilbas and Saigo (2004). Further, this integral is useful in the study of nuclear reaction rates in astrophysics, see Anderson et al. (1994), Haubold and Mathai (1986), Mathai and Haubold (1988) and Saxena et al. (2004), etc.

Following a similar procedure, it is not difficult to prove the next example.

Example 1.7. Prove that the Mellin–Barnes integral

$$J_{\nu}(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)}{\Gamma(1+\nu-s)}\left(\frac{1}{2}z\right)^{\nu-2s}\mathrm{d}s, \gamma > 0, \tag{1.54}$$

defines the Bessel function of the first kind, $J_\nu(z)$, defined by

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+\nu+k)k!}\left(\frac{z}{2}\right)^{\nu+2k}. \tag{1.55}$$

1.4 Some Identities of the H-Function

This section deals with certain basic properties of the H-function. Many authors have investigated various properties of this function, and the researches carried out by Braaksma (1964), Gupta (1965), Gupta and Jain (1966, 1968, 1969), Bajpai (1969a), Lawrynowicz (1969), Anandani (1969a, 1969b), Kilbas and Saigo (2004), Chaurasia (1976b) and Skibinski (1970) will be discussed here.

The results of this section follow as a consequence of the definition of the H-function (1.2) by the application of certain properties of gamma functions, hence their proofs are omitted.

Property 1.1. *The H-function is symmetric in the pairs $(a_1, A_1), \ldots, (a_n, A_n)$, likewise $(a_{n+1}, A_{n+1}), \ldots, (a_p, A_p)$; in $(b_1, B_1), \ldots, (b_m, B_m)$ and in $(b_{m+1}, B_{m+1}), \ldots, (b_q, B_q)$.*

Property 1.2. *If one of the (a_j, A_j), $j = 1, \ldots, n$ is equal to one of the (b_j, B_j), $j = m+1, \ldots, q$ or one of the (b_j, B_j), $j = 1, \ldots, m$ is equal to one of the (a_j, A_j), $j = n+1, \ldots, p$ then the H-function reduces to one of the lower order p and q, and n (or m) decrease by unity.*

Thus we have the following reduction formulae:

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(a_1,A_1)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right] = H_{p-1,q-1}^{m,n-1}\left[z\Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1})}^{(a_2,A_2),\ldots,(a_p,A_p)}\right], \tag{1.56}$$

provided $n \geq 1$ and $q > m$; and

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(b_1,B_1)}\right] = H_{p-1,q-1}^{m-1,n}\left[z\Big|_{(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1})}\right], \tag{1.57}$$

provided $m \geq 1$ and $p > n$.

Property 1.3. *There holds the formula:*

$$H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{q,p}^{n,m}\left[\frac{1}{z}\Big|_{(1-a_p,A_p)}^{(1-b_q,B_q)}\right]. \tag{1.58}$$

This is an important property of the H-function because it enables us to transform a H-function with $\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$ and $\arg z$ to one with $\mu < 0$ and $\arg\frac{1}{z}$ and vice versa. It also helps in deducing the asymptotic expansion for the H-function for the case $\mu < 0$ from the given result for this function for $\mu > 0$ and vice versa.

Property 1.4. *The following result holds:*

$$H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = k\ H_{p,q}^{m,n}\left[z^k\Big|_{(b_q,kB_q)}^{(a_p,kA_p)}\right], \tag{1.59}$$

where $k > 0$.

Property 1.5. *There holds the formula*

$$z^{\sigma} H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{(b_q+\sigma B_q,B_q)}^{(a_p+\sigma A_p,A_p)}\right], \tag{1.60}$$

where $\sigma \in C$.

Property 1.6. *The following relation holds:*

$$H_{p+1,q+1}^{m,n+1}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q),(r,\gamma)}^{(0,\gamma),(a_1,A_1),\ldots,(a_p,A_p)}\right] = (-1)^r H_{p+1,q+1}^{m+1,n}\left[z\Big|_{(r,\gamma),(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p),(0,\gamma)}\right], \tag{1.61}$$

where $p \le q, \gamma > 0$.

Property 1.7. *The following relation holds:*

$$H_{p+1,q+1}^{m+1,n}\left[z\Big|_{(1,\gamma),(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p),(1-r,\gamma)}\right]=(-1)^r H_{p+1,q+1}^{m,n+1}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q),(1,\gamma)}^{(1-r,\gamma),(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.62}$$

where $p \le q, \gamma > 0$.

Note 1.6. In the above results (1.58) to (1.62), the branches of the H-function are suitably chosen.

Property 1.8. *The multiplication formula for the H-function is given by:*

$$H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = (2\pi)^{(1-t)c^*} t^{\delta+1} H_{tp,tq}^{tm,tn}\left[(zt^{-\mu})^t\Big|_{(\Delta(t,b_q),B_q)}^{(\Delta(t,a_p),A_p)}\right], \tag{1.63}$$

where t is a positive integer, μ, δ and c^* are defined in (1.9), (1.10), and (1.22) respectively, and $(\Delta(t,\delta_r),\gamma_r)$ represents the sequence of parameters

$$\left(\frac{\delta_r}{t},\gamma_r\right), \left(\frac{\delta_r+1}{t},\gamma_r\right), \ldots, \left(\frac{\delta_r+t-1}{t},\gamma_r\right). \tag{1.64}$$

For similar results see Gupta and Jain (1969). The following properties of the H-function follow from the definition itself.

Property 1.9. *For $a, b, c \in C$, there holds the formulae:*

$$H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a,0),(a_2,A_2),\dots,(a_p,A_p)}\right] = \Gamma(1-a)H_{p-1,q}^{m,n-1}\left[z\Big|_{(b_q,B_q)}^{(a_2,A_2),\dots,(a_p,A_p)}\right], \quad (1.65)$$

where $\Re(a) < 1$ and $n \geq 1$;

$$H_{p,q}^{m,n}\left[z\Big|_{(b,0),(b_2,B_2),\dots,(b_q,B_q)}^{(a_1,A_1),\dots,(a_p,A_p)}\right] = \Gamma(b)H_{p,q-1}^{m-1,n}\left[z\Big|_{(b_2,B_2),\dots,(b_q,B_q)}^{(a_1,A_1),\dots,(a_p,A_p)}\right], \quad (1.66)$$

where $\Re(b) > 0$ and $m \geq 1$;

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\dots,(b_q,B_q)}^{(a_1,A_1),\dots,(a_{p-1},A_{p-1}),(a,0)}\right] = \frac{1}{\Gamma(a)}H_{p-1,q}^{m,n}\left[z\Big|_{(b_1,B_1),\dots,(b_q,B_q)}^{(a_1,A_1),\dots,(a_{p-1},A_{p-1})}\right], \quad (1.67)$$

where $\Re(a) > 0$ and $p > n$.

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\dots,(b_{q-1},B_{q-1}),(b,0)}^{(a_1,A_1),\dots,(a_p,A_p)}\right] = \frac{1}{\Gamma(1-b)}H_{p,q-1}^{m,n}\left[z\Big|_{(b_1,B_1),\dots,(b_{q-1},B_{q-1})}^{(a_1,A_1),\dots,(a_p,A_p)}\right], \quad (1.68)$$

where $\Re(b) < 1$ and $q > m$.

1.4.1 Derivatives of the H-Function

The following formulas immediately follow from the definition of the H-function and are useful in the study of fractional integrals and derivatives of the H-function.

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left\{z^{\rho-1}H_{p,q}^{m,n}\left[az^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\} &= z^{\rho-n-1}H_{p+1,q+1}^{m,n+1}\left[az\Big|_{(b_q,B_q),(1-\rho+n,\sigma)}^{(1-\rho,\sigma),(a_p,A_p)}\right]\\ &= (-1)^n z^{\rho-n-1}H_{p+1,q+1}^{m+1,n}\left[az^{\sigma}\Big|_{(1-\rho+n,\sigma),(b_q,B_q)}^{(a_p,A_p),(1-\rho,\sigma)}\right],\end{aligned} \quad (1.69)$$

where $a, \sigma \in C, \sigma > 0$.

Lawrynowich (1969) has given the following four formulae for the successive derivatives of the H-function:

$$\begin{aligned}&\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{z^{-(\gamma\frac{b_1}{B_1})}H_{p,q}^{m,n}\left[z^{\gamma}\Big|_{(b_1,B_1),\dots,(b_q,B_q)}^{(a_1,A_1),\dots,(a_p,A_p)}\right]\right\}\\ &\quad= \left(-\frac{\gamma}{B_1}\right)^r z^{-(r+\gamma\frac{b_1}{B_1})}H_{p,q}^{m,n}\left[z^{\gamma}\Big|_{(r+b_1,B_1),\dots,(b_q,B_q)}^{(a_1,A_1),\dots,(a_p,A_p)}\right],\end{aligned} \quad (1.70)$$

where $m \geq 1, \gamma = B_1$ for $r > 1$;

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{z^{-(\gamma\frac{b_q}{B_q})}H_{p,q}^{m,n}\left[z^{\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$=\left(\frac{\gamma}{B_q}\right)^r z^{-(r+\gamma\frac{b_q}{B_q})}H_{p,q}^{m,n}\left[z^{\gamma}\Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(r+b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \quad (1.71)$$

where $m < q, \gamma = B_q$ for $r > 1$;

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{z^{-(\gamma\frac{(1-a_1)}{A_1})}H_{p,q}^{m,n}\left[z^{-\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$=\left(-\frac{\gamma}{A_1}\right)^r z^{-(r+\gamma\frac{(1-a_1)}{A_1})}H_{p,q}^{m,n}\left[z^{-\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1-r,A_1),\ldots,(a_p,A_p)}\right], \quad (1.72)$$

where $n \geq 1, \gamma = A_1$ for $r > 1$;

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{z^{-(\gamma\frac{(1-a_p)}{A_p})}H_{p,q}^{m,n}\left[z^{-\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$=\left(\frac{\gamma}{A_p}\right)^r z^{-(r+\gamma\frac{(1-a_p)}{A_p})}H_{p,q}^{m,n}\left[z^{-\gamma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(a_p-r,A_p)}\right], \quad (1.73)$$

where $p > n, \gamma = A_p$ for $r > 1$.

The results (1.70) to (1.73) for $r = 1$ are immediate consequences of the differential formulae given by Anandani (1969a).

Remark 1.4. The results of Lawrynowicz cited above are in a compact form and are convenient for practical application.

Next we give three-term differentiation formulae for the H-function.

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$=\frac{\eta(a_1-1)}{A_1}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$+\frac{\eta}{A_1}H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1-1,A_1),(a_2,A_2),\ldots,(a_p,A_p)}\right], \quad (1.74)$$

where $n \geq 1$;

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$=\frac{\eta(a_p-1)}{A_p}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$-\frac{\eta}{A_p}H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(a_p-1,A_p)}\right], \quad (1.75)$$

where $n \leq p-1$;

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$=\frac{\eta b_1}{B_1}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$-\frac{\eta}{B_1}H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(1+b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.76}$$

where $m \geq 1$;

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$=\frac{\eta b_q}{B_q}\left\{H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$+\frac{\eta}{B_q}H_{p,q}^{m,n}\left[z^{\eta}\Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(b_q+1,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.77}$$

where $m \leq q-1$.

The above results can be proved with the help of the following formulae:

$$-A_1 s\Gamma(1-a_1-A_1 s) = (a_1-1)\Gamma(1-a_1-A_1 s) + \Gamma(2-a_1-A_1 s), \tag{1.78}$$

$$-\frac{A_p s}{\Gamma(a_p+A_p s)} = \frac{a_p-1}{\Gamma(a_p+A_p s)} - \frac{1}{\Gamma(a_p-1+A_p s)}, \tag{1.79}$$

$$-B_1 s\Gamma(b_1+B_1 s) = b_1\Gamma(b_1+B_1 s) - \Gamma(1+b_1+B_1 s), \tag{1.80}$$

and

$$-\frac{B_q s}{\Gamma(1-b_q-B_q s)} = \frac{b_q}{\Gamma(1-b_q-B_q s)} + \frac{1}{\Gamma(-b_q-B_q s)}, \tag{1.81}$$

which readily follow from the property of the gamma function

$$\Gamma(z+1) = z\Gamma(z). \tag{1.82}$$

Nair (1972, 1973) has given four formulae for the derivative of the H-function. His results are the extensions of the formulae proved earlier by Gupta and Jain (1968). One of the formulae proved by Nair (1972) is the following:

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}-c_1\right)\cdots\left(x\frac{\mathrm{d}}{\mathrm{d}x}-c_r\right)\left\{x^s H_{p,q}^{m,n}\left[zx^h\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= x^s H_{p+r,q+r}^{m,n+r}\left[zx^h\Big|_{(b_1,B_1),\ldots,(b_q,B_q),(c_1-s+1,h),\ldots,(c_r-s+1,h)}^{(c_1-s,h),\ldots,(c_r-s,h),(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.83}$$

where $h > 0$.

When $c_1 = c_2 = \cdots = c_r = 0$, (1.83) reduces to a result due to Gupta and Jain (1968, p. 191). Oliver and Kalla (1971) have derived four differentiation formulae for the H-function which extend the results of Anandani (1970c), which itself are the generalization of the results due to Goyal and Goyal (1967a). One of the results proved by Oliver and Kalla is the following:

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left\{H_{p,q}^{m,n}\left[(cx+d)^h\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]\right\}$$
$$= \frac{c^r}{(cx+d)^r} H_{p+1,q+1}^{m,n+1}\left[(cx+d)^h\Big|_{(b_1,B_1),\ldots,(b_q,B_q),(r,h)}^{(0,h),(a_1,A_1),\ldots,(a_p,A_p)}\right], \quad (1.84)$$

where c and d are complex numbers and h is real and positive.

Note 1.7. We note that partial derivatives of the H-function with respect to the parameters are investigated by Buschman (1974b).

1.5 Recurrence Relations for the H-Function

Gupta (1965) has obtained four recurrence formulae for the H-function by the method of integral transforms due to Meijer (1940, 1941). One of his results is given below.

$$(a_1 - a_2) H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right] = H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),(a_2-1,A_1),(a_3,A_3),\ldots,(a_p,A_p)}\right]$$
$$- H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1-1,A_1),(a_2,A_2),\ldots,(a_p,A_p)}\right], \quad (1.85)$$

where $n \geq 2$.

Anandani (1989) has given six recurrence relations for the H-function which follow as a consequence of the definition of the H-function (1.2). Two such results are enumerated below:

$$(b_1 A_1 - a_1 B_1 + B_1) H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]$$
$$= B_1 H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1-1,A_1),(a_2,A_2),\ldots,(a_p,A_p)}\right]$$
$$+ A_1 H_{p,q}^{m,n}\left[z\Big|_{(1+b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \quad (1.86)$$

where $m, n \geq 1$;

$$(b_q A_q - a_q B_q + B_q) H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]$$
$$= B_q H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_q-1,A_q),(a_2,A_2),\ldots,(a_p,A_p)}\right]$$
$$- A_q H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(b_q+1,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.87}$$

where $n \geq 1, 1 \leq m \leq q-1$.

For further results on recurrence relations of the H-function, see the work of Bora and Kalla (1971a), Jain (1967), Srivastava and Gupta (1970, 1971), Raina (1976), and Raina and Koul (1977). A set of contiguous relations for the H-function are given by Buschman (1974b).

1.6 Expansion Formulae for the H-Function

Expansion formulae for the H-function are given by Lawrynowich (1969), Raina (1979), and Kilbas and Saigo (2004). The four expansion formulae for the G-function due to Meijer (1941a) have been extended to H-functions by Lawrynowicz (1969) by using a method analogous to the one adopted by Meijer (1941a) for the G-function. The results are the following:

(i) Let m, n, p, and q be nonnegative integers such that $1 \leq m \leq q,\ 0 \leq n \leq p$. Further, let $A_j,\ j = 1, \ldots, p$ and $B_j,\ j = 1, \ldots, q$ be positive numbers and $a_j,\ j = 1, \ldots, p$ and $b_j,\ j = 1, \ldots, q$ be complex numbers satisfying the condition (1.6) and $\mu > 0$, where μ is defined in (1.9). Then if ω and η are complex numbers such that $\omega \neq 0$ and $\eta \neq 0$, then the following results hold:

$$H_{p,q}^{m,n}\left[\eta\omega\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]$$
$$= \eta^{\frac{b_1}{B_1}} \sum_{r=0}^{\infty} \frac{(1-\eta^{\frac{1}{B_1}})^r}{r!} H_{p,q}^{m,n}\left[\omega\Big|_{(r+b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.88}$$

where η is arbitrary for $m = 1$, and for $m > 1$, $|\eta^{\frac{1}{B_1}} - 1| < 1$, $\arg(\eta\omega) = B_1 \arg(\eta^{\frac{1}{B_1}}) + \arg\omega$ and $|\arg(\eta^{\frac{1}{B_1}})| < \frac{\pi}{2}$;

$$H_{p,q}^{m,n}\left[\eta\omega\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]$$
$$= \eta^{\left(\frac{b_q}{B_q}\right)} \sum_{r=0}^{\infty} \frac{(\eta^{\frac{1}{B_q}}-1)^r}{r!} H_{p,q}^{m,n}\left[\omega\Big|_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(r+b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right], \tag{1.89}$$

where $q > m$, $|\eta^{\frac{1}{B_q}} - 1| < 1$ $\arg(\eta\omega) = B_q \arg(\eta^{\frac{1}{B_q}}) + \arg\omega$, and $|\arg(\eta^{\frac{1}{B_q}}| < \frac{\pi}{2}$;

$$H^{m,n}_{p,q}\left[z\Big|^{(a_1,A_1),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right] = \eta^{\frac{(a_1-1)}{A_1}} \sum_{r=0}^{\infty} \frac{(1-\eta^{-\frac{1}{A_1}})^r}{r!} H^{m,n}_{p,q}\left[\omega\Big|^{(a_1-r,A_1),(a_2,A_2),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots(b_q,B_q)}\right], \tag{1.90}$$

where $n > 0$, $\Re(\eta^{\frac{1}{A_1}}) > \frac{1}{2}$, $\arg(\eta\omega) = A_1 \arg(\eta^{\frac{1}{A_1}}) + \arg\omega$ and $|\arg(\eta^{\frac{1}{A_1}})| < \frac{\pi}{2}$;

$$H^{m,n}_{p,q}\left[\eta\omega\Big|^{(a_1,A_1),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right] = \eta^{\frac{(a_p-1)}{A_p}} \sum_{r=0}^{\infty} \frac{(\eta^{-\frac{1}{A_p}}-1)^r}{r!} H^{m,n}_{p,q}\left[\omega\Big|^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(a_p-r,A_p)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right], \tag{1.91}$$

where $p > n$, $\Re(\eta^{\frac{1}{A_p}}) > \frac{1}{2}$, $\arg(\eta\omega) = A_p \arg(\eta^{\frac{1}{A_p}}) + \arg\omega$ and $|\arg(\eta^{\frac{1}{A_p}})| < \frac{\pi}{2}$.
By virtue of the following transformation formula for the Gauss hypergeometric function (Erdélyi et al. 1953, 2.10(1))

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a,b;a+b-c+1;1-z) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a,c-b;c-a-b;1-z), \end{aligned} \tag{1.92}$$

for $|\arg(1-z)| < \pi$ we find that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n H^{m+1,n}_{p+1,q+1}\left[z\Big|^{(a_p,A_p),(c+n,\gamma)}_{(b+n,\gamma),(b_q,B_q)}\right] \\ &= \frac{\Gamma(c-a-b)}{\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b-c+1)_n} \frac{(1-z)^n}{n!} H^{m+1,n}_{p+1,q+1}\left[z\Big|^{(a_p,A_p),(c-a,\gamma)}_{(c+n,\gamma),(b_q,B_q)}\right] \\ &\quad + \frac{\Gamma(a+b-c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(c-b)_n}{(c-a-b+1)_n} \frac{(1-z)^{c-a-b+n}}{n!} \\ &\quad \times H^{m+1,n}_{p+1,q+1}\left[z\Big|^{(a_p,A_p),(b,\gamma)}_{(c+n,\gamma),(b_q,B_q)}\right], \end{aligned} \tag{1.93}$$

where $a, b, c \in C$, $\gamma > 0$, $|\arg(1-z)| < \pi$, $\Re(c-a-b) > 0$ if $z = 1$.

1.7 Asymptotic Expansions

The behavior of the H-function for small and large values of the argument has been discussed by Braaksma (1964) in detail. Explicit power and power-logarithmic series expansions for the H-fucntion are given by Kilbas and Saigo (1999, 2004). In this section we present some of their results which are useful in applied problems. Asymptotic expansions of the H-function are discussed by Dixon and Ferrar (1936). Convergence of the Mellin–Barnes integrals are recently discussed by Paris and Kaminski (2001, p. 63).

Theorem 1.2. *Let α and μ be as given in (1.13) and (1.9) and let the condition (1.6) be satisfied. Then there holds the following results:*

(i) If $\mu \geq 0$ or $\mu < 0, \alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ then the H-function has either the asymptotic expansion at zero given by

$$H_{p,q}^{m,n}(z) = O(z^c),\ |z| \to 0,\ or \tag{1.94}$$

$$H_{p,q}^{m,n}(z) = O(z^c |\ln(z)|^{N-1}),\ |z| \to 0. \tag{1.95}$$

Here,

$$c = \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{B_j} \right], \tag{1.96}$$

and N is the order of the poles $\zeta_{j\nu}$ in (1.4) to which some other poles of $\Gamma(b_j + B_j s),\ j = 1, \ldots, m$ coincide. Also for $\mu < 0, \alpha = 0$

$$H_{p,q}^{m,n}(z) = O(z^\sigma),\ |z| \to 0, |\arg(z) \leq \epsilon^*, \tag{1.97}$$

$$\sigma = \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right], \tag{1.98}$$

and ϵ^* is a constant such that

$$0 < \epsilon^* < \frac{\pi}{2} \min_{1 \leq j \leq m; m+1 \leq k \leq q} (A_j, B_k). \tag{1.99}$$

(ii) If $\mu \leq 0$ or $\mu > 0, \alpha > 0$ then the H-function has either the asymptotic expansion at infinity given by

$$H_{p,q}^{m,n}(z) = O(z^d),\ |z| \to \infty,\ or \tag{1.100}$$

$$H_{p,q}^{m,n}(z) = O(z^d |\ln(z)|^{M-1}),\ |z| \to \infty, \tag{1.101}$$

$$d = \min_{1 \leq j \leq n} \left[\frac{\Re(a_j) - 1}{A_j} \right], \tag{1.102}$$

and M is the order of the poles $\omega_{\lambda k}$ in (1.5) to which some of the poles of $\Gamma(1-a_j-A_js)$, $j=1,\ldots,n$ coincide. Also for $\mu>0, \alpha=0$

$$H_{p,q}^{m,n}(z) = O(z^{\rho}),\ |z|\to\infty, |\arg(z)|\leq\epsilon, \tag{1.103}$$

$$\rho = \max_{1\leq j\leq n}\left[\frac{\Re(a_j)-1}{A_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right], \tag{1.104}$$

and ϵ is a constant such that

$$0<\epsilon<\frac{\pi}{2}\min_{n+1\leq j\leq p;1\leq k\leq m}(A_j, B_k). \tag{1.105}$$

For $n=0$ the H-function, for real argument x, vanishes exponentially for large x in certain cases. The case $m=0$ is also discussed. Let

$$\tau = \sum_{j=1}^{m} B_j - \sum_{j=n+1}^{p} A_j. \tag{1.106}$$

Theorem 1.3.

(i) Let $n=0, \alpha, \beta, \mu, \delta$ and τ be given by (1.13), (1.8), (1.9), (1.10), and (1.106) respectively. Further, let $\mu>0, \alpha\geq 0, \epsilon$ be a constant such that $0<\epsilon<\frac{\pi\mu}{2}$, and the condition (1.6) and $A_j(1-a_i+k)\neq A_i(1-a_j+\lambda), i\neq j, j=1,\ldots,n; k,\lambda\in N_0$ are satisfied then for real x there holds the following assertion: We have

$$H_{p,q}^{m,0}(x) = O\left(x^{[\Re(\delta)+\frac{1}{2}]/\mu}\right)\exp\left[\cos\left(\frac{\tau\pi}{\mu}\right)\mu\beta^{-\frac{1}{\mu}}x^{\frac{1}{\mu}}\right],\ x\to\infty. \tag{1.107}$$

In particular,

$$H_{p,q}^{q,0}(x) = O\left(x^{[\Re(\delta)+\frac{1}{2}]/\mu}\right)\exp\left[-\mu\beta^{-\frac{1}{\mu}}x^{\frac{1}{\mu}}\right],\ x\to\infty. \tag{1.108}$$

(ii) Let $m=0, \alpha, \beta, \mu,$ and δ be given by (1.13),(1.8),(1.9) and (1.10) respectively. Further, let $\mu<0, \alpha\geq 0; \epsilon^*$ be a constant such that $0<\epsilon^*<\frac{\pi|\mu|}{2}$, and the condition (1.6) and $B_j(b_i+k)\neq B_i(b_j+\lambda), i\neq j; i,j=1,\ldots,m; k,\lambda\in N_0$ are satisfied. Then for real x there holds the following assertion: We have

$$H_{p,q}^{0,n}(x) = O\left(x^{-[\Re(\delta)+\frac{1}{2}]/|\mu|}\right)\exp\left[\cos\left(\frac{\zeta\pi}{|\mu|}\right)|\mu|\beta^{\frac{1}{|\mu|}}x^{-\frac{1}{|\mu|}}\right], x\to 0+, \tag{1.109}$$

$$\zeta = \sum_{j=1}^{n} A_j - \sum_{j=m+1}^{q} B_j.$$

In particular,

$$H_{p,q}^{0,p}(x) = O\left(x^{-[\Re(\delta)+\frac{1}{2}]/|\mu|}\right)\exp\left[-|\mu|\beta^{\frac{1}{|\mu|}}x^{-\frac{1}{|\mu|}}\right],\ x \to 0+. \quad (1.110)$$

Remark 1.5. Power logarithmic expansions in particular cases of the H-function $H_{0,p}^{p,0}$ and $H_{p,p}^{p,0}$ are investigated by Mathai (1973).

1.8 Some Special Cases of the H-Function

Notation 1.5.

$$G(z) = G_{p,q}^{m,n}(z) = G_{p,q}^{m,n}\left(z\Big|_{b_q}^{a_p}\right) = G_{p,q}^{m,n}\left(z\Big|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right): \text{ Meijer's } G\text{-function}$$

or the G-function. (1.111)

Definition 1.5.

$$G(z) = G_{p,q}^{m,n}(z) = G_{p,q}^{m,n}\left(z\Big|_{b_q}^{a_p}\right) = G_{p,q}^{m,n}\left(z\Big|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right)$$

$$= \frac{1}{2\pi i}\int_L \frac{\left\{\prod_{j=1}^m \Gamma(b_j+s)\right\}\left\{\prod_{j=1}^n \Gamma(1-a_j-s)\right\}}{\left\{\prod_{j=m+1}^q \Gamma(1-b_j-s)\right\}\left\{\prod_{j=n+1}^p \Gamma(a_j+s)\right\}} z^{-s}\mathrm{d}s, \quad (1.112)$$

where $0 \le m \le q, 0 \le n \le q; a_j, j = 1,\ldots,p$ and $b_j, j = 1,\ldots,q$ are complex numbers and are such that

$$a_j - b_h \ne 0, 1, \ldots; j = 1,\ldots,n; h = 1,\ldots,m. \quad (1.113)$$

The parameters are such that the points

$$s = -(b_j + \nu), j = 1,\ldots,m; \nu \in N_0, \quad (1.114)$$

and

$$s = -(a_j - \nu - 1), j = 1,\ldots,n; \nu \in N_0, \quad (1.115)$$

are separated. Here L is the same contour taken for the H-function defined by (1.2).

A detailed and comprehensive account of the theory and applications of the G-function is available from the monographs written by Erdélyi et al. (1953, Sects. 5.3–5.6), Luke (1969), Mathai and Saxena (1973), Mathai (1993c), Prudnikov et al. (1990, Sects. 8.2 and 8.4). The G-function itself is a generalization of a number of known special functions occurring in applied mathematics and mathematical physics. Special cases of the G-function can be found in Erdélyi et al. (1953, Sect. 5.6), Luke (1969, Sects. 6.4, 6.5), Mathai and Saxena (1973a, Chap. II), and Mathai (1993c).

Notation 1.6. ${}_pF_q(z) = {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$: Generalized hypergeometric series.

Definition 1.6.

$$
{}_pF_q(z) = {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{1.116}
$$

where $(a)_k$ is the Pochhammer symbol defined in (1.42); $a_i, b_j \in C, i = 1, \ldots, p$; $j = 1, \ldots, q; b_j \neq -\nu, \nu \in N_0$.

Notation 1.7. $E(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)$: MacRobert's E-function (Erdélyi et al. 1953, p. 203).

Definition 1.7.

$$
\begin{aligned}
E(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) &= G_{q+1,p}^{p,1}\left[z\Big|_{\alpha_1,\ldots,\alpha_p}^{1,\beta_1,\ldots,\beta_q}\right] \\
&= \frac{1}{2\pi i} \int_L \frac{\Gamma(-s) \prod_{j=1}^{p} \Gamma(\alpha_j + s)}{\prod_{j=1}^{q} \Gamma(\beta_j + s)} z^{-s} \mathrm{d}s. \qquad (1.117)
\end{aligned}
$$

Notation 1.8. $J_\nu^\mu(z)$: Bessel–Maitland function or Maitland–Bessel function (Marichev, 1982, Eq. (8.3)).

Definition 1.8.

$$
J_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\nu + n\mu + 1)\, n!}. \tag{1.118}
$$

Notation 1.9. $J_{\nu,\lambda}^\mu(z)$: Generalized Bessel–Maitland function (Marichev 1983, (8.2)).

Definition 1.9.

$$
J_{\nu,\lambda}^\mu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu + n\mu + \lambda + 1)\Gamma(n + \lambda + 1)} \left(\frac{z}{2}\right)^{\nu+2\lambda+2n}. \tag{1.119}
$$

Notation 1.10. $Z_\rho^\nu(z)$: Krätzel function (Krätzel 1979).

Definition 1.10.

$$
Z_\rho^\nu(z) = \int_0^\infty t^{\nu-1} \exp\left[-t^\rho - \frac{z}{t}\right] \mathrm{d}t,\ \nu \in C, \rho > 0, \Re(z) > 0. \tag{1.120}
$$

Notation 1.11. $K_\nu(z)$: Modified Bessel function of the third kind or Macdonald function, see also Sect. 1.8.1.

Definition 1.11.

$$K_\nu(z) = \frac{1}{2}\int_0^\infty \exp[-\frac{z}{2}(t+\frac{1}{t})]t^{-\nu-1}\mathrm{d}t, \Re\nu < \frac{1}{2}, \tag{1.121}$$

$$= \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(\frac{2}{z}\right)^\nu \int_1^\infty \mathrm{e}^{-zt}(t^2-1)^{-\nu-\frac{1}{2}}\mathrm{d}t,\ \Re(z) > 0, \tag{1.122}$$

see Sect. 1.8.1 for more details.

Notation 1.12.

$${}_p\Psi_q(z) = {}_p\Psi_q\left[z\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right]$$: Wright generalized hypergeometric function (Wright (1935)).

Definition 1.12.

$${}_p\Psi_q\left[z\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p \Gamma(a_j+nA_j)}{\prod_{j=1}^q \Gamma(b_j+nB_j)}\frac{z^n}{n!}, \tag{1.123}$$

where $a_i, b_j \in C$ and $A_i, B_j \in R = (-\infty,\infty)$; $A_i, B_j \neq 0, i = 1,\ldots,p, j = 1,\ldots,q$; $\sum_{j=1}^q B_j - \sum_{j=1}^p A_j > -1$.

Notation 1.13. $\phi(a,b;z), {}_0\Psi_1(z)$: Wright function

Definition 1.13.

$$\phi(a,b;z) = {}_0\Psi_1\left[z\Big|_{(b,a)}\right] = \sum_{n=0}^\infty \frac{1}{\Gamma(an+b)}\frac{z^n}{n!},\ b,z\in C; a\in R, a\neq 0. \tag{1.124}$$

The H-function in the generalized form contains a vast number of analytic functions as special cases. These analytic functions appear in various problems arising in theoretical and applied branches of mathematics, statistics, and engineering sciences. We present here a few interesting special cases of the H-function, which may be useful for workers on integral transforms, fractional calculus, special functions, applied statistics, physical and engineering sciences, astrophysics, etc.

$$H_{0,1}^{1,0}\left[z\Big|_{(b,B)}\right] = B^{-1}z^{\frac{b}{B}}\exp\left(-z^{\frac{1}{B}}\right), \tag{1.125}$$

$$H_{1,1}^{1,1}\left[z\Big|^{(1-\nu,1)}_{(0,1)}\right] = \Gamma(\nu)(1+z)^{-\nu} = \Gamma(\nu){}_1F_0(\nu;\ ;-z), |z|<1 \tag{1.126}$$

$$H_{0,2}^{1,0}\left[\frac{z^2}{4}\Big|_{\left(\frac{a+\nu}{2},1\right),\left(\frac{a-\nu}{2},1\right)}\right] = \left(\frac{z}{2}\right)^a J_\nu(z), \tag{1.127}$$

where $J_\nu(z)$ is the ordinary Bessel function of the first kind, see also Sect. 1.8.1.

$$H_{0,2}^{2,0}\left[\frac{z^2}{4}\Big|_{\left(\frac{a+\nu}{2},1\right),\left(\frac{a-\nu}{2},1\right)}\right] = 2\left(\frac{z}{2}\right)^a K_\nu(z), \tag{1.128}$$

where $K_\nu(z)$ is the modified Bessel function of the third kind or Macdonald function, see also Sect. 1.8.1.

$$H_{1,3}^{2,0}\left[\frac{z^2}{4}\Big|_{\left(\frac{a-\nu}{2},1\right),\left(\frac{a+\nu}{2},1\right),\left(\frac{a-\nu-1}{2},1\right)}^{\left(\frac{a-\nu-1}{2},1\right)}\right] = \left(\frac{z}{2}\right)^a Y_\nu(z), \tag{1.129}$$

where $Y_\nu(z)$ is the modified Bessel function of the second kind or the Neumann function, see also Sect. 1.8.1.

$$H_{1,2}^{1,1}\left[z\Big|_{(0,1),(1-c,1)}^{(1-a,1)}\right] = \frac{\Gamma(a)}{\Gamma(c)}\Phi(a;c;-z) = \frac{\Gamma(a)}{\Gamma(c)}{}_1F_1(a;c;-z), \tag{1.130}$$

which are called the Kummer's confluent hypergeometric functions.

$$H_{2,2}^{1,2}\left[z\Big|_{(0,1),(1-c,1)}^{(1-a,1),(1-b,1)}\right] = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;-z), \tag{1.131}$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}{}_2F_1(b,a;c;-z), \tag{1.132}$$

which are called the Gauss' hypergeometric functions. The relation connecting H-function and MacRobert's E-function is given by

$$H_{q+1,p}^{p,1}\left[z\Big|_{(\alpha_1,1),\ldots,(\alpha_p,1)}^{(1,1),(\beta_1,1),\ldots,(\beta_q,1)}\right] = E(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z). \tag{1.133}$$

The relation connecting Whittaker function and the H-function is given by

$$H_{1,2}^{2,0}\left[\frac{z^2}{4}\Big|_{(\rho+m+\frac{1}{2}),(\rho-m+\frac{1}{2},1)}^{(\rho-k+1,1)}\right] = z^\rho e^{-\frac{z}{2}} W_{k,m}(z), \tag{1.134}$$

see also Sect. 1.8.1. We now give the special cases of the H-function which cannot be obtained from the G-function:

$$H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(0,\alpha)}^{(0,1)}\right] = E_\alpha(z), \tag{1.135}$$

where $E_\alpha(z)$ is the Mittag-Leffler function (Mittag-Leffler 1903).

$$H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(1-\beta,\alpha)}^{(0,1)}\right] = E_{\alpha,\beta}(z), \tag{1.136}$$

where $E_{\alpha,\beta}(z)$ is also the Mittag-Leffler function (Mittag-Leffler 1903).

$$\frac{1}{\Gamma(\gamma)}H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(1-\beta,\alpha)}^{(1-\gamma,1)}\right]=E_{\alpha,\beta}^{\gamma}(z),\Re(\gamma)>0, \tag{1.137}$$

where $E_{\alpha,\beta}^{\gamma}(z)$ is the generalized Mittag-Leffler function.

$$H_{0,2}^{1,0}\left[z\Big|_{(0,1),(-\nu,\mu)}\right]=J_{\nu}^{\mu}(z), \tag{1.138}$$

where $J_{\nu}^{\mu}(z)$ is the Bessel–Maitland function or Maitland Bessel function (see Marichev 1983, (8.3)).

$$H_{1,3}^{1,1}\left[\frac{z^2}{4}\Big|_{(\lambda+\frac{\nu}{2},1),(\frac{\nu}{2},1),(\mu(\lambda+\frac{\nu}{2})-\lambda-\nu,\mu)}^{(\lambda+\frac{\nu}{2},1)}\right]=J_{\nu,\lambda}^{\mu}(z), \tag{1.139}$$

where $J_{\nu,\lambda}^{\mu}(z)$ is the generalized Bessel–Maitland function (Marichev 1983, p. 128, (8.2)),

$$H_{p,q+1}^{1,p}\left[-z\Big|_{(0,1),(1-b_1,B_1),\dots,(1-b_q,B_q)}^{(1-a_1,A_1),\dots,(1-a_p,A_p)}\right]={}_p\Psi_q\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]$$

$$=\frac{1}{2\pi i}\int_L\frac{\Gamma(s)\prod_{j=1}^{p}\Gamma(a_j-A_js)}{\prod_{j=1}^{q}\Gamma(b_j-B_js)}(-z)^{-s}\mathrm{d}s, \tag{1.140}$$

where ${}_p\Psi_q(z)$ is the Wright generalized hypergeometric function (Wright 1935).

$$H_{0,2}^{2,0}\left[z\Big|_{(0,1),\left(\frac{\nu}{\rho},\frac{1}{\rho}\right)}\right]=\rho Z_{\rho}^{\nu}(z), z\in C, \rho>0, \nu\in C, \tag{1.141}$$

where $Z_{\rho}^{\nu}(z)$ is the Krätzel function (Krätzel, 1979). The following special cases of the H-function occur in the study of certain statistical distributions.

$$H_{2,2}^{2,0}\left[z\Big|_{(\alpha_1-1,1),(\alpha_2-1,1)}^{(\alpha_1+\beta_1-1,1),(\alpha_2+\beta_2-1,1)}\right]=\frac{z^{\alpha_2-1}(1-z)^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)}$$
$$\times{}_2F_1(\alpha_2+\beta_2-\alpha_1,\beta_1;\beta_1+\beta_2;1-z), |z|<1, \tag{1.142}$$

$$H_{1,1}^{1,0}\left[z\Big|_{(\alpha,1)}^{(\alpha+\frac{1}{2},1)}\right]=\pi^{-\frac{1}{2}}z^{\alpha}(1-z)^{-\frac{1}{2}}, |z|<1, \tag{1.143}$$

$$H_{2,2}^{2,0}\left[z\Big|_{(\alpha,1),(\alpha,1)}^{(\alpha+\frac{1}{3},1),(\alpha+\frac{2}{3},1)}\right]=z^{\alpha}{}_2F_1\left(\frac{2}{3},\frac{1}{3};1;1-z\right),\ \ |1-z|<1. \tag{1.144}$$

1.8.1 Some Commonly Used Special Cases of the H-Function

(i) Psi function

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}(\ln \Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{1.145}$$

$$= \int_0^\infty [t^{-1}\mathrm{e}^{-t} - (1-\mathrm{e}^{-t})^{-1}\mathrm{e}^{-tz}]\mathrm{d}t, \Re(z) > 0, \tag{1.146}$$

$$= -\gamma + (z-1)\sum_{k=0}^{\infty}[(k+1)(z+k)]^{-1}, \gamma \approx 0.5772156649\ldots, \tag{1.147}$$

(ii) Zeta function (Riemann zeta function)

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-\rho}, \Re(\rho) > 1, \tag{1.148}$$

$$\zeta(\rho, a) = \sum_{n=0}^{\infty}(n+a)^{-\rho}, \Re(\rho) > 1, a \neq 0, -1, -2, \ldots, \tag{1.149}$$

(iii) Whittaker functions

$$M_{\mu,\nu}(z) = z^{\nu+\frac{1}{2}}\mathrm{e}^{-z/2}{}_1F_1\left(\frac{1}{2} - \mu + \nu; 2\nu + 1; z\right) \tag{1.150}$$

$$= z^{\nu\frac{1}{2}}\mathrm{e}^{z/2}{}_1F_1\left(\frac{1}{2} + \mu + \nu; 2\nu + 1; -z\right) \tag{1.151}$$

$$= \frac{\Gamma(1+2\nu)}{\Gamma\left(\frac{1}{2}+\nu+\mu\right)\Gamma\left(\frac{1}{2}+\nu-\mu\right)}\mathrm{e}^{-z/2}z^{\nu+\frac{1}{2}}\int_0^1 \mathrm{e}^{-zt}t^{\nu-\mu-\frac{1}{2}}$$
$$\times (1-t)^{\nu+\mu-\frac{1}{2}}\mathrm{d}t, \Re\left(\frac{1}{2}+\nu\pm\mu\right) > 0, |\arg z| < \pi \tag{1.152}$$

$$= \frac{\Gamma(1+2\nu)}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)}\mathrm{e}^{-z/2}z^{\nu+\frac{1}{2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}$$
$$\times \frac{\Gamma(s)\Gamma\left(\frac{1}{2}+\nu-\mu-s\right)}{\Gamma(1+2\nu-s)}(-z)^{-s}\mathrm{d}s, |\arg z| < \pi/2, 2\nu \neq -1, -2, \ldots \tag{1.153}$$

$$W_{\mu,\nu}(z) = \frac{\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} M_{\mu,\nu}(z) + \frac{\Gamma(2\nu)}{\Gamma\left(\frac{1}{2}-\mu+\nu\right)} M_{\mu,-\nu}(z), \quad (1.154)$$

$$\frac{1}{2}-\mu \neq \nu \neq 0,-1,-2,\ldots,2\nu \neq 0,\pm 1,\ldots$$

$$= W_{\mu,-\nu}(z) \quad (1.155)$$

$$= \frac{z^{\mu}\mathrm{e}^{-z/2}}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)} \int_0^{\infty} \mathrm{e}^{-t} t^{\nu-\mu-\frac{1}{2}} (1+\frac{t}{z})^{\nu+\mu-\frac{1}{2}} \mathrm{d}t, \quad (1.156)$$

$$\Re\left(\frac{1}{2}+\nu-\mu\right) > 0, |\arg z| < \pi,$$

$$= \frac{z^{\mu}\mathrm{e}^{-z/2}}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)\Gamma\left(\frac{1}{2}-\nu-\mu\right)} \frac{1}{2\pi i} \int_{c-\imath\infty}^{c+\imath\infty} \Gamma(-s)$$

$$\times \Gamma\left(\frac{1}{2}+\nu-\mu+s\right)\Gamma\left(\frac{1}{2}-\nu-\mu+s\right) z^{-s} \mathrm{d}s \quad (1.157)$$

$$|\arg z| < \frac{3\pi}{2}, -\frac{1}{2}+\mu \pm \nu \neq 0,1,2,\ldots$$

(iv) Parabolic cylinder function

$$D_\nu(z) = 2^{\frac{\nu}{2}+\frac{1}{4}} z^{-\frac{1}{2}} W_{\frac{\nu}{2}+\frac{1}{4},\frac{1}{4}}\left(\frac{z^2}{2}\right) \quad (1.158)$$

$$= (-1)^n \mathrm{e}^{z^2/4} \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\mathrm{e}^{-\frac{z^2}{2}}\right) \quad (1.159)$$

$$= 2^{\frac{\nu}{2}+\frac{1}{4}} \mathrm{e}^{z/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(-\frac{1}{4}+s\right)\Gamma\left(\frac{1}{4}+s\right)}{\Gamma(s-\frac{\nu}{2}+\frac{1}{4})} \left(\frac{z^2}{2}\right)^{-s} \mathrm{d}s, \quad (1.160)$$

$$|\arg z| < \frac{\pi}{4}.$$

(v) Bessel and associated functions

$$J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)} = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(; 1+\nu; -\frac{z^2}{4}\right) \quad (1.161)$$

$$= \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{\nu+s}{2}\right)}{\Gamma\left(1+\frac{\nu-s}{2}\right)} \left(\frac{z}{2}\right)^{-s} \mathrm{d}s, -\Re(\nu) < 1, |\arg z| < \pi \quad (1.162)$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-s)}{\Gamma(\nu+s+1)} \left(\frac{z}{2}\right)^{\nu+2s} \mathrm{d}s, z > 0, \Re(\nu) > -1. \quad (1.163)$$

$$I_\nu(z) = \sum_{r=0}^{\infty} \frac{(z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)} = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(; 1+\nu; \frac{z^2}{4}\right) \quad (1.164)$$

$$= \mathrm{e}^{-i\nu\pi/2} J_\nu(z\mathrm{e}^{i\pi/2}), -\pi < \arg z \leq \pi/2. \quad (1.165)$$

$$I_m\left(\frac{z}{2}\right) = \frac{2^{-2m}z^{-\frac{1}{2}}}{\Gamma(m+1)}M_{0,m}(z). \tag{1.166}$$

$$Y_\nu(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(s-\frac{\nu}{2}\right)\Gamma\left(s+\frac{\nu}{2}\right)}{\Gamma\left(s-\frac{\nu+1}{2}\right)\Gamma\left(\frac{3+\nu}{2}-s\right)}\left(\frac{z^2}{4}\right)^{-s}\mathrm{d}s \tag{1.167}$$

$$-3 < \Re(\nu) < -1,$$

$$K_\nu(z) = \left(\frac{2z}{\pi}\right)^{-\frac{1}{2}}W_{0,\nu}(2z) \tag{1.168}$$

$$= \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(s-\frac{\nu}{2}\right)\left(\frac{z^2}{4}\right)^{-s}\mathrm{d}s, |\arg z| < \frac{\pi}{2}. \tag{1.169}$$

(vi) Struve's function

$$H_\nu(z) = \sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{\nu+2r+1}}{\Gamma\left(r+\frac{3}{2}\right)\Gamma\left(\nu+r+\frac{3}{2}\right)} \tag{1.170}$$

$$= \frac{(z/2)^{\nu+1}}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}{}_1F_2\left(1;\frac{3}{2},\nu+\frac{3}{2};-\frac{z^2}{4}\right). \tag{1.171}$$

(vii) Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}{}_2F_1\left(-n,n+\lambda;\alpha+1;\frac{1-x}{2}\right) \tag{1.172}$$

$$= \frac{(-1)^n(\beta+1)_n}{n!}{}_2F_1\left(-n,n+\lambda;\beta+1;\frac{1+x}{2}\right) \tag{1.173}$$

$$= \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\} \tag{1.174}$$

$$= 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k, \lambda = \alpha+\beta+1. \tag{1.175}$$

(viii) Shifted Jacobi polynomial

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1). \tag{1.176}$$

(ix) Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n = P_n^{(0,0)}(x). \tag{1.177}$$

(x) Gegenbauer polynomial

$$C_n^{(\alpha+\frac{1}{2})} = \frac{(2\alpha+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\alpha)}(x). \tag{1.178}$$

(xi) Chebyshev polynomials

$$T_n(x) = \frac{n!}{(1/2)_n} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) \tag{1.179}$$

$$= \cos(n \cos^{-1} x). \tag{1.180}$$

$$T_n^*(x) = T_n(2x-1). \tag{1.181}$$

$$U_n(x) = \frac{(n+1)!}{(3/2)_n} P_n^{(\frac{1}{2},\frac{1}{2})}(x). \tag{1.182}$$

$$U_n^*(x) = U_n(2x-1). \tag{1.183}$$

(xii) Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}) \tag{1.184}$$

$$= \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x) \tag{1.185}$$

$$= \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right). \tag{1.186}$$

$$L_n^{(0)}(x) = L_n(x). \tag{1.187}$$

(xiii) Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}). \tag{1.188}$$

$$H_{e_n}(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}). \tag{1.189}$$

1.9 Generalized Wright Functions

In this section, generalized Wright function is studied. Its existence conditions are presented. In the preceding section the representations of the generalized Wright function in terms of the Mellin–Barnes integral and the H-function were given. Conditions for such representations are proved by Kilbas et al. (2002), also see Kilbas et al. (2006).

1.9.1 Existence Conditions

Existence conditions for the generalized Wright function are given by Braaksma (1964, p. 326), also see Kilbas et al. (2002). In this section we will prove the existence conditions for the generalized Wright function. The main result is given in the form of the following:

Theorem 1.4. *Let $p, q \in N_0$. Further, let $a_i, b_j \in C$ and $A_i, B_j \in R_+, i = 1, \ldots, p; j = 1, \ldots, q$*

(i) If $\mu > -1$ then the series in (1.190) is absolutely convergent for all $z \in C$.
(ii) If $\mu = -1$ then the series in (1.190) is absolutely convergent for all values of $|z| < \beta$ and for $|z| = \beta, \Re(\delta) > \frac{1}{2}$ where μ and δ are defined in (1.9) and (1.10) respectively.

Proof 1.2. Equation (1.190) is a power series

$$ {}_p\Psi_q\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \sum_{n=0}^{\infty} c_n z^n, \tag{1.190}$$

$$ c_n = \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i n)}{\prod_{j=1}^{q} \Gamma(b_j + B_j n)\, n!}, n \in N_0. \tag{1.191}$$

In order to investigate the asymptotic behavior of c_n when $n \to \infty$ we use the Stirling formula for the gamma function (1.25) to obtain the following relations:

$$ \Gamma(a_i + nA_i) \sim P_i \left(\frac{n}{\mathrm{e}}\right)^{nA_i} A_i^{nA_i} n^{a_i - \frac{1}{2}},\ P_i = (2\pi)^{\frac{1}{2}} A_i^{a_i - \frac{1}{2}} \mathrm{e}^{-a_i}, \tag{1.192}$$

as $n \to \infty$ for $i = 1, \ldots, p$;

$$ \Gamma(b_j + B_j n) \sim Q_j \left(\frac{n}{\mathrm{e}}\right)^{nB_j} B_j^{nB_j} n^{b_j - \frac{1}{2}}, Q_j = (2\pi)^{\frac{1}{2}} B_j^{b_j - \frac{1}{2}} \mathrm{e}^{-b_j}, \tag{1.193}$$

as $n \to \infty$ for $j = 1, \ldots, q$; and

$$ n! \sim (2\pi)^{\frac{1}{2}} \left(\frac{n}{\mathrm{e}}\right)^{n} n^{\frac{1}{2}} \mathrm{e}, n \to \infty. \tag{1.194}$$

Using the results (1.192), (1.193), and (1.194) into (1.191) it yields the estimate for c_n in the form

$$ c_n \sim R \left(\frac{n}{\mathrm{e}}\right)^{-n(\mu+1)} \left\{ \left[\prod_{j=1}^{p} A_j^{A_j}\right] \left[\prod_{j=1}^{q} B_j^{-B_j}\right] \right\}^{n} n^{-[\delta + \frac{1}{2}]},\ n \to \infty, \tag{1.195}$$

where μ and δ are defined in (1.17) and (1.18) respectively and

$$R = (2\pi)^{\frac{(p-q-1)}{2}} \frac{\prod_{j=1}^{p}(A_j^{a_j-\frac{1}{2}}\mathrm{e}^{-a_j})}{\mathrm{e}\prod_{j=1}^{q}(B_j^{b_j-\frac{1}{2}}\mathrm{e}^{-b_j})}. \tag{1.196}$$

The theorem now follows from the known convergence principles of the power series in (1.190). □

Corollary 1.1. *Let $p, q \in N_0$. Let $a_i, b_j \in C, A_i, B_j \in R_+, i = 1, \ldots, p; j = 1, \ldots, q$ be such that the condition $\mu > -1$ is satisfied. Then the generalized Wright function ${}_p\Psi_q(z)$ is an entire function of z, where μ is defined in (1.9).*

Corollary 1.2. *Let a be real and $b \in C$ in the Wright function $\phi(a; b; z)$ of (1.124).*

(i) If $a > -1$ then the series in (1.124) is absolutely convergent for all $z \in C$.
(ii) If $a = -1$ then the series in (1.124) is absolutely convergent for all $|z| < 1$ and for $|z| = 1, \Re(\beta) > 1$ where μ is defined in (1.9).

Corollary 1.3. *If $a > -1$ and $b \in C$ then the Wright function $\phi(a, b; z)$ defined by (1.124) is an entire function of z.*

Corollary 1.4. *If $\mu > -1$ and $\nu \in C$ then the Bessel–Maitland function $J_\nu^\mu(z)$ defined by (1.118) is an entire function of z.*

1.9.2 Representation of Generalized Wright Function

Notation 1.14.

$${}_2R_1(a, b; c, \omega; \mu; z): \text{ Dotsenko function (Dotsenko 1991, 1993)} \tag{1.197}$$

Definition 1.14.

$${}_2R_1(a, b; c, \omega; \mu; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k\frac{\omega}{\mu})}{\Gamma(c+k\frac{\omega}{\mu})} \frac{z^k}{k!} \tag{1.198}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} {}_2\Psi_1\left[z\Big|^{(a,1),(b,\frac{\omega}{\mu})}_{(c,\frac{\omega}{\mu})}\right]. \tag{1.199}$$

The existence of the generalized Wright function ${}_p\Psi_q(z)$ defined by means of the Mellin–Barnes integral (1.140) is given by the following results which yield different conditions for the representation (1.140) with the contours $L = L_{-\infty}, L = L_{+\infty}$ and $L = L_{i\gamma\infty}$. By following a procedure similar to that adopted in proving the existence conditions of the H-function in Theorem 1.1, the following theorems can be established on the contours $L_{\infty}, L_{-\infty}$ and $L_{i\gamma\infty}$ (defined in Sect. 1.1). For a detailed proof of these theorems, one can refer to Kilbas, Saigo, and Trujillo (2002) and also to a recent article by Kilbas et al. (2006).

Theorem 1.5. *Let $p, q \in N_0$. Let $a_i, b_j \in C$ and $A_i, B_j \in R_+, i = 1, \ldots, p; j = 1, \ldots, q$ and be such that the conditions $\frac{a_i+k}{A_i} \neq -v; k, v \in N_0, i = 1, \ldots, p$ and $(a_i + k)A_j \neq (a_j + m)A_i, i \neq j, j = 1, \ldots, p; k, m \in N_0$ be satisfied. Let either of the following conditions hold:*

$$\mu > -1,\ z \neq 0, \tag{1.200}$$

$$\mu = -1,\ 0 < |z| < \beta, \tag{1.201}$$

$$\mu = -1, |z| = \beta, \Re(\delta) > \frac{1}{2}. \tag{1.202}$$

Then there exists the generalized Wright function ${}_p\Psi_q(z)$ defined by means of the Mellin–Barnes integral (1.140), where the path of integration $L = L_{-\infty}$ separates all poles given in $s = -v, v \in N_0$ to the left and all poles given by $s = \frac{a_i+k}{A_i}, i = 1, \ldots, n; k \in N_0$ to the right.

Theorem 1.6. *Let $p, q \in N_0, a_i, b_j \in C$ and $A_i, B_j \in R_+, i = 1, \ldots, p; j = 1, \ldots, q$ and be such that the conditions on the parameters in Theorem 1.5 are satisfied. Let either of the following conditions hold:*

$$\mu < -1,\ z \neq 0, \tag{1.203}$$

$$\mu = -1, |z| > \beta, \tag{1.204}$$

$$\mu = -1, |z| = \beta, \Re(\delta) > \frac{1}{2}. \tag{1.205}$$

Then there exists the generalized Wright function ${}_p\Psi_q(z)$ defined by means of Mellin–Barnes integral (1.140), where the path of integration $L = L_{+\infty}$ separates all poles as stated in Theorem 1.5.

Theorem 1.7. *Let $p, q \in N_0, a_i, b_j \in C$ and $A_i, B_j \in R_+, i = 1, \ldots, p; j = 1, \ldots, q$ and be such that the conditions on the parameters as stated in Theorem 1.5 be satisfied. Let either of the following conditions hold:*

$$\mu < 1,\ |\arg(-z)| < \frac{(1-\mu)\pi}{2}, z \neq 0, \tag{1.206}$$

$$\mu = 1, (1+\mu)\gamma + \frac{1}{2} < \Re(\delta), \arg(-z) = 0, z \neq 0. \tag{1.207}$$

Then there exists the generalized Wright function ${}_p\Psi_q(z)$ defined by means of Mellin–Barnes integral (1.140), where the path of integration $L = L_{i\gamma\infty}$ separates all poles as stated as in the Theorem 1.5.

If we combine the Theorems 1.5–1.7 then we arrive at the following theorem given by Kilbas et al. (2006, p. 125), which gives the conditions under which the generalized Wright function can be represented as an H-function by (1.140).

Theorem 1.8. *Let $p, q \in N_0, a_i, b_j \in C$ and $A_i, B_j \in R_+, i = 1, \ldots, p; j = 1, \ldots, q$ and be such that the conditions in Theorem 1.5 be satisfied, and let $\gamma \in R$. Let L be the contour which separates all poles as given in Theorem 1.5. Further, let either of the following conditions hold:*

$$(i) \quad L = L_{-\infty} \text{ and either (1.200), (1.201) or (1.202) holds} \tag{1.208}$$

$$(ii) \quad L = L_{+\infty} \text{ and either (1.203), (1.204) or (1.205) holds} \tag{1.209}$$

$$(iii) \quad L = L_{\gamma\infty} \text{ and either (1.206), or (1.207) holds} \tag{1.210}$$

Then the generalized Wright function ${}_p\Psi_q(z)$ defined by (1.123) is represented as an H-function by (1.140).

The utility and importance of the generalized Wright function is realized in recent years due to its occurrence in certain problems of applied character. This function is in the proximity of the H-function so its utility is further increased. Nearly all the Mittag-Leffler functions and their generalizations can be expressed in terms of this function; in this connection one can refer to the paper by Kilbas et al. (2002). Various properties of the Wright function are studied by many authors in a series of papers, some of which are enumerated below.

Wright (1933) showed the application of the results obtained for the function $\phi(a, b; z)$ defined by (1.124) to the asymptotic theory of partitions. Dotsenko (1991) developed fractional relations for the Wright function. Asymptotic relations and distribution of the zeros of this function $\phi(a, b; z)$ are investigated by Luchko (2000, 2001). Application of this function in operational calculus is given by Mikusinski (1959) and in integral transform of Hankel type by Gajic and Stankovic (1976) and Stankovic (1970). Mainardi (1994) derived the solution of fractional diffusion-wave equation in terms of the Wright function. In this connection, the interested reader can also refer to the book by Podlubny (1999, Sect. 4.12) and to the survey paper Mainardi (1997). Scale-variant solutions of some partial differential equations of fractional order are given in terms of the special cases of the generalized Wright function by Buckwar and Buckwar and Luchko (1998), Luchko and Gorenflo (1998) and Gorenflo et al. (2000). Analytic properties of the Wright function with applications are obtained by Gorenflo et al. (1999). Existence conditions and representations of the generalized Wright function in terms of Mellin–Barnes integrals and the H-function are obtained by Kilbas et al. (2002). Wright function representations of the Krätzel function are investigated recently by Kilbas et al. (2006). Generalized Wright function has been used in the study of generalized gamma functions by Srivastava et al. (2003). Generalized Wright function as a kernel of an integral transform is recently studied by Saxena et al. (2006). Analytical continuation formulae and asymptotic formulae for the generalized Wright function are investigated by Kilbas et al. (2006).

Exercises

1.1. Prove that if $\Re(\delta) > 0$, then

$$\text{(i)}\quad f(x;\delta,\alpha,\gamma,1) = 2\left(\frac{\gamma x}{\alpha}\right)^{\frac{\delta}{2}} K_\delta[2(\alpha\gamma x)^{\frac{1}{2}}],$$

$$\text{(ii)}\quad f(x;\delta,\alpha,\gamma,-1) = \Gamma(\delta)(\alpha+\gamma x)^{-\delta}, \Re(\alpha+\gamma x) > 0, |\frac{\gamma x}{\alpha}| < 1,$$

$$\text{(iii)}\quad f(x;\delta,\alpha,\gamma,-\frac{1}{2}) = 2^{1-\delta}\Gamma(2\delta)\alpha^{-\delta}\exp\left(\frac{\alpha^{-1}\gamma^2 x}{8}\right) D_{-2\delta}[(2\alpha)^{-\frac{1}{2}}\gamma x],$$

$$\text{(iv)}\quad f(x;\delta,\alpha,\gamma,-2) = \Gamma(\delta)(2\gamma x)^{-\frac{\delta}{2}}\exp[-\frac{\alpha^2}{\delta\gamma x}]D_{-\delta}[\alpha(2\gamma x)^{-\frac{1}{2}}],$$

where $f(x;\delta,\alpha,\gamma,\phi) = \alpha^{-\delta} H_{0,2}^{2,0}[\alpha^{\phi}\gamma x|(\delta,\phi),(0,1)]$ (Buschman 1974a).

1.2. Prove that

$$B_1 z^{-b_1} H_{p,q}^{1,n}\left[z^{B_1}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]$$
$$= \sum_{v=0}^{\infty}\frac{(-z)^v}{v!}\frac{\prod_{j=1}^{n}\Gamma\left[1-a_j+A_j\left(\frac{b_1+v}{B_1}\right)\right]}{\left\{\prod_{j=2}^{q}\Gamma\left[1-b_j+B_j\left(\frac{b_1+v}{B_1}\right)\right]\right\}\left\{\prod_{j=n+1}^{p}\Gamma\left[a_j-A_j\left(\frac{b_1+v}{B_1}\right)\right]\right\}}.$$

(Braaksma 1964, p. 279)

1.3. Prove that

$$\text{(i)}\quad z^r\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{H_{p,q}^{m,n}\left[x^{\delta}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\} = H_{p+1,q+1}^{m,n+1}\left[z^{\delta}\Big|_{(b_q,B_q),(r,\delta)}^{(0,\delta),(a_p,A_p)}\right],$$

$$\text{(ii)}\quad z^r\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{H_{p,q}^{m,n}\left[z^{-\delta}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\} = (-1)^r H_{p+1,q+1}^{m,n+1}\left[z^{-\delta}\Big|_{(b_q,B_q),(1,\delta)}^{(1-r,\delta),(a_p,A_p)}\right],$$

giving the conditions of validity of the result. Hint: use the formulae

$$z^r\frac{\mathrm{d}^r}{\mathrm{d}z^r}(z^{s\delta}) = \frac{\Gamma(1+s\delta)}{\Gamma(1+s\delta-r)}z^{s\delta},$$

and

$$z^r\frac{\mathrm{d}^r}{\mathrm{d}z^r}(z^{-s\delta}) = \frac{(-1)^r\Gamma(r+s\delta)}{\Gamma(s\delta)}z^{-s\delta}.$$

Show that

$$\frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{z^\lambda H_{p,q}^{m,n}\left[\beta z^\delta\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\} = z^{\lambda-r} H_{p+1,q+1}^{m,n+1}\left[\beta z^\delta\Big|_{(b_q,B_q),(r-\lambda,\delta)}^{(-\lambda,\delta),(a_p,A_p)}\right].$$

(Anandani 1970)

1.4. Establish the following identities:

(i) $$H_{p+1,q+1}^{m,n+1}\left[z\Big|_{(b_q,B_q),(\alpha+r,\delta)}^{(\alpha,\delta),(a_p,A_p)}\right] = (-1)^r H_{p+1,q+1}^{m+1,n}\left[z\Big|_{(\alpha+r,\delta),(b_q,B_q)}^{(a_p,A_p),(\alpha,\delta)}\right].$$

(Anandani 1970, p. 191)

(ii) $$H_{2,4}^{4,0}\left[z\Big|_{(0,1),(\frac{1}{2},1),(b,1),(-b,1)}^{(\frac{1}{2}+a,1),(\frac{1}{2}-a,1)}\right] = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} W_{a,b}(2z^{\frac{1}{2}})W_{-a,b}(2z^{\frac{1}{2}}),$$

where $W_{a,b}(z)$ and $W_{-a,b}(z)$ are Whittaker functions.

(iii) $$H_{p+2,q+2}^{m,n+2}\left[z\Big|_{(b_q,B_q),(\alpha-\sigma-\nu,h),(-1-\beta-\sigma-\nu,h)}^{(-\sigma,h),(\alpha-\sigma,h),(a_p,A_p)}\right]$$
$$= (-1)^\nu H_{p+2,q+2}^{m+1,n+1}\left[z\Big|_{(\alpha-\sigma-\nu,h),(b_q,B_q),(-1-\beta-\sigma-\nu,h)}^{(-\sigma,h),(a_p,A_p),(\alpha-\sigma,h)}\right],$$

(iv) $$H_{p+1,q+1}^{m+1,n}\left[x\Big|_{(\alpha-\beta,h),(b_q,B_q)}^{(a_p,A_p),(\alpha-\beta-1,h)}\right]$$
$$= H_{p+1,q+1}^{m+1,n}\left[x\Big|_{(\alpha+2,h),(b_q,B_q)}^{(a_p,A_p),(\alpha+1,h)}\right] - (\beta+2)H_{p,q}^{m,n}\left[x\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]$$

(Anandani 1969).

1.5. Prove that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}x - c_1\right)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}x}x - c_r\right)\left\{x^\delta H_{p,q}^{m,n}\left[zx^h\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\}$$
$$= x^\delta H_{p+r,q+r}^{m,n+r}\left[zx^h\Big|_{(b_q,B_q),(c_r-\delta,h),\ldots,(c_1-\delta,h)}^{(c_r-\delta-1,h),\ldots,(c_1-\delta-1,h),(a_p,A_p)}\right],$$

where $h > 0$ and the symbol $\frac{\mathrm{d}}{\mathrm{d}x}x$ indicates that the function of x in front of it is first multiplied by x and then the product is differentiated with respect to x. Hence deduce the following result:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}x - c\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}x - c + e\right)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}x}x - c + (r-1)e\right)$$
$$\times\left\{x^{\delta e+c-1} H_{p,q}^{m,n}\left[zx^{he}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\right\}$$
$$= e^r x^{\delta e+c-1} H_{p+1,q+1}^{m,n+1}\left[zx^{he}\Big|_{(b_q,B_q),(1-\delta,h)}^{(1-r-\delta,h),(a_p,A_p)}\right],$$

provided $e \neq 0, h > 0$.

(Nair 1972)

1.6. Establish the following differentiation formulae:

$$\text{(i)}\quad \frac{d^r}{dx^r} H_{p,q}^{m,n}\left[(cx+d)^h\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{(-c)^r}{(cx+d)^r} H_{p+1,q+1}^{m+1,n}\left[(cx+d)^h\Big|_{(r,h),(b_q,B_q)}^{(a_p,A_p),(0,h)}\right],$$

$$\text{(ii)}\quad \frac{d^r}{dx^r} H_{p,q}^{m,n}\left[\frac{1}{(cx+d)^h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{c^r}{(cx+d)^r} H_{p+1,q+1}^{m,n+1}\left[\frac{1}{(cx+d)^h}\Big|_{(1,h),(b_q,B_q)}^{(a_p,A_p),(1-r,h)}\right],$$

$$\text{(iii)}\quad \frac{d^r}{dx^r} H_{p,q}^{m,n}\left[\frac{1}{(cx+d)^h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{(-c)^r}{(cx+d)^r} H_{p+1,q+1}^{m,n+1}\left[\frac{1}{(cx+d)^h}\Big|_{(b_q,B_q),(1,h)}^{(1-r,h),(a_p,A_p)}\right],$$

where c and d are complex numbers, r is a positive integer and $h > 0$. (Oliver and Kalla 1971).

1.7. Prove the following results:

$$\text{(i)}\quad H_{p,q}^{m,n}\left[z\lambda^{\sigma}\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,\sigma),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right] = \lambda^{a_1-1}\sum_{r=0}^{\infty}\frac{1}{r!}\left(1-\frac{1}{\lambda}\right)^r \times H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1-r,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right],$$

where $1 \le n \le p-1$, $\mu > 0, \sigma > 0$ and λ and z are complex numbers.

$$\text{(ii)}\quad (a_p-\mu a_1)H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(1+a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right] = H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(1+a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right] + \mu H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p+1,\mu\sigma)}\right],$$

where $1 \le n \le p-1$ and $\mu > 0$.

$$\text{(iii)}\quad H_{p+1,q+1}^{m+1,n}\left[x\Big|_{(a_1+\nu+1,\sigma),(b_1,B_1),\ldots,(b_q,B_q)}^{(1+a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma),(a_1+\nu,\sigma)}\right] = \nu H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(1+a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right] - H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}\right],$$

where $1 \le n \le p-1$ and $\mu > 0$.

(iv) $$\left[x^{1-\frac{1}{\mu}}\frac{\mathrm{d}}{\mathrm{d}x}x^{\frac{(a_p-1)}{\mu}}\right]^r H^{m,n}_{p,q}\left[zx^{-\sigma}\Big|^{(a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p,\mu\sigma)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right]$$
$$= \left(\frac{1}{\mu}\right)^r x^{\frac{(a_p-r-1)}{\mu}}$$
$$\times H^{m,n}_{p,q}\left[zx^{-\sigma}\Big|^{(a_1,\sigma),(a_2,A_2),\ldots,(a_{p-1},A_{p-1}),(a_p-r,\mu\sigma)}_{(b_1,B_1),\ldots,(b_q,B_q)}\right],$$
where $1 \le n \le p-1$ and $\mu > 0$.

(Srivastava and Gupta 1970)

Hint: The above results can be proved by representing the H-functions on the right by their Mellin–Barnes representations, taking the common factors out and then combining the terms.

1.8. Let
$$d(b_1, a_p - k) = \det\begin{bmatrix} b_1 & a_p - k \\ B_1 & A_p \end{bmatrix},$$
in which the first row of the determinant is written by our notation. The second row of the determinant is always to be completed with the appropriate A's and B's corresponding to the a's and b's of the first row. Further, we employ the notation $H[b_1 + 1]$ to denote the contiguous function in which b_1 is replaced by $b_1 + 1$, but with all other parameters left unchanged. Similar meanings hold for all other contiguous H-functions occurring in this problem. In the following results H will denote the H-function. Prove the following relations of contiguity for the H-function.

$$A_p H[b_1 + 1] - B_1[a_p - 1] = d(b_1, a_p - 1)H. \tag{1.211}$$
$$A_p H[a_1 - 1] + A_1 H[a_p - 1] = -d(a_1 - 1, a_p - 1)H. \tag{1.212}$$

$$B_q H[a_1 - 1] - A_1 H[b_q + 1] = -d(a_1 - 1, b_q)H. \tag{1.213}$$
$$B_q H[b_1 + 1] + B_1 H[b_q + 1] = d(b_1, b_q)H. \tag{1.214}$$
$$A_1 H[b_1 + 1] + B_1 H[a_1 - 1] = d(b_1, a_1 - 1)H. \tag{1.215}$$
$$B_q H[a_p - 1] + A_p H[b_q + 1] = d(a_p - 1, b_q)H. \tag{1.216}$$

$$B_q H[b_1 + 1] - B_1 H[b_2 + 1] = d(b_1, b_2)H. \tag{1.217}$$
$$A_2 H[a_1 - 1] - A_1 H[a_2 - 1] = -d(a_1 - 1, a_2 - 1)H. \tag{1.218}$$

$$A_{p-1}H[a_p-1]-A_pH[a_{p-1}-1]=d(a_p-1,a_{p-1}-1)H. \tag{1.219}$$
$$B_{q-1}H[b_q+1]-B_qH[b_{q-1}+1]=-d(b_q,b_{q-1})H. \tag{1.220}$$

$$d(a_p-1,b_q)H[a_1-1]-d(b_q-a_1-1)H[a_p-1] \\ =-d(a_1-1,a_p-1)H[b_q+1]. \tag{1.221}$$

$$d(a_p-1,b_q)H[b_1+1]+d(b_q,b_1)H[a_p-1] \\ =d(b_1,a_p-1)H[b_q+1]. \tag{1.222}$$
$$d(a_1-1,b_q)H[b_1+1]-d(b_q,b_1)H[a_1-1] \\ =d(b_1,a_1-1)H[b_q+1]. \tag{1.223}$$

$$d(a_1-1,b_q)H[b_1+1]-d(b_q,b_1)H[a_1-1] \\ =d(b_1,a_1-1)H[b_q+1]. \tag{1.224}$$
$$d(a_1-1,a_p-1)H[b_1+1]-d(a_p-1,b_1)H[a_1-1] \tag{1.225}$$
$$=-d(b_1,a_1-1)H[a_p-1]. \tag{1.226}$$

$$d(b_2,b_3)H[b_1+1]+d(b_3,b_1)H[b_2+1) \\ =-d(b_1,b_2)H[b_3+1]. \tag{1.227}$$
$$d(a_2-1,a_3-1)H[a_1-1]+d(a_3-1,a_2-1)H[a_2-1] \\ =-d(a_1-1,a_2-1)H[a_3-1]. \tag{1.228}$$

$$d(a_{p-1}-1,a_{p-2}-1)H[a_p-1]+d(a_{p-2}-1,a_p-1)H[a_{p-1}-1] \\ =-d(a_p-1,a_{p-1}-1)H[a_{p-2}-1]. \tag{1.229}$$
$$d(b_{q-1},b_{q-2})H[b_q+1]+d(b_q-2,b_q)H[b_{q-1}+1] \\ =-d(b_q,b_{q-1})H[b_{q-2}+1]. \tag{1.230}$$

$$d(a_p-1,b_1)H[a_{p-1}-1]+d(b_1,a_{p-1}-1)H[a_p-1] \\ =-d(a_{p-1}-1,a_p-1)H[b_1+1]. \tag{1.231}$$
$$d(b_q,a_1-1)H[b_{q-1}+1]+d(a_1-1,b_{q-1})H[b_q+1] \\ =-d(b_{q-1},b_q)H[a_1-1]. \tag{1.232}$$

$$\begin{aligned} d(a_2-1,a_p-1)H[a_1-1]+d(a_p-1,a_1-1)H[a_2-1] \\ = d(a_1-1,a_2-1)H[a_p-1]. \end{aligned} \tag{1.233}$$

$$\begin{aligned} d(b_2,b_q)H[b_1+1]+d(b_q,b_1)H[b_2+1] \\ = d(b_1,b_2)H[b_q+1]. \end{aligned} \tag{1.234}$$

$$\begin{aligned} d(a_{p-1}-1,a_1-1)H[a_p-1]+d(a_1-1,a_p-1)H[a_{p-1}-1] \\ = d(a_{p-1},a_{p-1}-1)H[a_1-1]. \end{aligned} \tag{1.235}$$

$$\begin{aligned} d(b_{q-1},b_1)H[b_q+1]+d(b_1,b_q)H[b_{q-1}+1] \\ = d(b_q,b_{q-1})H[b_1+1]. \end{aligned} \tag{1.236}$$

$$\begin{aligned} d(a_2-1,b_q)H[a_1-1]+d(b_q,a_1-1)H[a_2-1] \\ = -d(a_1-1,a_2-1)H[b_q+1]. \end{aligned} \tag{1.237}$$

$$\begin{aligned} d(b_2,a_{p-1})H[b_1+1]+d(a_{p-1},b_1)H[b_2+1] \\ = -d(b_1,b_2)H[a_p-1]. \end{aligned} \tag{1.238}$$

$$\begin{aligned} d(a_2-1,b_1)H[a_1-1]+d(b_1,a_1-1)H[a_2-1] \\ = d(a_1-1,a_2-1)H[b_1+1]. \end{aligned} \tag{1.239}$$

$$\begin{aligned} d(b_2,a_1-1)H[b_1+1]+d(a_1-1,b_1)H[b_2+1] \\ = d(b_1,b_2)H[a_1-1]. \end{aligned} \tag{1.240}$$

$$\begin{aligned} d(a_{p-1}-1,b_q)H[a_p-1]+d(b_q,a_p-1)H[a_{p-1}-1] \\ = d(a_p-1,a_{p-1}-1)H[b_q+1]. \end{aligned} \tag{1.241}$$

$$\begin{aligned} d(b_{q-1},a_p-1)H[b_q+1]+d(a_p-1,b_q)H[b_{q-1}+1] \\ = d(b_q,b_{q-1})H[a_p-1]. \end{aligned} \tag{1.242}$$

(Buschman 1972)

Hint: First establish the basic relations (1.211) and (1.212) given above and then derive all the others from two of them and using the transformation formula of $H(x)$ going to $H(\frac{1}{x})$.

1.9. Establish the following results associated with the Mellin transforms of the partial derivatives of the H-function with respect to their parameters.

$$\text{(i)} \quad M\left\{\frac{\partial}{\partial b_1}H_{p,q}^{m,n}(x)\right\} = \chi(-s)\psi(b_1+B_1 s),\ m>0$$

$$\text{(ii)} \quad M\left\{\frac{\partial}{\partial a_1}H_{p,q}^{m,n}(x)\right\} = -\chi(-s)\psi(1-a_1-A_1 s), n>0$$

(iii) $M\left\{\dfrac{\partial}{\partial a_p}H_{p,q}^{m,n}(x)\right\} = -\chi(-s)\psi(a_p + A_p s), n < p$

(iv) $M\left\{\dfrac{\partial}{\partial b_q}H_{p,q}^{m,n}(x)\right\} = \chi(-s)\psi(1 - b_q - B_q s), m < q$

(v) $M\left\{\dfrac{\partial}{\partial B_1}H_{p,q}^{m,n}(x)\right\} = s\chi(-s)\psi(b_1 + B_1 s), m > 0$

(vi) $M\left\{\dfrac{\partial}{\partial A_1}H_{p,q}^{m,n}(x)\right\} = -s\chi(-s)\psi(1 - a_1 - A_1 s), n > 0$

(vii) $M\left\{\dfrac{\partial}{\partial A_p}H_{p,q}^{m,n}(x)\right\} = -s\chi(-s)\psi(a_p + A_p s), n < p$

(viii) $M\left\{\dfrac{\partial}{\partial B_q}H_{p,q}^{m,n}(x)\right\} = s\chi(-s)\psi(1 - b_q - B_q s), m < q$

where M denotes the Mellin transform, ψ is the psi-function and $\chi(s)$ is given as $\Theta(s)$ in (1.3). (Buschman 1974a, p. 151).

1.10. Prove that

$$H_{1,2}^{1,1}\left[z\Big|_{(a,A),(0,1)}^{(a,A)}\right] = A^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k z^{\frac{(k+a)}{A}}}{\Gamma\left(1+\frac{(k+a)}{A}\right)},$$

where $A > 0$.

1.11. Prove that

$$H_{2,1}^{1,1}\left[z\Big|_{(1-a,A)}^{(1-a,A),(1,1)}\right] = A^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k z^{-\frac{(a-k-1)}{A}}}{\Gamma\left(1+\frac{(a-k-1)}{A}\right)},$$

where $A > 0$.

1.12. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}z}H_{1,2}^{1,1}\left[z\Big|_{(a,A),(0,1)}^{(a,A)}\right] = H_{1,2}^{1,1}\left[z\Big|_{(a-A,A),(0,1)}^{(a-A,A)}\right],\ A > 0.$$

1.13. Prove that

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\mathrm{d}s = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)}.$$

1.14. Prove that

$$Z_\rho^\nu(z) = -\frac{1}{\rho} H_{1,1}^{1,1}\left[z\middle|_{(0,1)}^{(1-\frac{\nu}{\rho},-\frac{1}{\rho})}\right], z \in C, z \neq 0, \rho < 0, \Re(\nu) < 0.$$

1.15. Evaluate

$$f(z) = \frac{1}{2\pi i}\int_C \Gamma(s-a) z^{-s} \mathrm{d}s,$$

where C is a loop which embraces all the poles of $\Gamma(s-a)$ at the points $s = a - \nu, \nu \in N_0$.

1.16. Prove that the Mellin–Barnes integral (Paris and Kaminski 2001, p. 113)

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(-s)}{s+a} z^{a+s} \mathrm{d}s,$$

defines the incomplete gamma function $\gamma(a,z)$ defined by $\gamma(a,z) = \int_0^z t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$, where $|\arg(z)| < \frac{\pi}{2}$ and the contour C separates the poles at $s = -\nu, \nu \in N_0$ from the pole $s = -a$ (a is not a positive integer).

1.17. Prove that the Wright function (or Dotsenko function) ${}_2R_1(a,b;c;\omega,\mu;z)$ can be expressed by the Mellin–Barnes integral (Kilbas et al. 2006, p. 123) in the form

$${}_2R_1(a,b;c;\omega,\mu;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-\frac{\omega s}{\mu})}{\Gamma(c-\frac{\omega s}{\mu})}(-z)^{-s}\mathrm{d}s,$$

where the contour of integration $L = L_{-\infty}$ separates all poles of $\Gamma(s)$ to the left and all the poles of $\Gamma(1-s)$ and $\Gamma(b-\frac{\omega s}{\mu})$ to the right.

1.18. Prove that (Braaksma 1964, p. 289)

$$G_{p,q}^{q,0}\left[x\middle|_{(b_q)}^{(a_p)}\right] = \frac{(2\pi)^{\frac{1}{2}(a-1)}}{\sqrt{(a)}} x^{\frac{(1-b)}{a}} \mathrm{e}^{-ax^{\frac{1}{a}}}[1 + O(x^{-\frac{1}{a}})], \text{ as } x \to \infty,$$

$$a = q - p, b = \sum_{i=1}^{p} a_i - \sum_{j=1}^{q} b_j + \frac{1}{2}(q-p+1).$$

1.19. Prove that the function $\lambda_\gamma^{(n)}(z)$ defined by the integral

$$\lambda_\gamma^{(n)}(z) = \frac{(2\pi)^{\frac{(n-1)}{2}}\sqrt{n}}{\Gamma(\gamma+1-\frac{1}{n})}\left(\frac{z}{n}\right)^{n\gamma}\int_1^\infty (t^n-1)^{\gamma-\frac{1}{n}}\mathrm{e}^{-zt}\mathrm{d}t,$$

for $n \in N, \Re(\gamma) > \frac{1}{n} - 1, \Re(z) > 0$ can be expressed in terms of the H-function as

$$H_{1,2}^{2,0}\left[z\Big|_{(n\gamma,1),(0,\frac{1}{n})}^{(\gamma+1-\frac{1}{n},\frac{1}{n})}\right] = (2\pi)^{\frac{(1-n)}{2}} n^{n\gamma+\frac{1}{2}} \lambda_{\gamma}^{(n)}(z).$$

1.20. Prove that the function $\lambda_{\gamma,\sigma}^{(\beta)}(z)$ defined by the integral

$$\lambda_{\gamma,\sigma}^{(\beta)}(z) = \frac{\beta}{\Gamma(\gamma + 1 - \frac{1}{\beta})} \int_1^{\infty} (t^{\beta} - 1)^{\gamma-\frac{1}{\beta}} \mathrm{e}^{-zt} \mathrm{d}t,$$

for $\beta > 0, \Re(\gamma) > \frac{1}{\beta} - 1, \Re(z) > 0, \sigma \in C$, can be expressed in terms of the H-function as

$$H_{1,2}^{2,0}\left[z\Big|_{(n\gamma,1),(-\gamma-\frac{\sigma}{\beta},\frac{1}{\beta})}^{(1-\frac{(\sigma+1)}{\beta},\frac{1}{\beta})}\right] = \lambda_{\gamma,\sigma}^{(\beta)}(z).$$

1.21. Prove the following results:

$$\lambda_{\gamma}^{(2)}(z) = 2\left(\frac{z}{2}\right)^{\gamma} K_{-\gamma}(z),$$

and

$$\lambda_{\gamma,0}^{2}(z) = \frac{2}{\sqrt{\pi}}\left(\frac{2}{z}\right)^{\gamma} K_{-\gamma}(z),$$

where $K_{-\gamma}(z)$ is the modified Bessel function of the third kind.

Notation 1.15. Multi-index Mittag-Leffler functions: $E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z)$.

Definition 1.15. Let $m = 1$ be an integer, $\rho_1, \ldots, \rho_m > 0$ and $\mu_1, \ldots, \mu_m$ be arbitrary real numbers. By means of "multi-indices" $(\rho_i), (\mu_i)$, the so-called multi-index (m-tuple, multiple) Mittag-Leffler functions are introduced (Kiryakova 2000, p. 244) as

$$E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})}. \tag{1.243}$$

1.22. Prove that the multi-index Mittag-Leffler functions in Definition 1.15 can be expressed as follows:

$$E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z) = {}_1\Psi_m\left[z\Big|_{(\mu_1,\frac{1}{\rho_1}),\ldots,(\mu_m,\frac{1}{\rho_m})}^{(1,1)}\right]$$
$$= H_{1,m+1}^{1,1}\left[-z\Big|_{(0,1),(1-\mu_1,\frac{1}{\rho_1}),\ldots,(1-\mu_m,\frac{1}{\rho_m})}^{(0,1)}\right].$$

1.23. For the multi-index function $E_{\left(\frac{1}{\rho_1}\right),(\mu_i)}(z)$ prove the following result (Saxena et al. 2003, p. 369): For $\rho_i > 0, \mu_i > 0, i = 1, \ldots, m, r \in N$ there holds the formula

$$z^r E_{\left(\frac{1}{\rho_i}\right),(\mu_i + \frac{r}{\rho_i})}(z) = E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z) - \sum_{h=0}^{r-1} \frac{z^h}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{h}{\rho_j})}.$$

1.24. Prove the following asymptotic estimates for the Mittag-Leffler function $E_\alpha(z)$. For $0 < \alpha < 2$ show that

$$E_\alpha(z) \sim \begin{cases} \frac{1}{\alpha}\exp(z^{\frac{1}{\alpha}}) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}, |\arg(z)| < \frac{3}{2}\pi\alpha \\ -\sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}, |\arg(-z)| < \frac{1}{2}\pi(2-\alpha) \end{cases},$$

as $|z| \to \infty$. Further, show that for $\alpha > 2$ the following asymptotic estimate holds:

$$E_\alpha(z) \sim \frac{1}{\alpha} \sum_{r=-N}^{N} \exp\{z^{\frac{1}{2}} \mathrm{e}^{\frac{2\pi i r}{\alpha}}\} - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}, -\pi < \arg(z) \leq \pi,$$

as $|z| \to \infty$, where $N = \left[\frac{1}{2}\alpha - \frac{1}{2}\right]$, (Paris and Kaminski 2001, p. 189).

Chapter 2
H-Function in Science and Engineering

2.1 Integrals Involving H-Functions

This chapter deals with integrals involving H-functions. We propose to present the results for Mellin, Laplace, Hankel, Bessel, and Euler transforms of the H-functions. Further, on account of the importance and considerable popularity achieved by fractional calculus, that is, the calculus of fractional integrals and fractional derivatives of arbitrary real or complex orders, during the last four decades due to its applications in various fields of science and engineering, such as fluid flow rheology, diffusive transport akin to diffusion, electric networks and probability, the discussion of H-function is more relevant. In this connection, one can refer to the work of Phillips (1989, 1990), Bagley (1990), Bagley and Torvik (1986) and Somorjai and Bishop (1970) and the book by Podlubny (1999). In the present book, fractional integration and fractional differentiation of the H-functions will be discussed. A long list of papers on integrals of the H-functions is available from the bibliography of the books by Mathai and Saxena (1978), Srivastava et al. (1982), Prudnikov et al. (1990) and Kilbas and Saigo (2004).

2.2 Integral Transforms of the H-Function

2.2.1 Mellin Transform

In order to present the results of this section, a few notations and definitions are given first

Notation 2.1. $M\{f(t) : s\}, f^*(s)$, Mellin transform of f with respect to a parameter s.

Notation 2.2. $M^{-1}\{f^*(s); x\}$: Inverse Mellin transform

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Definition 2.1. The Mellin transform of a function $f(t)$, denoted by $f^*(s)$, is defined by

$$f^*(s) = M[f(t); s] = \int_0^\infty t^{s-1} f(t) \mathrm{d}t, \quad t > 0, \tag{2.1}$$

provided that the integral converges. The inverse Mellin transform is given by the contour integral

$$f(x) = M^{-1}\{f^*(s); x\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s) x^{-s} \mathrm{d}s. \tag{2.2}$$

If $f^*(s)$ is analytic in the relevant strip then $f(x)$ is uniquely determined by $f^*(s)$ by using the formula (2.2).

2.2.2 Illustrative Examples

Example 2.1. Find the Mellin transform of Gauss hypergeometric function ${}_2F_1$.

Solution 2.1. By definition (2.1), we have to evaluate the integral

$$I = \int_0^\infty t^{s-1} {}_2F_1(a, b : c : -t) \mathrm{d}t,$$

where $a, b, c \in C$, $\min\{\Re(a), \Re(b)\} > \Re(s) > 0$.

If we use Euler integral representation of the hypergeometric function then the given integral becomes

$$\begin{aligned} I &= \int_0^\infty t^{s-1} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1} (1-u)^{c-b-1} (1+tu)^{-a} \mathrm{d}u \mathrm{d}t \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1} (1-u)^{c-b-1} \int_0^\infty \frac{t^{s-1}}{(1+tu)^a} \mathrm{d}t \mathrm{d}u \\ &= \frac{\Gamma(s)\Gamma(a-s)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-s-1} (1-u)^{c-b-1} \mathrm{d}u = \frac{\Gamma(s)\Gamma(a-s)\Gamma(c)\Gamma(b-s)}{\Gamma(a)\Gamma(b)\Gamma(c-s)}, \end{aligned} \tag{2.3}$$

for $\Re(s) > 0, \Re(a-s) > 0, \Re(b-s) > 0, \Re(c-s) > 0$. The interchange of the order of integration in the above steps is justified under the conditions given along with the integral. This completes the solution of Example 2.1.

Example 2.2. Find the inverse Mellin transform of the right side in (2.3).

Solution 2.2. By virtue of the results (2.2) and (2.3), we find that

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)} (-z)^{-s} \mathrm{d}s, \quad (2.4) $$

where $a,b,c \in C$, $\min\{\Re(a),\Re(b)\} > \gamma > 0$ and $c \neq 0,-1,-2,\cdots.;|arg(-z)| < \pi$. The path of integration separates the poles at $s = a+m,\ s = b+m$ from the poles at $s = -m, m \in N_0$.

Example 2.3. Prove that the Mellin transform of the generalized Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$, defined by (1.47), is given by

$$ M\left\{E^{\gamma}_{\alpha,\beta}(-z);s\right\} = \frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\gamma)\Gamma(\beta-\alpha s)}, \quad (2.5) $$

where $\Re(s) > 0, \Re(\gamma - s) > 0; \alpha \in R^+, \beta, \gamma \in C, \beta \neq 0,-1,-2,\ldots$ and when $\Gamma(\gamma)$ is defined.

The result (2.5) follows from Example 1.5, and (2.2).

Note 2.1. From (2.5), we see that the Mellin transforms of the Mittag Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, defined by (1.44) and (1.45) respectively, are given by

$$ M\left\{E_\alpha(-z);s\right\} = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-as)}, 0 < \Re(s) < 1, \quad (2.6) $$

and

$$ M\{E_{\alpha,\beta}(-z);s\} = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-as)}, 0 < \Re(s) < 1. \quad (2.7) $$

In what follows, the H-functions considered satisfy the condition Eq. (1.6) and α, β, μ and δ have the values given in (1.13), (1.8), (1.9), and (1.10), respectively.

2.2.3 Mellin Transform of the H-Function

In view of the Mellin inversion formula (see, Titchmarsh (1986), Sect. 1.5) the Mellin transform of the H-function follows from the Definition 1.1). We have

$$ \int_0^\infty x^{s-1} H^{m,n}_{p,q}\left[ax \left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right] \mathrm{d}x $$

$$ = a^{-s} \frac{\left[\prod_{j=1}^{m}\Gamma(b_j+B_j s)\right]\left[\prod_{j=1}^{n}\Gamma(1-a_j-A_j s)\right]}{\left[\prod_{j=m+1}^{q}\Gamma(1-b_j-B_j s)\right]\left[\prod_{j=n+1}^{p}\Gamma(a_j+A_j s)\right]}, \quad (2.8) $$

where $a, s \in C; -\min_{1\le j\le m} \Re\left(\frac{b_j}{B_j}\right) < \Re(s) < \max_{1\le i\le n}\left[\frac{1-\Re(a_j)}{A_j}\right]$, $|\arg a| < \frac{1}{2}\pi\alpha$, $\alpha > 0$. Further, $\mu\Re(s) + \Re(\delta) < -1$, when $\alpha = 0, \arg a = 0$ and $a \neq 0$.

2.2.4 Mellin Transform of the G-Function

If we set $A_j = B_j = 1$, for all i and j and use the identity (1.112), we obtain the Mellin transform of the G-function.

$$\int_0^\infty x^{s-1} G_{p,q}^{m,n}\left[ax\Big|_{(b_q)}^{(a_p)}\right] \mathrm{d}x$$
$$= a^{-s} \frac{\left[\prod_{j=1}^{m} \Gamma(b_j + s)\right]\left[\prod_{j=1}^{n} \Gamma(1 - a_j - s)\right]}{\left[\prod_{j=m+1}^{q} \Gamma(1 - b_j - s)\right]\left[\prod_{j=n+1}^{p} \Gamma(a_j + s)\right]}, \tag{2.9}$$

where $a, s \in C, -\min_{1\le j\le m} \Re(b_j) < \Re(s) < 1 - \max_{1\le i\le n} \Re(a_i)$, $|\arg a| < \frac{1}{2}\pi c^*, c^* > 0$ and c^* is defined in (1.22).

2.2.5 Mellin Transform of the Wright Function

$$\int_0^\infty x^{s-1} \; {}_p\Psi_q\left[{}^{(a_p,A_p)}_{(b_q,B_q)}\Big| - ax\right] \mathrm{d}x = a^{-s} \frac{\Gamma(s)\left[\prod_{j=1}^{p} \Gamma(a_j - A_j s)\right]}{\left[\prod_{j=1}^{q} \Gamma(b_j - B_j s)\right]}, \tag{2.10}$$

where $s \in C, \Re(s) > 0, \Re(a_j - A_j s) > 0, j = 1, \cdots, p$, $|\arg a| < \frac{1}{2}\pi b$, $b = 1 + \sum_{i=1}^{p} A_i - \sum_{j-1}^{q} B_j$; $\mu > -1$ and μ is defined in (1.9).

2.2.6 Laplace Transform

Notation 2.3. $F(s) = L\{f(t); s\} = (Lf)(s)$: Laplace transform of $f(t)$ with parameter s

Notation 2.4. $L^{-1}\{F(s); t\}$: Inverse Laplace transform

Definition 2.2. The Laplace transform of a function $f(t)$, denoted by $F(s)$, is defined by the integral equation

$$F(s) = L\{f(t); s\} = (L\ f)(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t, \tag{2.11}$$

where $\Re(s) > 0$, which may be symbolically written as

$$F(s) = L\{f(t); s\} \text{ or } f(t) = L^{-1}\{F(s); t\},$$

provided that the function $f(t)$ is continuous for $t \geq 0$, it being tacitly assumed that the integral in (2.11) exists.

Definition 2.3. The inverse Laplace transform is given by the contour integral

$$f(t) = L^{-1}\{F(s); t\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{e}^{st} F(s) \mathrm{d}s. \tag{2.12}$$

2.2.7 Illustrative Examples

Example 2.4. Find the Laplace transform of the Mittag-Leffler function $x^{\beta-1} E_{\alpha,\beta}$ (ax^{α}).

Solution 2.3. We have

$$\begin{aligned} L\{t^{\beta-1} E_{\alpha,\beta}(ax^{\alpha}); s\} &= \int_0^{\infty} \mathrm{e}^{-sx} x^{\beta-1} E_{\alpha,\beta}(ax^{\alpha}) \mathrm{d}x \\ &= \int_0^{\infty} x^{\beta-1} \mathrm{e}^{-sx} \sum_{k=0}^{\infty} \frac{a^k x^{\alpha k}}{\Gamma(ak+\beta)} \mathrm{d}x \\ &= \sum_{k=0}^{\infty} \frac{a^k}{\Gamma(ak+\beta)} \int_0^{\infty} \mathrm{e}^{-sx} x^{ak+\beta-1} \mathrm{d}x \\ &= \frac{s^{\alpha-\beta}}{s^{\alpha}-a}, \Re(\alpha) > 0, \Re(\beta) > 0, |as^{-\alpha}| < 1. \end{aligned} \tag{2.13}$$

Note 2.2. We note from the above result that

$$L\{E_{\alpha}(ax^{\alpha}; s)\} = \frac{s^{\alpha-1}}{s^{\alpha}-a}, \tag{2.14}$$

where $a, s, \alpha \in C$, $\Re(\alpha) > 0, \Re(s) > 0$, and $|as^{-\alpha}| < 1$.

Example 2.5. Find the inverse Laplace transform of $s^{-\beta}(1 - as^{-\alpha})^{-\gamma}$.

Solution 2.4. We have

$$L^{-1}\{s^{-\beta}(1 - as^{-\alpha})^{-\gamma}; x\} = L^{-1}\left\{\sum_{k=0}^{\infty} \frac{(\gamma)_k a^k s^{-ak-\beta}}{k!}; x\right\}.$$

Applying the formula

$$L^{-1}\{s^{-\rho}; x\} = \frac{x^{\rho-1}}{\Gamma(\rho)}, \rho, s \in C, \Re(s) > 0, \Re(\rho) > 0, \tag{2.15}$$

the above line reduces to

$$x^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k(ax^{\alpha})^k}{\Gamma(\alpha k+\beta)k!} = x^{\beta-1}E^{\gamma}_{\alpha,\beta}(ax^{\alpha}), \tag{2.16}$$

where $\alpha, a, \beta, \gamma \epsilon C, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, |as^{-\alpha}| < 1$ is the generalized Mittag-Leffler function defined in (1.46).

Remark 2.1. When $\gamma = 1$, Example 2.5 gives the interesting transform pair

$$L^{-1}\{s^{-\beta}(1-as^{-\alpha})^{-1}; x\} = x^{\beta-1}E_{\alpha,\beta}(at^{\alpha}), \tag{2.17}$$

where $\alpha, \beta, a \epsilon C, \Re(\alpha) > 0, \Re(\beta) > 0$, and $|as^{-\alpha}| < 1$. For $\beta = 1$, (2.17) reduces to

$$L^{-1}\{s^{-1}(1-as^{-\alpha})^{-1}; x\} = E_{\alpha}(ax^{\alpha}), \tag{2.18}$$

where $a,\ \alpha \epsilon C, \Re(\alpha) > 0, |as^{-\alpha}| < 1$.

2.2.8 Laplace Transform of the *H*-Function

Let either $\alpha > 0, |\arg a| < \frac{1}{2}\pi\alpha$ or $\alpha = 0$ and $\Re(\delta) < -1$. Further assume that $\alpha > 0;\ \rho, \alpha, s \epsilon C,\ \sigma > 0$, satisfy the condition

$$\Re(\rho) + \sigma \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > 0 \text{ for } \alpha > 0 \text{ or } \alpha = 0, \mu \ge 0; \text{ and}$$

$$\Re(\rho) + \sigma \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j} + \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > 0,$$

for $\alpha = 0$ and $\mu < 0$. Then for $\Re(s) > 0$, there holds the formula

$$L\left\{x^{\rho-1}H^{m,n}_{p,q}\left[ax^{\sigma}\left|\begin{matrix}(a_p, A_p)\\ (b_q; b_q)\end{matrix}\right.\right]; s\right\} = s^{-\rho}H^{m,n+1}_{p+1,q}\left[as^{-\sigma}\left|\begin{matrix}(1-\rho,\sigma),(a_p,A_p)\\ (b_q, B_q)\end{matrix}\right.\right], \tag{2.19}$$

for $\Re(s) > 0, s \in C$. (2.19) can be established by virtue of the definition of the H-function (1.2) and the well-known gamma function formula.

2.2.9 Inverse Laplace Transform of the H-Function

Due to the importance and utility of inverse Laplace transforms of special functions in physical problems, we present the inverse Laplace transform of the H-function in this section.

By virtue of the cancelation law for the H-function (1.56), the result (2.19) can be written in the form

$$L\left\{x^{\rho-1}H_{p,q+1}^{m,n}\left[ax^{\sigma}\Big|_{(b_q;b_q),(1-\rho,\sigma)}^{(a_p,A_p)}\right];s\right\} = s^{-\rho}H_{p,q}^{m,n}\left[as^{-\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right], \quad (2.20)$$

If we use the property of the H-function from Mathai and Saxena (1978, p. 4, Eq. (1.38)) then the desired result follows:

$$L^{-1}\left\{s^{-\rho}H_{p,q}^{m,n}\left[as^{\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right];t\right\} = t^{\rho-1}H_{p+1,q}^{m,n}\left[at^{-\sigma}\Big|_{(b_q,B_q)}^{(a_p,A_p),(\rho,\sigma)}\right], \quad (2.21)$$

where $\rho, a, s \in C, \Re(s) > 0, \sigma > 0, \Re(\rho)+\sigma \max_{1\leq i\leq n}\left[\frac{1}{A_i} - \frac{\Re(a_i)}{A_i}\right] > 0, |\arg a| < \frac{1}{2}\pi\theta, \theta = \alpha - \sigma$. Two interesting special cases of (2.21), which are applicable in fractional diffusion problems, are given below. If we use the identity

$$H_{0,1}^{1,0}\left[x\Big|_{(\alpha,1)}\right] = x^{\alpha}\mathrm{e}^{-x}, \quad (2.22)$$

we obtain

$$L^{-1}\{s^{-\rho}\exp(-as^{\sigma});t\} = t^{\rho-1}H_{1,1}^{1,0}\left[at^{-\sigma}\Big|_{(0,1)}^{(\rho,\sigma)}\right], \quad (2.23)$$

where $\Re(s) > 0, \Re(a) > 0, \sigma > 0$. Further, if we employ the identity

$$H_{0,2}^{2,0}\left[x\Big|_{(\frac{\nu}{2},1),(-\frac{\nu}{2},1)}\right] = 2K_{\nu}(2x^{\frac{1}{2}}), \quad (2.24)$$

we obtain

$$2L^{-1}\{s^{-\rho}K_{\nu}(as^{\sigma});x\} = x^{\rho-1}H_{1,2}^{2,0}\left[\frac{a^2x^{-2\sigma}}{4}\Big|_{(\frac{\nu}{2},1),(-\frac{\nu}{2},1)}^{(\rho,2\sigma)}\right], \quad (2.25)$$

where $\Re(s) > 0, \Re(a^2) > 0, \sigma > 0$, and $K_{\nu}(x)$ is the modified Bessel function of the third kind or Macdonald function.

Remark 2.2. It will not be out of place to mention here that one-sided Lévy stable density can be obtained from the above result by virtue of the identity (Mathai and Saxena 1973a)

$$K_{\pm\frac{1}{2}}(x) = \left[\frac{\pi}{2x}\right]^{\frac{1}{2}} \mathrm{e}^{-x}, \tag{2.26}$$

and can be conveniently expressed in terms of the Laplace transform

$$\int_0^\infty \mathrm{e}^{-sx}\Phi_\rho(x)\mathrm{d}x = \exp(-s^\rho), \rho, s \in C,\ \Re(s) > 0, \Re(\rho) > 0, \tag{2.27}$$

where

$$\Phi_\rho(x) = \frac{1}{\rho} H_{1,1}^{1,0}\left[\frac{1}{x}\Bigg|_{\left(\frac{1}{\rho},\frac{1}{\rho}\right)}^{(1,1)}\right],\ \rho > 0. \tag{2.28}$$

This result is obtained earlier by Schneider and Wyss (1989) by following a different procedure. Asymptotic expansion of $\Phi_\alpha(x)$ is given by Schneider (1986).

2.2.10 Laplace Transform of the G-Function

In what follows, the G-functions involved satisfy the existence conditions. When $A_i = B_j = 1$ for all i and j, the H-function reduces to a G-function and consequently we arrive at the following result:

$$L\left\{x^{\rho-1}G_{p,q}^{m,n}\left[ax^\sigma\Bigg|_{b_q}^{a_p}\right];s\right\} = s^{-\rho}H_{p+1,q}^{m,n+1}\left[as^{-\sigma}\Bigg|_{(b_q,1)}^{(1-\rho,\sigma),(a_p,1)}\right], \tag{2.29}$$

where $\rho, s \in C, \Re(s) > 0, \sigma > 0, \Re(\rho) + \sigma \min_{1\le j\le m}\Re(b_j) > 0, |\arg\alpha| < \frac{1}{2}c^*, c^* > 0, c^*$ is defined in (1.21).

If we set $\sigma = \frac{k}{\lambda}, k, \lambda \in N$ in (2.29), we arrive at a result given by Saxena (1960, p. 402):

$$\begin{aligned} &L\left\{x^{\rho-1}G_{p,q}^{m,n}\left[ax^{\frac{k}{\lambda}}\Bigg|_{b_q}^{a_p}\right];s\right\} \\ &= s^{-\rho}(2\pi)^{(1-\lambda)c^*+\frac{1}{2}(1-k)}\lambda^{\delta+1}k^{\rho-\frac{1}{2}} \\ &\times H_{\lambda p+k,\lambda q}^{\lambda m,\lambda n+k}\left[\frac{k^k a^\lambda s^{-k}}{\lambda^{(q-p)\lambda}}\Bigg|_{\Delta(\lambda;b_1),\ldots,\Delta(\lambda;b_q)}^{\Delta(k;1-\rho),\Delta(\lambda,a_1),\ldots,\Delta(\lambda;a_p)}\right], \end{aligned} \tag{2.30}$$

where $\rho, s \in C, \Re(s) > 0, \Re(\rho) + \left(\frac{k}{\lambda}\right)\min_{1\le j\le m}\Re(b_j) > 0, |\arg a| < \frac{1}{2}\pi c^*, c^* > 0, c^*$ is defined in equation (1.21) and the existence conditions of the G-function are satisfied. Here, $\Delta(k; b)$ represents the sequence

$$\frac{b}{k}, \frac{b+1}{k}, \ldots, \frac{b+k-1}{k}, k \in N.$$

Several special cases of the general result (2.30) can be obtained by using the tables of the special cases of the G-function (Mathai and Saxena 1973a; Mathai 1993c) but for brevity one interesting case is presented here, associated with the Whittaker function, given by Saxena (1960, p. 404, Eq. (15))

$$L\left\{x^{\rho-1}\exp\left(-\frac{1}{2}ax^{-\frac{k}{\lambda}}\right)W_{\tau,\nu}(ax^{-\frac{k}{\lambda}});s\right\} = s^{-\rho}(2\pi)^{\frac{1}{2}(2-k-\lambda)}\lambda^{\tau+\frac{1}{2}}k^{\rho-\frac{1}{2}} \times G^{2\lambda+k,0}_{\lambda,2\lambda+k}\left[\frac{a^{\lambda}s^{k}}{\lambda^{\lambda}k^{k}}\Bigg|^{\Delta(\lambda;1-\tau)}_{\Delta(2\lambda;1\pm 2\nu),\Delta(k;\rho)}\right], \tag{2.31}$$

where $\rho, s \in C, \Re(a) > 0, \Re(s) > 0$. One interesting particular case of (2.31) can be obtained by using the identity

$$W_{0,\pm\frac{1}{2}}(x) = \exp\left(-\frac{1}{2}x\right).$$

That is,

$$L\left\{x^{\rho-1}\exp(-ax^{-\frac{k}{\lambda}});s\right\} = s^{-\rho}(2\pi)^{\frac{1}{2}(2-k-\lambda)}k^{\rho-\frac{1}{2}}\lambda^{\frac{1}{2}} \times G^{\lambda+k,0}_{0,\lambda+k}\left[\frac{a^{\lambda}s^{k}}{\lambda^{\lambda}k^{k}}\Bigg|_{\Delta(\lambda;0),\Delta(k;\rho)}\right], \tag{2.32}$$

where $\rho, s \in C, \Re(a) > 0, \Re(s) > 0$.

Remark 2.3. The result (2.32) is very useful in problems of physics. Regarding its application in nuclear and neutrino astrophysics, one can refer to the monograph of Mathai and Haubold (1988). An alternative derivation of this result based on statistical techniques is given by Mathai (1971).

2.2.11 K-Transform

Notation 2.5. $R_\nu\{f(x);p\}$: K-Transform

Definition 2.4. The transform defined by the following integral equation

$$R_\nu\{f(x);p\} = g(p;\nu) = \int_0^\infty (px)^{\frac{1}{2}}K_\nu(px)f(x)\mathrm{d}x, \tag{2.33}$$

is called the K-transform with p as a complex parameter.

This transform was defined by Meijer (1940) who obtained its inversion formula and representation theorems. Its inversion formula is given by

$$G(p) = \frac{1}{\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (px)^{\frac{1}{2}} I_v(xp) g(p) \mathrm{d}p, \tag{2.34}$$

where $I_v(x)$ is Bessel function of the first kind defined by

$$I_v(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{v+2k}}{k!\Gamma(v+k+1)}. \tag{2.35}$$

2.2.12 K-Transform of the H-Function

Let us assume that either $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0$ and $\Re(\delta) < -1$. Further assume that $\alpha > 0; \rho, v, a, b \in C, \sigma > 0$ satisfy the condition

$$\Re(\rho) + |\Re(v)| + \sigma \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j} \right] > 0,$$

for $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$; and

$$\Re(\rho) + |\Re(v)| + \sigma \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] > 0,$$

for $\alpha = 0$ and $\mu < 0$. Then for $\Re(a) > 0, \sigma > 0$ there holds the formula

$$\int_0^\infty x^{\rho-1} K_v(ax) H_{p,q}^{m,n} \left[bx^\sigma \middle|_{(b_q,B_q)}^{(a_p,A_p)} \right] \mathrm{d}x$$
$$= 2^{\rho-2} a^{-\rho} H_{p+2,q}^{m,n+2} \left[b \left(\frac{2}{a}\right)^\sigma \middle|_{(b_q,B_q)}^{(1-\frac{1}{2}(\rho-v),\frac{\sigma}{2}),(1-\frac{1}{2}(\rho+v),\frac{\sigma}{2}),(a_p,A_p)} \right]. \tag{2.36}$$

The result (2.36) can be computed directly from the definition of the H-function (1.2) if we use the formula (Mathai and Saxena 1973, p. 78)

$$\int_0^\infty x^{\rho-1} K_v(ax) \mathrm{d}x = 2^{\rho-2} a^{-\rho} \Gamma\left(\frac{\rho \pm v}{2}\right), \tag{2.37}$$

where $\Re(s) > |\Re(v)|, \Re(a) > 0$. If we apply the definition (1.2) of the H-function to the given integral then we have

$$\int_0^\infty x^{\rho-1} K_\nu(ax) H_{p,q}^{m,n}\left[bx^\sigma \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] \mathrm{d}x$$

$$= \frac{1}{2\pi i}\int_{L_{i\gamma\infty}} \Theta(s) b^{-s} \int_0^\infty x^{\rho-s\sigma-1} K_\nu(ax)\mathrm{d}x\mathrm{d}s$$

$$= 2^{\rho-2} a^{-\rho} \frac{1}{2\pi i}\int_{L_{i\gamma\infty}} \Gamma\left(\frac{\rho \pm \nu - \sigma s}{2}\right) \Theta(s) b^{-s} \left(\frac{2}{a}\right)^{-\sigma s} \mathrm{d}s,$$

and the result (2.36) readily follows from the definition of the H-function (1.2).

Remark 2.4. When $\nu = \pm\frac{1}{2}$ in (2.36) then by virtue of the identity (2.26) one can obtain the Laplace transform of the H-function with argument $bx^\sigma, \sigma > 0$.

2.2.13 Varma Transform

Notation 2.6. $V(f,k,m,s)$: Varma transform

Definition 2.5. Varma transform is defined by the integral equation

$$V(f,k,m;s) = \int_0^\infty (sx)^{m-\frac{1}{2}} \exp\left(-\frac{1}{2}sx\right) W_{k,m}(sx) f(x)\mathrm{d}x, \quad \Re(s) > 0, \quad (2.38)$$

where $W_{k,m}$ represents a Whittaker function, defined by

$$W_{k,m}(z) = \sum_{m,-m} \frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-k-m\right)} M_{k,m}(z), \quad (2.39)$$

where the summation symbol indicates that the expression following it, a similar expression with m replaced by $-m$ is to be added. For the definition of $M_{k,m}(z)$ see, Sect. 1.8.1. This transform is introduced by Varma (1951), who gave some inversion formulae for this transform.

2.2.14 Varma Transform of the H-Function

Let $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0$ and $\Re(\delta) < -1$; further, $\nu, a, k, b\rho \in C, \sigma > 0$,

$$\Re(\rho) + |\Re(\nu)| + \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > -\frac{1}{2},$$

for $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$ and

$$\Re(\rho) + |\Re(\nu)| + \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{3}}{\mu}\right] > -\frac{1}{2},$$

for $\alpha = 0$ and $\mu < 0$ then for $\Re(a) > 0$, the following result holds:

$$\int_0^\infty x^{\rho-1} \exp\left(-\frac{1}{2}ax\right) W_{k,\nu}(ax) H_{p,q}^{m,n}\left[bx^\sigma \middle|_{(b_q,B_q)}^{(a_p,A_p)}\right] \mathrm{d}x$$

$$= a^{-\rho} H_{p+2,q+1}^{m,n+2}\left[\frac{b}{a^\sigma} \middle|_{(b_q,B_q),(k-\rho,\sigma)}^{(\frac{1}{2}-\nu-\rho,\sigma),(\frac{1}{2}+\nu-\rho,\sigma),(a_p,A_p)}\right], \tag{2.40}$$

which can be computed directly from the definition of the H-function (1.2) and from the following formula (Mathai and Saxena 1973, p. 79):

$$\int_0^\infty x^{\rho-1} \exp\left(-\frac{1}{2}ax\right) W_{k,\nu}(ax)\mathrm{d}x = a^{-\rho}\frac{\Gamma(\rho+\nu+\frac{1}{2})\Gamma(\rho-\nu+\frac{1}{2})}{\Gamma(1-k+\rho)}, \tag{2.41}$$

where $\Re(a) > 0, \Re(\rho \pm \nu) > -\frac{1}{2}$.

Remark 2.5. It is interesting to observe that for $k = \nu + \frac{1}{2}$ the Varma transform defined by (2.38) reduces to the Laplace transform (2.11) by virtue of the identity

$$W_{\nu+\frac{1}{2},\pm\nu}(x) = x^{\nu+\frac{1}{2}} \exp\left(-\frac{1}{2}x\right). \tag{2.42}$$

Consequently the Laplace transform of the H-function (2.19) can be derived from the result (2.40) by taking $k = \nu+\frac{1}{2}$. Certain properties of the Varma transform involving Meijer's G-functions and Whittaker functions are investigated by Saxena in a series of papers in Saxena (1960, 1962, 1964).

2.2.15 *Hankel Transform*

Notation 2.7. $H_\nu\{f(x);\rho\}$: Hankel transform of order ν of $f(x)$.

Definition 2.6. The Hankel transform of a function $f(x)$, denoted by $g(p;\nu)$ or in short by simply $g(p)$ is defined as

$$g(p;\nu) = \int_0^\infty (px)^{\frac{1}{2}} J_\nu(px) f(x)\mathrm{d}x,\ p > 0. \tag{2.43}$$

The inverse Hankel transform is given by

$$f(x) = \int_0^\infty (xp)^{\frac{1}{2}} J_\nu(xp) g(p)\mathrm{d}p,\ \ \Re(\nu) > -1. \tag{2.44}$$

Remark 2.6. This transform is self-reciprocal. It is used in solving problems of applied mathematics and physical sciences.

2.2.16 Hankel Transform of the H-Function

Suppose that $\alpha > 0$ or $\alpha = \mu = 0$ and $\Re(\delta) < -1$. Then if $\rho, v, b \in C, \sigma > 0$ satisfy the conditions

$$\Re(\rho) + \Re(v) + \sigma \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j} \right] > -1,$$

and

$$\Re(\rho) + \sigma \max_{1 \le j \le n} \left[\frac{1}{A_j} - \frac{\Re(a_j)}{A_j} \right] < \frac{3}{2},$$

for $\Re(v) > -\frac{1}{2}$. Then for $a, b > 0$ there holds the formula

$$\int_0^\infty x^{\rho-1} J_v(ax) H_{p,q}^{m,n} \left[bx^\sigma \middle|_{(b_q,B_q)}^{(a_p,A_p)} \right] \mathrm{d}x$$
$$= \frac{2^{\rho-1}}{a^\rho} H_{p+2,q}^{m,n+1} \left[b \left(\frac{2}{a} \right)^\sigma \middle|_{(b_q,B_q)}^{(1-\frac{(\rho+v)}{2}, \frac{\sigma}{2}),(a_p,A_p),(1-\frac{(\rho-v)}{2}, \frac{\sigma}{2})} \right]. \tag{2.45}$$

The result (2.45) can be established with the help of the definition of the H-function (1.2) and the formula

$$\int_0^\infty x^{\lambda-1} J_v(ax) \mathrm{d}x = 2^{\lambda-1} a^{-\lambda} \frac{\Gamma\left(\frac{\lambda+v}{2}\right)}{\Gamma\left(1 + \frac{v-\lambda}{2}\right)}, \tag{2.46}$$

where $a > 0, -\Re(v) < \Re(\lambda) < \frac{3}{2}$. If we use the definition of the H-function (1.2) and the result (2.46), then

$$\int_0^\infty x^{\rho-1} J_v(ax) H_{p,q}^{m,n} \left[bx^\sigma \middle|_{(b_q,B_q)}^{(a_p,A_p)} \right] \mathrm{d}x$$
$$= \frac{1}{2\pi i} \int_{L_{i\gamma\infty}} \Theta(s) b^{-s} \int_0^\infty x^{\rho-\sigma s-1} J_v(ax) \mathrm{d}x \mathrm{d}s$$
$$= 2^{\rho-1} a^{-\rho} \frac{1}{2\pi i} \int_{L_{i\gamma\infty}} \Theta(s) \frac{\Gamma\left(\frac{\rho+v-\sigma s}{2}\right)}{\Gamma\left(1 + \frac{v-\rho+\sigma s}{2}\right)} b^{-s} \left(\frac{2}{a} \right)^{-\sigma s} \mathrm{d}s.$$

Interpreting the above result with the help of (1.2), the result (2.45) readily follows. When $v = \pm\frac{1}{2}$ then by using the identities

$$J_{\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x} \right)} \sin(x), \tag{2.47}$$

and

$$J_{-\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos(x), \tag{2.48}$$

we arrive at the following results which provide the sine and cosine transforms of the H-function

$$\int_0^\infty x^{\rho-1}\sin(ax)H_{p,q}^{m,n}\left[bx^\sigma\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= \frac{2^{\rho-1}\sqrt{\pi}}{a^\rho}H_{p+2,q}^{m,n+1}\left[b\left(\frac{2}{a}\right)^\sigma\Big|_{(b_q,B_q)}^{\left(\frac{(1-\rho)}{2},\frac{\sigma}{2}\right),(a_p,A_p),\left(\frac{(2-\rho)}{2},\frac{\sigma}{2}\right)}\right], \tag{2.49}$$

where $a, \alpha, \sigma > 0, \rho, b \in C$; $|\arg b| < \frac{1}{2}\pi\alpha$;

$$\Re(\rho) + \sigma \min_{1\le j\le m}\Re\left(\frac{b_j}{B_j}\right) > -1; \Re(\rho) + \sigma \max_{1\le j\le n}[\frac{(a_j-1)}{A_j}] < 1.$$

$$\int_0^\infty x^{\rho-1}\cos(ax)H_{p,q}^{m,n}\left[bx^\sigma\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x = \frac{2^{\rho-1}\sqrt{\pi}}{a^\rho}$$
$$\times H_{p+2,q}^{m,n+1}\left[b\left(\frac{2}{a}\right)^\sigma\Big|_{(b_q,B_q)}^{\left(\frac{(2-\rho)}{2},\frac{\sigma}{2}\right),(a_p,A_p),\left(\frac{1-\rho}{2},\frac{\sigma}{2}\right)}\right], \tag{2.50}$$

where $a, \alpha, \sigma > 0, \rho, b \in C$; $|\arg b| < \frac{1}{2}\pi\alpha$;

$$\Re(\rho) + \sigma \min_{1\le j\le m}\Re\left(\frac{b_j}{B_j}\right) > 0; \Re(\rho) + \sigma \max_{1\le j\le n}\left[\frac{(a_j-1)}{A_j}\right] < 1.$$

2.2.17 Euler Transform of the H-Function

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}H_{p,q}^{m,n}\left[bx^k(t-x)^\tau\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= t^{\rho+\sigma-1}H_{p+2,q+1}^{m,n+2}\left[bt^{k+\tau}\Big|_{(b_q,B_q),(1-\rho-\sigma,k+\tau)}^{(1-\rho,k),(1-\sigma,\tau),(a_p,A_p)}\right], \tag{2.51}$$

Let $\alpha > 0$ and $|\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha > 0, \Re(\delta) < -1$. The result (2.51) holds provided the parameters $\rho, \sigma, b \in C, k$ and $\tau > 0$ satisfy

$$\Re(\rho) + k \min_{1 \le j \le m}\left[\Re\left(\frac{b_j}{B_j}\right)\right] > 0,\ \Re(\rho) + \tau \min_{1 \le j \le m}\left[\Re\left(\frac{b_j}{B_j}\right)\right] > 0,$$

for $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$, and

$$\Re(\rho) + k \min_{1 \le j \le m} \Re\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu}\right] > 0, \Re(\rho)$$
$$+\tau \min_{1 \le j \le m} \Re\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu}\right] > 0,$$

for $\alpha = 0$ and $\mu < 0$. The result (2.51) can be proved in the same way if we use the well-known beta function formula

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\alpha, \beta). \tag{2.52}$$

where $\min\{\Re(\alpha), \Re(\beta)\} > 0$. The result (2.51) has been given by Goyal (1969). As $\tau \to 0$ in (2.51), it yields the Euler transform of the H-function:

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} H_{p,q}^{m,n}\left[bx^k \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= t^{\rho+\sigma-1}\Gamma(\sigma) H_{p+1,q+1}^{m,n+1}\left[bt^k \Big|_{(b_q,B_q),(1-\rho-\sigma,k)}^{(1-\rho,k),(a_p,A_p)}\right], \tag{2.53}$$

which holds under the same condition as given with the result (2.51) with $\tau = 0$. By an obvious change of variable in (2.53) we arrive at its companion the integral

$$\int_t^\infty x^{\rho-1}(x-t)^{\sigma-1} H_{p,q}^{m,n}\left[bx^k \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= t^{\rho+\sigma-1}\Gamma(\sigma) H_{p+1,q+1}^{m+1,n}\left[bt^k \Big|_{(1-\rho-\sigma,k),(b_q,B_q)}^{(a_p,A_p),(1-\rho,k)}\right]; \tag{2.54}$$

which holds for $\rho, b, \sigma \in C, \Re(\sigma) > 0, \rho \in C$ and $k > 0$; either $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \Re(\delta) < -1$ and the following conditions are satisfied:

$$k \max_{1 \le j \le n}\left[\frac{\Re(a_j) - 1}{A_j}\right] + \Re(\rho) + \Re(\sigma) < 1,$$

for $\alpha > 0, |\arg b| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \leq 0$; and

$$k \max_{1\leq i\leq n}\left[\frac{\Re(a_i)-1}{A_i}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] + \Re(\rho) + \Re(\sigma) < 1,$$

for $\alpha = 0$ and $\mu < 0$.

2.3 Mellin Transform of the Product of Two H-Functions

$$\int_0^\infty x^{s-1} H_{p,q}^{m,n}\left[zx^\sigma\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] H_{p_1,q_1}^{m_1,n_1}\left[\eta x\Big|_{(e_{q_1},E_{q_1})}^{(d_{p_1},D_{p_1})}\right]\mathrm{d}x$$
$$= \eta^{-s} H_{p+q_1,q+p_1}^{m+n_1,n+m_1}\left[z\eta^{-\sigma}\Big|_{(b_m,B_m),(1-d_{p_1}-sD_{p_1},\sigma D_{p_1}),(b_{m+1},B_{m+1}),\ldots,(b_q,B_q)}^{(1-e_{q_1}-sE_{q_1},\sigma E_{q_1}),(a_p,A_p)}\right],$$

where $\eta, s, z, \in C, \sigma > 0, \alpha > 0, \mu > 0, |\arg z| < \frac{1}{2}\pi\alpha, |\arg\eta| < \frac{1}{2}\pi k, k > 0$;

$$k = \sum_{i=1}^{n_1} D_i - \sum_{i=n+1}^{p_1} D_i - \sum_{i=1}^{m_1} E_i - \sum_{i=m+1}^{q_1} E_i > 0,$$

$$-\sigma \min_{1\leq h\leq m}\left[\frac{\Re(b_h)}{B_h}\right] - \min_{1\leq j\leq m_1}\left[\frac{\Re(e_j)}{E_j}\right] < \Re(s)$$
$$< \sigma \max_{1\leq j\leq n}\left[\frac{1-\Re(a_j)}{A_j}\right] + \max_{1\leq j\leq n_1}\left[\frac{1-\Re(d_j)}{D_j}\right].$$

Remark 2.7. For the applications of this result in the theory of statistical distributions, see the work of Mathai and Saxena (1969). It can be established with the help of the definition of the H-function and the result (2.8).

2.3.1 Eulerian Integrals for the H-Function

In this section, certain Eulerian integrals for the H-functions will be evaluated in terms of the H-function of two variables. In order to present the results, we need the definition of the H-function of two complex variables introduced earlier by Mittal and Gupta (1972). The analysis developed here is based on the work of Saxena and Nishimoto (1994), Saigo and Saxena (1998). To unify and extend the existing results on Riemann-Liouville fractional integrals available in the literature, certain new Eulerian integrals associated with the H-function are investigated by Saxena and Nishimoto (1994). The importance of the derived results lies in the fact that a table of Riemann-Liouville fractional integrals can be prepared by using the tables of the special cases of the H-function given in the monograph by Mathai and Saxena (1978, pp. 145–151). Further special cases of these integrals can be used in studying statistical density functions.

Notation 2.8.

$$H\begin{bmatrix} x \\ y \end{bmatrix} = H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix}(a_i;\alpha_i,A_i)_1,p_1:(c_i,\gamma_i)_1,p_2;(e_i,E_i)1,p_3 \\ (b_j;\beta_j,B_j)_1,q_1:(d_j,\delta_j)_1,q_2;(f_j,F_j)1,q_3\end{matrix}\right]: \quad (2.55)$$

The H-function of two variables.

Definition 2.7. (Srivastava et al. 1982, pp. 82–83; also see Srivastava and Panda 1976)

$$H\begin{bmatrix} x \\ y \end{bmatrix} = H^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix}(a_i;\alpha_i,A_i)1,p_1:(c_i,\gamma_i)1,p_2;(e_i,E_i)1,p_3 \\ (b_j;\beta_j,B_j)1,q_1:(d_j,\delta_j)1,q_2;(f_j,F_j)1,q_3\end{matrix}\right] \quad (2.56)$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi(s,t)\phi_1(s)\phi_2(t)x^s y^t \mathrm{d}s\mathrm{d}t, \quad (2.57)$$

where x and y are not equal to zero. For convenience the parameters $(a_i;\alpha_i,A_i)_1,p_1$ and $(c_i,\gamma_i)_1,p_2$ will abbreviate the sequence of the parameters $(a_1;\alpha_1,A_1),\ldots,(a_{p_1};\alpha_{p_1},A_{p_1})$ and $(c_1,\gamma_1),\ldots,(c_{p_2},\gamma_{p_2})$ respectively, and similar meanings hold for the other parameters $(b_j;\beta_j,B_j)_1,q_1$ and $(d_j,\delta_j)_1,q_2$, etc. Here

$$\phi(s,t) = \frac{\prod_{i=1}^{n_1}\Gamma(1-a_i+\alpha_i s+A_i t)}{[\prod_{i=n_1+1}^{p_1}\Gamma(a_i-\alpha_i s-A_i t)][\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)]} \quad (2.58)$$

$$\phi_1(s) = \frac{[\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)][\prod_{i=1}^{n_2}\Gamma(1-c_i+\gamma_i s)]}{[\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)][\prod_{i=n_2+1}^{p_2}\Gamma(c_i-\gamma_i s)]}, \quad (2.59)$$

$$\phi_2(t) = \frac{[\prod_{j=1}^{m_3}\Gamma(f_j-F_j t)][\prod_{i=1}^{n_3}\Gamma(1-e_i+E_i t)]}{[\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j t)][\prod_{i=n_3+1}^{p_3}\Gamma(e_i-E_i t)]}. \quad (2.60)$$

It is assumed that all the poles of the integrand are simple. An empty product is interpreted as unity. Further, we suppose that all the parameters a_i, b_j, c_i, d_j, e_i and f_j be complex numbers and associated coefficients $\alpha_i, A_i, \beta_j, B_j, \gamma_i, \delta_j, E_i$ and F_j be real and positive for the standardization purposes, such that

$$\rho_1 = \sum_{i=1}^{p_1} \alpha_i + \sum_{i=1}^{p_2} \gamma_i - \sum_{j=1}^{q_1} \beta_j - \sum_{j=1}^{q_2} \delta_j \le 0, \tag{2.61}$$

$$\rho_2 = \sum_{i=1}^{p_1} A_i + \sum_{i=1}^{p_2} E_i - \sum_{j=1}^{q_1} B_j - \sum_{j=1}^{q_2} F_j \le 0, \tag{2.62}$$

$$\Omega_1 = - \sum_{i=n_1+1}^{p_1} \alpha_i - \sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{m_2} \delta_j - \sum_{j=m_2+1}^{p_2} \delta_j + \sum_{i=1}^{n_2} \gamma_i - \sum_{i=n_2+1}^{p_2} \gamma_i > 0, \tag{2.63}$$

$$\Omega_2 = - \sum_{i=n_1+1}^{p_1} A_i - \sum_{j=1}^{q_1} B_j + \sum_{j=1}^{m_3} F_j - \sum_{j=m_3+1}^{p_3} F_j + \sum_{i=1}^{n_3} E_i - \sum_{i=n_3+1}^{p_3} E_i > 0. \tag{2.64}$$

It can be seen that the contour integral (2.56) converges absolutely under the conditions (2.61)–(2.64) and defines an analytic function of two complex variables x and y inside the sectors given by

$$|\arg x| < \frac{1}{2}\pi\Omega_1 \quad \text{and} \quad |\arg y| < \frac{1}{2}\pi\Omega_2, \tag{2.65}$$

the points $x = 0$ and $y = 0$ being tacitly excluded.

The conditions given here from (2.61) to (2.65) are the sufficient conditions for the convergence of the Mellin–Barnes double integral (2.57), for details the reader is referred to the book by Srivastava et al. (1982).

Remark 2.8. In a series of papers Buschman (1978) has given a detailed analysis of the sufficient conditions for the convergence of H-function of two variables of a general character. Simple criteria are provided for the determination of the convergence of certain double Mellin–Barnes integrals in terms of their parameters by Hai et al. (1992). A systematic and comprehensive account of the double Mellin–Barnes type integrals or rather H-function of two variables can be found in the book by Hai and Yakubovich (1992).

2.3.2 Fractional Integration of a H-Function

Theorem 2.1. *If* $\min\{\Re(\alpha), \Re(\beta)\} > 0, \alpha, \beta, \gamma \in C, \mu \ge 0, b \ne a, \left|\frac{(a-b)c}{ac+d}\right| < 1,$ $|\arg(d+cb)/(d+ca)| < \pi; \phi, \eta > 0, |\arg k| < \frac{1}{2}\pi\phi$, *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} H_{p,q}^{m,n}\left[k(cx+d)^{-\eta}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H_{1,0:p,q+1;1,2}^{0,1:m,n;1,1}\left[\begin{matrix}\frac{k}{(ac+d)^{\eta}}\\ \frac{c(b-a)}{(ac+d)}\end{matrix}\Bigg|\begin{matrix}(1+\gamma:\eta,1):-\\ -\\ (a_1,A_1),\ldots,(a_p,A_p);(1-\alpha,1)\\ (b_1,B_1),\ldots,(b_a,B_q),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right], \tag{2.66}$$

$$\phi = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0.$$

Proof 2.1. To establish (2.66) we express the H-function in terms of the contour integral (1.2), interchange the order of integration, which is permissible due to absolute convergence of the integrals involved in the process, and evaluate the x-integral by means of the integral (Prudnikov et al. 1986, p. 301):

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma}\mathrm{d}x$$
$$= (ac+d)^{\gamma}(b-a)^{\alpha+\beta-1}B(\alpha,\beta)\,{}_2F_1\left(\alpha,-\gamma;\alpha+\beta;\frac{c(a-b)}{(ac+d)}\right), \tag{2.67}$$

where $\Re(\alpha)>0, \Re(\beta)>0, |c(a-b)/(ac+d)|<1, |\arg(((d+bc)/(ac+d))|<1$, the integral transforms into the form

$$(b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\frac{1}{2\pi i}\int_L \Theta(s)(ac+d)^{s\eta}{}_2F_1\left(\alpha,-\gamma-s\eta;\alpha+\beta;\frac{c(a-b)}{ac+d}\right)\mathrm{d}s.$$

If we now employ the formula

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma_i\infty}\frac{\Gamma(-s)\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}(-z)^s\mathrm{d}s, \tag{2.68}$$

where $|\arg(-z)| < \pi$ and the path of integration separates the poles at $s = 0,1,2,\ldots$ from the poles at $s=-a-n, s=-b-n, n=0,1,\ldots$, the result readily follows from (2.57). □

On applying the identity (1.58), we obtain the following theorem:

Theorem 2.2. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0$, $\alpha,\beta,\gamma \in C, \mu \geq 0, b \neq a$, $|[(a-b)c]/(ac+d)| < 1$, $|\arg(d+cb)/(d+ca)| < \pi$, $\phi,\eta > 0$, $|\arg k| < \frac{1}{2}\pi\phi$, *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} H_{p,q}^{m,n}\left[k(cx+d)^{\eta}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H_{1,0:q,p+1;1,2}^{0,1:n,m;1,1}\left[\begin{matrix} k^{-1}(ac+d)^{-\eta} \\ \frac{c(b-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):- \\ - \\ (1-b_1,B_1),\ldots,(1-b_q,B_q);(1-\alpha,1) \\ (1-a_1,A_1),\ldots,(1-a_p,A_p),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right]. \tag{2.69}$$

When $A_i = B_j = 1$ for all i and j, then we obtain the following corollaries from the above theorems, involving Meijer G-function.

Corollary 2.1. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0, \alpha,\beta,\gamma \in C, \mu \geq 0, b \neq a$, $|[(a-b)c]/(ac+d)| < 1$, $|\arg(d+cb)/(d+ca)| < \pi$; $c^* > 0, |\arg k| < \frac{1}{2}\pi c^*$, *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} G_{p,q}^{m,n}\left[k(cx+d)^{-\eta}\Big|_{b_q}^{a_p}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H_{1,0:p,q+1;1,2}^{0,1:m,n;1,1}\left[\begin{matrix} \frac{k}{(ac+d)^{\eta}} \\ \frac{c(b-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):- \\ - \\ (a_1,1),\ldots,(a_p,1);(1-\alpha,1) \\ (b_1,1),\ldots,(b_q,1),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right]. \tag{2.70}$$

Corollary 2.2. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0, \alpha,\beta,\gamma \in C, \mu \geq 0, b \neq a$, $|[(a-b)c]/(ac+d)| < 1$, $|\arg(d+cb)/(d+ca)| < \pi$; $c^* > 0, |\arg k| < \frac{1}{2}\pi c^*$, *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} G_{p,q}^{m,n}\left[k(cx+d)^{\eta}\Big|_{(b_q)}^{(a_p)}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H_{1,0:q,p+1;1,2}^{0,1:n,m;1,1}\left[\begin{matrix} \frac{k^{-1}}{(ac+d)^{\eta}} \\ \frac{c(b-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):- \\ - \\ (1-b_1,1),\ldots,(1-b_q,1):(1-\alpha,1) \\ (1-a_1,1),\ldots,(1-a_p,1),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right], \tag{2.71}$$

where c^* *is defined in (1.22).*

On the other hand, if we use the identity (Mathai and Saxena 1978, p. 4) then we arrive at the following corollaries associated with Wright generalized hypergeometric functions.

Corollary 2.3. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0, \alpha,\beta,\gamma \in C, \mu \geq 0, b \neq a, |[c(a-b)]/(ac+d)| < 1$, $|\arg(d+cb)/(d+ca)| < \pi$; $\Omega = \mu+1 > 0, \eta > 0$ *and* $|\arg k| < \frac{1}{2}\pi\Omega$ *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma}\,{}_p\Psi_q\left[-k(cx+d)^{-\eta}\Big|_{(b_1,B_1),\dots,(b_q,B_q)}^{(a_1,A_1),\dots,(a_p,A_p)}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H_{1,0:p,q+1;1,2}^{0,1:1,p;1,1}\left[\begin{matrix}\frac{k}{(ac+d)^{\eta}}\\ \frac{c(b-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):-\\ -\\ (1-a_1,A_1),\dots,(1-a_p,A_p):(1-\alpha,1)\\ (0,1),(1-b_1,B_1),\dots,(1-b_q,B_q),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right], \quad (2.72)$$

where μ is defined in (1.9).

Corollary 2.4. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0, \alpha,\beta,\gamma \in C, \mu \geq 0, k \neq 0, b \neq a,$ $|[c(a-b)]/(ac+d)| < 1$, $|\arg(d+cb)/(d+ca)| < \pi$*;* $\Omega = \mu+1 > 0, \eta > 0$ *and* $|\arg k| < \frac{1}{2}\pi\Omega$ *then there holds the formula*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma}\,{}_p\Psi_q\left[-k(cx+d)^{\eta}\Big|_{(b_1,B_1),\dots(b_q,B_q)}^{(a_1,A_1),\dots,(a_p,A_p)}\right]\mathrm{d}x$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\Gamma(\beta)$$
$$\times H_{1,0:q+1,p+1;1,2}^{0,1:p,1;1,1}\left[\begin{matrix}\frac{1}{k(ac+d)^{\eta}}\\ \frac{c(b-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):-\\ -\\ (1,1),(b_1,B_1),\dots,(b_q,B_q):(1-\alpha,1)\\ (a_1,A_1),\dots,(a_p,A_p),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right], \quad (2.73)$$

where μ is defined in (1.9).

When $d=0$, (2.66), (2.69) give rise to the following theorems:

Theorem 2.3. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0, \alpha,\beta,\gamma \in C, \mu \geq 0, b \neq a, a \neq 0,$ $|1-\frac{b}{a}| < 1$, $|\arg(b/a)| < \pi$, $\phi,\eta > 0, |\arg k| < \frac{1}{2}\pi\phi$*, then there holds the formula*

$$\int_a^b x^{\gamma}(x-a)^{\alpha-1}(b-x)^{\beta-1}H_{p,q}^{m,n}\left[kx^{-\eta}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= a^{\gamma}(b-a)^{\alpha+\beta-1}\Gamma(\beta)$$
$$\times H_{1,0:p,q+1;1,2}^{0,1:m,n;1,1}\left[\begin{matrix}\frac{k}{a^{\eta}}\\ \frac{b}{a}-1\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):-\\ -\\ (a_p,A_p);(1-\alpha,1)\\ (b_q,B_q),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right]. \quad (2.74)$$

Theorem 2.4. *If* $\min\{\Re(\alpha),\Re(\beta)\} > 0, \alpha,\beta,\gamma \in C, \mu \geq 0, b \neq a, a \neq 0, k \neq 0,$ $|1-\frac{b}{a}| < 1, |\arg(b/a)| < \pi$*;* $\phi,\eta > 0$, $|\arg k| < \frac{1}{2}\pi\phi$*, then there holds the formula*

$$\int_a^b x^\gamma (x-a)^{\alpha-1}(b-x)^{\beta-1} H_{p,q}^{m,n}\left[kx^\eta \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= a^\gamma (b-a)^{\alpha+\beta-1}\Gamma(\beta)$$
$$\times H_{1,0:q,p+1;1,2}^{0,1:n,m;1,1}\left[\begin{matrix}\frac{1}{ka^\eta}\\ \frac{b}{a}-1\end{matrix}\left|\begin{matrix}(1+\gamma:\eta,1):-\\ -\\ (1-b_1,B_1),\ldots,(1-b_q,B_q);(1-\alpha,1)\\ (1-a_1,A_1),\ldots,(1-a_p,A_p),(1+\gamma,\eta);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right]. \quad (2.75)$$

Alternative form of Theorem 2.1. *Let*

$$f(z) = (z-a)^{\beta-1}(cz+d)^\gamma H_{p,q}^{m,n}[k(cz+d)^{-\nu}],$$

then there holds the formula

$$_aD_z^{-\alpha}[f(z)] = (z-a)^{\alpha+\beta-1}(ac+d)^\gamma$$
$$\times H_{1,0:p,q+1;1,2}^{0,1:m,n;1,1}\left[\begin{matrix}\frac{k}{(ac+d)^\nu}\\ \frac{c(z-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\nu,1):-\\ -\\ (a_1,A_1),\ldots,(a_p,A_p);(1-\beta,1)\\ (b_1,B_1),\ldots,(b_q,B_q),(1+\gamma,\nu);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right], \quad (2.76)$$

under the conditions stated along with (2.66) with b replaced by z and η replaced by $+\nu$, where $_0D_z^{-\alpha}$ is the fractional integral operator, see Chap. 3 for a discussion of fractional integrals and fractional derivatives.

Alternative form of Theorem 2.2. *Let*

$$f(z) = (z-a)^{\beta-1}(cz+d)^\gamma H_{p,q}^{m,n}[k(cz+d)^{\nu}],$$

then there holds the formula

$$_aD_z^{-\alpha}[f(z)] = (z-a)^{\alpha+\beta-1}(ac+d)^\gamma$$
$$\times H_{1,0:q,p+1;1,2}^{0,1:n,m;1,1}\left[\begin{matrix}\frac{1}{k(ac+d)^\nu}\\ \frac{c(z-a)}{ac+d}\end{matrix}\left|\begin{matrix}(1+\gamma:\nu,1):-\\ -\\ (1-b_1,B_1),\ldots,(1-b_q,B_q);(1-\beta,1)\\ (1-a_1,A_1),\ldots,(1-a_p.A_p),(1+\gamma,\nu);(0,1),(1-\alpha-\beta,1)\end{matrix}\right.\right], \quad (2.77)$$

under the conditions stated along with (2.69) with b replaced by z and η replaced by $+\nu$.

It may be mentioned here that for generalization of the results of this section, one can refer to the papers by Saxena and Saigo (1998), Saigo and Saxena (1999, 1999a, 2001) and Srivastava and Hussain (1995).

Remark 2.9. On the integration of H-functions with respect to their parameters, see the works of Nair (1973), Nair and Nambudiripad (1973), Anandani (1970b), Taxak (1971). Golas (1968) and Pendse (1970). Integration of products of generalized Legendre functions and H-functions with respect to a parameter is discussed by Anandani (1970b, 1971d).

2.4 H-Function and Exponential Functions

The following integrals are evaluated by Bajpai (1970) with the help of the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{\alpha-1}\exp(i\beta\theta)\mathrm{d}\theta = \frac{\pi\Gamma(\alpha)}{2^{\alpha-1}\Gamma\left(\frac{\alpha+\beta+1}{2}\right)\Gamma\left(\frac{\alpha-\beta+1}{2}\right)}, \tag{2.78}$$

where $\Re(\alpha) > 0$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{k+\lambda-2}\exp[i(k-\lambda)\theta]H_{p,q}^{m,n}\left[z(\mathrm{e}^{i\theta}\cos\theta)^{-h}\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]\mathrm{d}\theta$$
$$= \frac{\pi}{2^{k+\lambda-2}\Gamma(\lambda)}H_{p+1,q+1}^{m+1,n}\left[2^h z\middle|\begin{matrix}(a_p,A_p),(k,h)\\(k+\lambda-1,h),(b_q,B_q)\end{matrix}\right], \tag{2.79}$$

where $\Re(k+\lambda-h\frac{a_i}{A_i}) > 1-\frac{h}{A_i}, i = 1,\ldots,n, h > 0, k,\lambda \in C, \mu \geq 0, \alpha > 0$, $|\arg z| < \frac{1}{2}\pi\alpha$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{k+\lambda-2}\exp[i(k-\lambda)\theta]H_{p,q}^{m,n}\left[z\mathrm{e}^{ih\theta}(\sec\theta)^{h}\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]\mathrm{d}\theta$$
$$= \frac{\pi}{2^{k+\lambda-2}\Gamma(k)}H_{p+1,q+1}^{m+1,n}\left[2^h z\middle|\begin{matrix}(a_p,A_p),(\lambda,h)\\(k+\lambda-1,h),(b_q,B_q)\end{matrix}\right], \tag{2.80}$$

where $\Re(k+\lambda-h\frac{a_i}{A_i}) > 1-\frac{h}{A_i}, i = 1,\ldots,n, h > 0, k,\lambda \in C, \mu \geq 0, \alpha > 0$, $|\arg z| < \frac{1}{2}\pi\alpha$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{k+\lambda-2}\exp[i(k-\lambda)\theta]H_{p,q}^{m,n}\left[z(\sec\theta)^{2h}\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]\mathrm{d}\theta$$
$$= \frac{\pi}{2^{k+\lambda-2}}H_{p+2,q+1}^{m+1,n}\left[2^{2h} z\middle|\begin{matrix}(a_p,A_p),(k,h),(\lambda,h)\\(k+\lambda-1,2h),(b_q,B_q)\end{matrix}\right], \tag{2.81}$$

where $\Re(k+\lambda-h\frac{a_i}{A_i}) > 1-\frac{h}{A_i}, i = 1,\ldots,n, h > 0, k,\lambda \in C, \mu \geq 0, \alpha > 0$, $|\arg z| < \frac{1}{2}\pi\alpha$. By means of the following integral (Nielson 1906, p. 158)

$$\int_0^{\pi}(\sin t)^{\alpha}\mathrm{e}^{-\beta t}\mathrm{d}t=\frac{\pi\mathrm{e}^{-\frac{\pi\beta}{2}}\Gamma(\alpha+1)}{2^{\alpha}\Gamma\left(1+\frac{\alpha+i\beta}{2}\right)\Gamma\left(1+\frac{\alpha-i\beta}{2}\right)},\tag{2.82}$$

where $\Re(\alpha)>-1$, Saxena (1971a) has established the following results:

(i) Let $\alpha>0, |\arg z|<\frac{1}{2}\pi\alpha$ or $\alpha=0, \Re(\delta)<-1$ then there holds the formula

$$\int_0^{\pi}(\sin\theta)^{\gamma-1}\mathrm{e}^{-\eta\theta}H_{p,q}^{m,n}\left[z(\sin\theta)^{2h}\middle|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}\theta$$
$$=\frac{\pi}{2^{\gamma-1}}\exp\left(-\frac{\pi\eta}{2}\right)H_{p+1,q+2}^{m,n+1}\left[\frac{z}{4^h}\middle|_{(b_q,B_q),\left(\frac{1-\gamma+i\eta}{2},h\right),\left(\frac{1-\gamma-i\eta}{2},h\right)}^{(1-\gamma,2h),(a_p,A_p)}\right],\tag{2.83}$$

where $\gamma,\eta\in C, h>0$ are such that $\Re(\gamma)+2h\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right]>0$ for $\alpha>0$, $|\arg z|<\frac{1}{2}\pi\alpha$ or $\alpha=0, \mu\ge 0$, and $\Re(\gamma)+2h\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]$ >0 for $\alpha=0$ and $\mu<0$.

(ii)

$$\int_0^{\pi}(\sin\theta)^{\gamma-1}\mathrm{e}^{-\eta\theta}H_{p,q}^{m,n}\left[z\mathrm{e}^{i2h\theta}\middle|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}\theta$$
$$=\frac{\pi\Gamma(\gamma)}{2^{\gamma-1}}\exp\left(-\frac{\pi\eta}{2}]\right)H_{p+1,q+1}^{m,n}\left[\frac{z\mathrm{e}^{i\pi h}}{4^h}\middle|_{(b_q,B_q),\left(\frac{1-\gamma-i\eta}{2},h\right)}^{(a_p,A_p),\left(\frac{1+\gamma-i\eta}{2},h\right)}\right],\tag{2.84}$$

where $\gamma,\eta\in C, h>0, |\arg z|<\frac{1}{2}\pi\alpha, \alpha>0, \Re(\gamma)>0$.

(iii) Let $\alpha>0, \Re(\gamma)>0$ or $\alpha=0, \Re(\delta)<-1$ then

$$\int_0^{\pi}(\sin\theta)^{\gamma-1}\mathrm{e}^{-\eta\theta}H_{p,q}^{m,n}\left[z(\sin\theta)^{2\lambda}\mathrm{e}^{i2h\theta}\middle|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}\theta$$
$$=\frac{\pi}{2^{\gamma-1}}\exp\left(-\frac{\pi\eta}{2}\right)H_{p+1,q+2}^{m,n+1}\left[\frac{z\mathrm{e}^{i\pi h}}{4^{\lambda}}\middle|_{(b_q,B_q),\left(\frac{1-\gamma-i\eta}{2},\lambda+h\right),\left(\frac{1-\gamma+i\eta}{2},\lambda-h\right)}^{(1-\gamma,2\lambda),(a_p,A_p)}\right],\tag{2.85}$$

holds for $h>0, \lambda>h, \gamma,\eta\in C$, such that $\Re(\gamma)+2\lambda\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right]>0$ for $\alpha>0$ or $\alpha=0$ and $\mu\ge 0$; and $\Re(\gamma)+2\lambda\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]>0$ for $\alpha=0$ and $\mu<0$. When $h=1$ and $A_i=B_j=1$ for all i and j, then the H-function reduces to a G-function and from the results (2.83) and (2.85) we find that

$$\int_0^{\pi} (\sin\theta)^{\gamma-1} \mathrm{e}^{-\eta\theta} G_{p,q}^{m,n}\left[z \sin^2\theta \,\middle|\, \begin{matrix}(a_p)\\(b_q)\end{matrix}\right] \mathrm{d}\theta$$
$$= \sqrt{\pi}\exp\left(-\frac{\pi\eta}{2}\right) G_{p+2,q+2}^{m,n+2}\left[z \,\middle|\, \begin{matrix}\frac{1-\gamma}{2}, \frac{2-\lambda}{2}, a_p\\ b_q, \frac{1-\gamma+i\eta}{2}, \frac{1-\gamma-i\eta}{2}\end{matrix}\right], \tag{2.86}$$

where $\Re(\gamma) + 2\min_{1\le j\le m}\Re(b_j) > 0, \gamma, \eta \in C, c^* > 0, |\arg z| < \frac{1}{2}\pi c^*$, where c^* is defined in (1.22) and

$$\int_0^{\pi} (\sin\theta)^{\gamma-1} \mathrm{e}^{-\eta\theta} G_{p,q}^{m,n}\left[z\mathrm{e}^{2i\theta} \,\middle|\, \begin{matrix}a_p\\b_q\end{matrix}\right] \mathrm{d}\theta$$
$$= \frac{\pi}{2^{\gamma-1}}\Gamma(\gamma)\exp\left(-\frac{\pi\eta}{2}\right) G_{p+1,q+1}^{m,n}\left[z\mathrm{e}^{i\pi} \,\middle|\, \begin{matrix}a_p, \frac{1+\gamma-i\eta}{2}\\ (b_q), \frac{1-\gamma-i\eta}{2}\end{matrix}\right], \tag{2.87}$$

where $\Re(\gamma) > 0, |\arg z| < \frac{1}{2}\pi c^*, c^* > 0.$

2.5 Legendre Function and the H-Function

Let $\rho, z \in C, \alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \Re(\mu) < -1$. Further, let $\rho \in C, k > 0$ satisfy the conditions

$$\Re(\rho) + k \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > \frac{1}{2}|\Re(\mu)|$$

for $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$ and

$$\Re(\rho) + k \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu}\right] > \frac{1}{2}|\Re(\mu)|,$$

for $\alpha = 0, \mu < 0$ then there holds the formula (Singh and Varma 1972)

$$\int_{-1}^{1} (1-x^2)^{\rho-1} P_{\nu}^{\lambda}(x) H_{p,q}^{m,n}\left[z(1-x^2)^k \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right] \mathrm{d}x$$
$$= \frac{2^{\lambda}\pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right)\Gamma\left(\frac{1-\nu-\lambda}{2}\right)}$$
$$\times H_{p+2,q+2}^{m,n+2}\left[z \,\middle|\, \begin{matrix}(1-\rho\pm\frac{\lambda}{2}, k), (a_p, A_p)\\ (b_q, B_q), (1-\rho+\frac{\nu}{2}, k), (-\rho-\frac{\nu}{2}, k)\end{matrix}\right]. \tag{2.88}$$

For a definition of $P^{\lambda}_{nu}(x)$ see Sect. 1.8.1. On making use of finite difference operator E (Milne-Thomson 1933, p. 33 with $\omega = 1$), which has the following properties:

$$E_a f(a) = f(a+1) \tag{2.89}$$

$$E_a^n f(a) = E_a[E_a^{n-1} f(a)]. \tag{2.90}$$

Singh and Varma (1972) have further shown that

$$\begin{aligned}
&\int_{-1}^{1} (1-x^2)^{\rho-1} P_\nu^\lambda(x)\,{}_U F_V(\alpha_1,\ldots,\alpha_U;\beta_1,\ldots,\beta_V;c(1-x^2)^d)\\
&\times H_{p,q}^{m,n}\left[z(1-x^2)^k \middle|\begin{matrix}(a_p,A_p)\\ (b_q.B_q)\end{matrix}\right] \mathrm{d}x\\
&= \frac{2^\lambda \pi}{\Gamma\left(\frac{2+\nu-\lambda}{2}\right)\Gamma\left(\frac{1-\nu-\lambda}{2}\right)} \sum_{r=0}^{\infty} \frac{(\alpha_1)_r \cdots (\alpha_U)_r}{(\beta_1)_r \cdots (\beta_V)_r} \frac{c^r}{r!}\\
&\times H_{p+2,q+2}^{m,n+2}\left[z \middle|\begin{matrix}(1-\rho-rd\pm\frac{\lambda}{2},k),(a_p,A_p)\\ (b_q,B_q),(1-\rho-rd+\frac{\nu}{2},k),(-\rho-rd-\frac{\nu}{2},k)\end{matrix}\right],
\end{aligned} \tag{2.91}$$

which holds under the conditions given with the result along with the conditions that k and d are positive integers, $U < V$ or $U = V+1$ and $|c| < 1$ and none of $\beta_j, j = 1,\ldots,V$ is a negative integer or zero. In case $\lambda = 0$ and $\nu = \lambda$, where λ is a positive integer, then the result (2.91) reduces to

$$\begin{aligned}
&\int_{-1}^{1} (1-x^2)^{\rho-1} P_\lambda(x)\,{}_U F_V(\alpha_1,\ldots,\alpha_U;\beta_1,\ldots,\beta_V;c(1-x^2)^d)\\
&\times H_{p,q}^{m,n}\left[z(1-x^2)^k \middle|\begin{matrix}(a_p,A_p)\\ (b_q,B_q)\end{matrix}\right] \mathrm{d}x\\
&= \frac{\pi}{\Gamma\left(\frac{2+\lambda}{2}\right)\Gamma\left(\frac{1-\lambda}{2}\right)} \sum_{r=0}^{\infty} \frac{(\alpha_1)_r \cdots (\alpha_U)_r}{(\beta_1)_r \cdots (\beta_V)_r} \frac{c^r}{r!}\\
&\times H_{p+2,q+2}^{m,n+2}\left[z \middle|\begin{matrix}(1-\rho-rd,k),(1-\rho-rd,k),(a_p,A_p)\\ (b_q,B_q),(1-\rho-rd+\frac{\lambda}{2},k),(-\rho-rd-\frac{\lambda}{2},k)\end{matrix}\right],
\end{aligned} \tag{2.92}$$

where $P_\lambda(x)$ is the Legendre polynomial and the conditions of the validity are the same as stated in (2.91) with $\lambda = 0$ and ν replaced by λ.

2.6 Generalized Laguerre Polynomials

From the integral (Mathai and Saxena 1973, p. 76) it can be easily shown that

$$\int_0^\infty x^\gamma \mathrm{e}^{-x} L_k^{(\sigma)}(x) H_{p,q}^{m,n}\left[zx^\eta \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= \frac{(2\pi)^{\frac{(1-\eta)}{2}}\eta^{\gamma+k+\frac{1}{2}}}{k!}$$
$$\times H_{p+2\eta,q+\eta}^{m+\eta,n+\eta}\left[z\eta^\eta \Big|_{(\Delta(\eta;\sigma-\gamma+k),1),(b_q,B_q)}^{\Delta(\eta;-\gamma),1),(\Delta(\eta;\sigma-\gamma);1),(a_p,A_p)}\right], \quad (2.93)$$

where η is a positive integer, either $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \Re(\delta) < -1$. Further, the parameters $z, \gamma, \sigma \in C$ are such that $\Re(\gamma) + \eta \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > -1$ for $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha=0, \mu \ge 0$; and $\Re(\gamma)+\eta\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > -1$ for $\alpha = 0$ and $\mu < 0$. For a definition of Laguerre polynomials see Sect. 1.8.1.

Remark 2.10. Solutions of certain integral equations involving general H-function were developed by Galué et al. (1993). It is interesting to observe that the results given earlier by Kalla and Kiryakova (1990) for the Erdélyi–Kober and Weyl operators follow easily from the results of this section.

Exercises

2.1. Prove that

$$\int_0^t x^{\rho-1}(t-x)^{c-1}{}_2F_1(a,b;c;1-\frac{x}{t})H_{p,q}^{m,n}\left[dx^\sigma\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$
$$= t^{\rho+c-1}\Gamma(c)H_{p+2,q+2}^{m,n+2}\left[dt^\sigma\Big|_{(b_q,B_q),(1+a-c-\rho,\sigma),(1+b-c-\rho,\sigma)}^{(1-\rho,\sigma),(1+a+b-c-\rho,\sigma),(a_p,A_p)}\right], \quad (2.94)$$

where either $\alpha > 0, |\arg d| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \Re(\delta) < -1$ and the parameters $\rho, a, b, c, d \in C, \Re(c) > 0, \sigma > 0$ be such that $\Re(\rho + c - a - b) + \min_{1\le k\le m}\left[\frac{\Re(b_j)}{B_j}\right] > 0$ for $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha$ or $\alpha = 0, \mu \ge 0$ and $\Re(\rho + c - a - b) + \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > 0$ for $\alpha = 0$ and $\mu < 0$.

2.2. Establish the following integrals:

(i)

$$\prod_{r=1}^{t}\int_0^1 x_r^{\alpha_r-1}(1-x_r)^{-\frac{1}{2}}T_{n_r}(2x_r-1)H_{p,q}^{m,n}\left[\frac{z}{(x_1x_2\cdots x_t)^h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x_r$$
$$=\pi^{\frac{t}{2}}H_{p+2t,q+2t}^{m+2t,n}\left[z\left|\begin{matrix}(a_p,A_p),(\alpha_1-n_1+\frac{1}{2},h),(\alpha_1+n_1+\frac{1}{2},h)\\ \cdots(\alpha_t-n_t+\frac{1}{2},h),(\alpha_t+n_t+\frac{1}{2},h)\\ (b_q,B_q),(\alpha_1,h),(\alpha_1+\frac{1}{2},h),\ldots\\ (\alpha_t,h),(\alpha_t+\frac{1}{2},h)\end{matrix}\right.\right],$$

where $z,\alpha_r\in C$, either $\alpha>0,|\arg z|<\frac{1}{2}\pi\alpha$ or $\alpha>0,\Re(\delta)<-1$. [For a definition of $T_n(x)$ see Sect. 1.8.1]. Further, the parameter h is such that $\Re(\alpha_r)>h\max_{1\le j\le n}\left[\frac{\Re(a_j)-1}{A_j}\right]>0, r=1,\ldots,t$ for $\alpha>0$ or $\alpha=0,\mu\le 0$ and $\Re(\alpha_r)+\min_{1\le j\le m}\left[\frac{\Re(a_j)-1}{A_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]>0, r=1,\ldots,t$ for $\alpha=0$ and $\mu>0$.

(ii)

$$\prod_{r=1}^{t}\int_0^1 x_r^{\alpha_r-1}(1-x_r)^{-\frac{1}{2}}T_{n_r}(2x_r-1)H_{p,q}^{m,n}\left[z(x_1x_2\cdots x_t)^h\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x_r$$
$$=\pi^{\frac{t}{2}}H_{p+2t,q+2t}^{m,n+2t}\left[z\left|\begin{matrix}(1-\alpha_1,h),\left(\frac{1}{2}-\alpha_1,h\right),\\ \ldots,(1-\alpha_t,h),\left(\frac{1}{2}-\alpha_t,h\right),(a_p,A_p)\\ (b_q,B_q),\left(\frac{1}{2}-\alpha_1-n_1,h\right),\left(\frac{1}{2}-\alpha_1+n_1,h\right),\ldots\\ \left(\frac{1}{2}-\alpha_t-n_t,h\right),\left(\frac{1}{2}-\alpha_t+n_t,h\right)\end{matrix}\right.\right],$$

where $z,\alpha_r\in C$, either $\alpha>0,|\arg z|<\frac{1}{2}\pi\alpha$ or $\alpha=0,\Re(\delta)<-1$. Further, the parameter $h>0$ is such that $\Re(\alpha_r)+h\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right]>-1$, $r=1,\ldots,t$ for $\alpha>0$ or $\alpha=0,\mu\ge 0$, and $\Re(\rho+c-a-b)+\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]>0, r=1,\ldots,t$ for $\alpha=0$ and $\mu<0$. Hint: use the integral (Prudnikov et al. 1990, p. 681, Eq. (8.4.31.1))

2.3. Let $\alpha,\beta,\gamma\in C$, either $\alpha>0,|\arg y|<\frac{1}{2}\pi\alpha$ or $\alpha=0,\Re(\delta)<-1$. Further, let $\eta\ge 0, b\ne a,\left|\frac{(b-a)c}{ac+d}\right|<1,\left|\frac{y(b-a)^{\lambda+\eta}}{(ac+d)^{\nu}}\right|<1,|\arg(d+cb)/(d+ca)|<\pi$ be such that $\Re(\alpha)+\lambda\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right]>0,\Re(\alpha)+\eta\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right]>0$ for $\alpha>0$ or $\alpha=0,\mu\ge 0$ and $\Re(\alpha)+\lambda\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]>0,\Re(\alpha)+\eta\min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j},\frac{\Re(\delta)+\frac{1}{2}}{\mu}\right]>0$ for $\alpha=0,\mu<0$ then there holds the formula

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} H_{p,q}^{m,n}\left[y(x-a)^{\lambda}(b-x)^{\eta}(cx+d)^{-\nu}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$

$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}$$

$$\times H_{2,1:p+1,q+1;0,1}^{0,2:m,n+1;1,0}\left[\begin{matrix}\frac{y(b-a)^{\lambda+\eta}}{(ac+d)^{\nu}} \\ \frac{c(b-a)}{ac+d}\end{matrix}\,\middle|\,\begin{matrix}(1-\alpha;\lambda,1),(1+\gamma;\nu,1):- \\ - \\ (a_1,A_1),\ldots,(a_p,A_p);(1-\beta,\eta) \\ (1-\alpha-\beta;\lambda+\eta,1);(b_1,B_1),\ldots,(b_q,B_q),(1+\gamma,\nu);(0,1)\end{matrix}\right].$$

Hence or otherwise show that

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}x^{\gamma} H_{p,q}^{m,n}\left[yx^{-\nu}(x-a)^{\lambda}(b-x)^{\eta}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}x$$

$$= (b-a)^{\alpha+\beta-1}a^{\gamma}$$

$$\times H_{2,1:p+1,q+1;0,1}^{0,2:m,n+1;1,0}\left[\begin{matrix}\frac{y(b-a)^{\lambda+\eta}}{a^{\nu}} \\ \frac{c(b-a)}{a}\end{matrix}\,\middle|\,\begin{matrix}(1-\alpha;\lambda,1),(1+\gamma;\nu,1):- \\ - \\ (a_1,A_1),\ldots,(a_p,A_p);(1-\beta,\eta);- \\ (1-\alpha-\beta;\lambda+\eta,1):(b_1,B_1),\ldots,(b_q,B_q),(1+\gamma,\nu);(0,1)\end{matrix}\right],$$

and give its conditions of validity (Saxena and Saigo 1998).

2.4. *Notation 2.9.* F_3: Appell function of the third kind

Definition 2.8. The Appell function of the third kind is defined in the form

$$F_3(a,a',b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m!n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1(a',b';c';y)\frac{x^m}{m!},$$

where $\max\{|x|,|y|\} < 1$. Prove the following result:

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\eta}\mathrm{d}t$$

$$= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\eta}B(\alpha,\beta)$$

$$\times F_3\left(a,\beta,-\gamma,-\eta;\alpha+\beta;-\frac{(b-a)u}{au+v},\frac{(b-a)y}{by+z}\right),$$

where for convergence

$$\max\left\{\left|\frac{(b-a)u}{au+v}\right|,\left|\frac{(b-a)y}{by+z}\right|\right\} < 1; b \neq a, \min\{\Re(\alpha),\Re(\beta)\} > 0.$$

2.5. Show that

$$\int_{-1}^{1}(1+t)^{\rho-1}(1-t)^{\lambda-1}P_{\nu}^{(\alpha,\beta)}\left[1-\frac{\sigma y}{2}(1-t)\right]H_{p,q}^{m,n}\left[z(1-t)^{h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}t$$
$$=\frac{2^{\lambda+\rho-1}(\alpha+1)_{\nu}\Gamma(\rho)}{\nu!}\sum_{r=0}^{\nu}\frac{(-\nu)_r}{r!}\frac{(1+\alpha+\beta_1)_r}{(1+\alpha)_r}\left(\frac{\sigma y}{2}\right)^{r}$$
$$\times H_{p+1,q+1}^{m,n+1}\left[2^{h}z\Big|_{(b_q,B_q),(1-\lambda-\rho-\nu,h)}^{(1-\lambda-r,h),(a_p,A_p)}\right],$$

where $z,\lambda,\rho\in C, h>0, \mu\geq 0, \alpha>0, |\arg z|<\frac{1}{2}\pi\alpha$; $\Re(\rho)>0, \Re(\lambda)+h\min_{1\leq j\leq m}\left[\frac{\Re(b_j)}{B_j}\right]>0$.

2.6. Prove that

$$\int_{0}^{\infty}t^{-\rho}J_{\nu}(t)J_{w}(t)H_{p,q}^{m,n}\left[at^{2h}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}t$$
$$=2^{-\rho}H_{p+4,q+1}^{m+1,n+1}\left[zt^{2h}\Big|_{(\rho,2h),(b_q,B_q)}^{\left(\frac{1+\rho-w-\nu}{2},h\right),(a_p,A_p),\left(\frac{1+\rho+\nu\pm w}{2},h\right),\left(\frac{p+w-\nu+1}{2},h\right)}\right],$$

where $J_{\nu}(\cdot)$ is the ordinary Bessel function, $h>0, \rho,\nu,\omega\in C, \mu\geq 0, \alpha>0$, $|\arg a|<\frac{1}{2}\pi\alpha$, $\Re(w+\nu-\rho+2h)\min_{1\leq j\leq m}\left[\frac{\Re(b_j)}{B_j}\right]>-1$ and $\Re(\rho)>2h\max_{1\leq i\leq n}\left[\frac{\Re(a_i)-1}{A_i}\right]$.

Chapter 3
Fractional Calculus

3.1 Introduction

The subject of fractional calculus deals with the investigations of integrals and derivatives of any arbitrary real or complex order, which unify and extend the notions of integer-order derivative and n-fold integral. It has gained importance and popularity during the last four decades or so, mainly due to its vast potential of demonstrated applications in various seemingly diversified fields of science and engineering, such as fluid flow, rheology, diffusion, relaxation, oscillation, anomalous diffusion, reaction-diffusion, turbulence, diffusive transport akin to diffusion, electric networks, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures and others. In this connection, one can refer to Caputo (1967), Glöckle and Nonnenmacher (1991), Mainardi (1995, 1996), Mainardi and Tomirotti (1997), Metzler et al. (1994), and monographs by Podlubny (1999), Dzherbashyan (1966), Oldham and Spanier (1974), Miller and Ross (1993), Hilfer (2000), Kilbas et al. (2006) and references therein.

The importance of this subject further lies in the fact that during the last three decades, three international conferences dedicated exclusively to fractional calculus and its applications were held in the University of New Haven in 1974, University of Strathclyde, Glasgow, Scotland in 1984, and the third in Nihon University in Tokyo, Japan in 1989 in which various workers presented their investigations dealing with the theory and applications of fractional calculus (see, for details, Ross (1975), McBride and Roach (1985), and Nishimoto (1991)). The works of Srivastava and Owa (1989), Kalia (1993), Rusev et al. (1995, 1997), Gaishun et al. (1996) also deal especially with the subject of fractional calculus.

A comprehensive account of fractional calculus and its applications can be found in the monographs written by Kiryakova (1994), McBride (1985), Oldham and Spanier (1974), Miller and Ross (1993), and Ross (1975). In particular, the five volumes work published recently by Nishimoto (1984, 1987, 1989, 1991, 1996) contains an interesting account of the theory and applications of fractional calculus in a number of areas of mathematical analysis, such as ordinary and partial differential equations, summation of series, special functions, etc.

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This chapter deals with the definition and basic properties of various operators of fractional integration and fractional differentiation of arbitrary order. Among the various operators studied, it involves the Riemann–Liouville fractional integration operators, Riemann–Liouville fractional differentiation operators, Weyl operators, Kober operators, Saigo operators, etc. Besides the basic properties of these operators, their behavior under Laplace, Fourier, and Mellin transforms are also presented. Application of Riemann–Liouville fractional calculus operators in the solution of kinetic equations, fractional reaction, fractional diffusion and fractional reaction–diffusion equations, etc. are demonstrated. The results are mostly derived in a closed form in terms of the H-functions and Mittag-Leffler functions suitable for numerical computation.

3.2 A Brief Historical Background

In order to give a meaning to the notation $\frac{d^n y}{dx^n}$ for the nth order derivative, when n is any number: fractional, irrational or complex, fractional calculus came into existence. In fact G.A. l'Hopital wrote to G. W. Leibnitz to know the meaning of $\frac{d^n y}{dx^n}$, when $n = \frac{1}{2}$. Leibnitz replied in a letter of 30 September 1695 to l'Hopital that " $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx : x}$, an apparent paradox from which one day useful consequences will be drawn". The name "fractional calculus" is probably due to l'Hopital's question "what if n is $\frac{1}{2}$?" In another letter of Leibniz to J. Wallis dated 28 May 1697, Leibniz discusses Wallis' infinite product for π, mentions differential calculus and uses the notation $d^{\frac{1}{2}}y$ to denote a derivative of order $\frac{1}{2}$.

Lacroix (1819) observed that

$$\frac{d^m}{dx^m}x^n = \frac{n!}{(n-m)!}x^{n-m}, \; n \in N = 1, 2, 3, \ldots; \;\; m \in N_0 = N \cup \{0\}; \;\; n \geq m. \tag{3.1}$$

Since $n! = \Gamma(n+1)$ and $(n-m)! = \Gamma(n-m+1)$, the above equation was written by Lacroix (1819) in terms of the gamma function in the form

$$\frac{d^m}{dx^m}x^n = \frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m}, \tag{3.2}$$

and then set $m = \frac{1}{2}$ and $n = 1$ to obtain

$$\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = \frac{2x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}.$$

During the eighteenth century, several mathematicians have contributed to the development of fractional calculus, which includes Fourier (1822), Abel (1823–1826), Liouville (1822–1837), and Riemann (1847).

Gruñwald (1867) defined the differintegration in terms of the following infinite series:

$$\frac{d^q f}{[d(x-a)]^q} = \lim_{N\to\infty}\left\{\frac{[(x-a)/N]^{-q}}{\Gamma(-q)}\sum_{k=0}^{N-1}\frac{\Gamma(k-q)}{\Gamma(k+1)} f\left(x-k\left[\frac{x-a}{N}\right]\right)\right\}, \tag{3.3}$$

where q is arbitrary. The above definition was further generalized by Post (1930) to the form

$$\frac{d^n f}{dx^n} = \lim_{\delta x\to 0}\left\{(\delta x)^{-n}\sum_{k=0}^{n}(-1)^k \binom{n}{k} f(x-k\delta x)\right\}, \ n \in N_0, \tag{3.4}$$

where,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The theory of fractional calculus by complex integral transformations approach has been developed by many mathematicians including Augustin–Louis Cauchy (1789–1857) and Edward Goursat (1858–1936). Further, Sonin in 1869 wrote a paper entitled "On differentiation with arbitrary index" from which the present definition of the Riemann–Liouville operator appears to follow. Letnikov (1872) in his four papers presented an explanation of the main concept of theory of differentiation of an arbitrary index which provides extension of Sonin's work. A detailed account of the origin of the Riemann–Liouville definition and its applications can be found in the monograph of Miller and Ross (1993). The works of Davis (1927, 1936), Love (1936–1996), Erdélyi (1939–1965), Kober (1940), Riesz (1949), Gelfand and Shilov (1959–1964), and Caputo (1969) may also be mentioned in this connection.

A chronological bibliography of fractional calculus given by Ross is available from the monograph of Oldham and Spanier (1974, pp. 1–15). Ross (1975) has also given a brief history and exposition of the fundamental theory of fractional calculus.

3.3 Fractional Integrals

Notation 3.1. ${}_aI_x^n, {}_aD_x^{-n}; n \in N_0$: Fractional integral of integer order n.

Definition 3.1.

$$({}_aI_x^n f)(x) = {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)dt, x > a, \tag{3.5}$$

where $n \in N_0$.

We begin our study by introducing a fractional integral of order n in the form (Cauchy formula):

$$({}_aD_x^{-n} f)(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)\mathrm{d}t. \tag{3.6}$$

It will be shown that the above integral can be expressed in terms of n-fold integral, that is,

$$({}_aD_x^{-n} f)(x) = \int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} \mathrm{d}x_3 \cdots \int_a^{x_{n-1}} f(t)\mathrm{d}t, \tag{3.7}$$

Proof 3.1. When $n = 2$, then using the well-known Dirichlet formula, namely

$$\int_a^b \mathrm{d}x \int_a^x f(x,y)\mathrm{d}y = \int_a^b \mathrm{d}y \int_y^b f(x,y)\mathrm{d}x \tag{3.8}$$

Equation (3.7) becomes

$$\begin{aligned} \int_a^x \mathrm{d}x_1 \int_a^{x_1} f(t)\mathrm{d}t &= \int_a^x \mathrm{d}t\, f(t) \int_t^x \mathrm{d}x_1 \\ &= \int_a^x (x-t) f(t)\mathrm{d}t. \end{aligned} \tag{3.9}$$

This shows that the twofold integral can be reduced to a simple integral with the help of Dirichlet formula. For $n = 3$, the integral in (3.7) gives

$$\begin{aligned} ({}_aD_x^{-3} f)(x) &= \int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} f(t)\mathrm{d}t \\ &= \int_a^x \mathrm{d}x_1 \left[\int_a^{x_1} \mathrm{d}x_2 \int_a^{x_2} f(t)\mathrm{d}t \right]. \end{aligned} \tag{3.10}$$

Using the result (3.9) the integrals within big bracket simplify to yield

$$({}_aD_x^{-3} f)(x) = \int_a^x \mathrm{d}x_1 \left[\int_a^{x_1} (x_1 - t) f(t)\mathrm{d}t \right]. \tag{3.11}$$

If we use (3.8), then the above line reduces to

$$({}_aD_x^{-3} f)(x) = \int_a^x \mathrm{d}t\, f(t) \left[\int_t^x (x_1 - t)\mathrm{d}x_1 \right] = \int_a^x \frac{(x-t)^2}{2!} f(t)\mathrm{d}t, \tag{3.12}$$

□

Continuing this process, we finally obtain

$$({}_aD_x^{-n}f)(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f(t)\mathrm{d}t = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1}f(t)\mathrm{d}t. \tag{3.13}$$

It is evident that the last integral in (3.13) is meaningful for any number n provided its real part is greater than zero.

3.3.1 Riemann–Liouville Fractional Integrals

Notation 3.2. ${}_aI_x^{\alpha}$, ${}_aD_x^{-\alpha}$; I_{a+}^{α} : Riemann–Liouville left-sided fractional integral of order α.

Notation 3.3. ${}_xI_b^{\alpha}, {}_xD_b^{-\alpha}; I_{b-}^{\alpha}$: Riemann–Liouville right-sided fractional integral of order α.

Notation 3.4. $L(a,b)$: Space of Lebesgue measurable real or complex valued functions.

Definition 3.2. $L(a,b)$ consists of Lebesgue measurable real or complex valued function $f(x)$ on $[a,b]$:

$$L(a,b) = \left\{ f : ||f||_1 = \int_a^b |f(t)|\mathrm{d}t < +\infty \right\}. \tag{3.14}$$

Definition 3.3. Let $f(x) \in L(a,b), \alpha \in C, \Re(\alpha) > 0$, then

$${}_aI_x^{\alpha} f(x) = {}_aD_x^{-\alpha} f(x) = I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, x > a, \tag{3.15}$$

is called the Riemann–Liouville left-sided fractional integral of order α.

Definition 3.4. Let $f(x) \in L(a,b), \alpha \in C, \Re(\alpha) > 0$, then

$${}_xI_b^{\alpha} f(x) = {}_xD_b^{-\alpha} f(x) = I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, x < b, \tag{3.16}$$

is called the Riemann–Liouville right-sided fractional integral of order α.

3.3.2 Basic Properties of Fractional Integrals

Proposition 3.1. *Fractional integrals obey the following property:*

$$\begin{aligned} ({}_aI_x^{\alpha}\, {}_aI_x^{\beta}\varphi)(x) &= ({}_aI_x^{\alpha+\beta}\varphi)(x) = ({}_aI_x^{\beta}\, {}_aI_x^{\alpha}\varphi)(x). \\ ({}_xI_b^{\alpha}\, {}_xI_b^{\beta}\varphi)(x) &= ({}_xI_b^{\alpha+\beta}\varphi)(x) = ({}_xI_b^{\beta}\, {}_xI_b^{\alpha}\varphi)(x). \end{aligned} \tag{3.17}$$

Proof 3.2. By virtue of the definition (3.14) and the Dirichlet formula (3.8), it follows that

$$\begin{aligned}({}_aI_x^\alpha \;\; {}_aI_x^\beta \varphi)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\mathrm{d}t}{(x-t)^{1-\alpha}} \frac{1}{\Gamma(\beta)} \int_a^t \frac{\varphi(u)\mathrm{d}u}{(t-u)^{1-\beta}} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \mathrm{d}u\varphi(u) \int_u^x \frac{\mathrm{d}t}{(x-t)^{1-\alpha}(t-u)^{1-\beta}}, \qquad (3.18)\end{aligned}$$

If we use the substitution $y = \frac{t-u}{x-u}$, the value of the second integral is

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)(x-u)^{1-\alpha-\beta}} \int_0^1 y^{\beta-1}(1-y)^{\alpha-1}\mathrm{d}y = \frac{(x-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},$$

which when substituted in (3.18) yields the first part of (3.17). The second part can be similarly established. In particular,

$${}_aI_x^{n+\alpha} f(x) = ({}_aI_x^n \;\; {}_aI_x^\alpha f)(x), \quad n \in N, \Re(\alpha) > 0, \qquad (3.19)$$

which shows that the n-fold differentiation

$$\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} I_x^{n+\alpha} f\right)(x) = {}_aI_x^\alpha f(x), n \in N, \Re(\alpha) > 0, \qquad (3.20)$$

for all x. When $\alpha = 0$, we obtain

$$({}_aI_x^0 f)(x) = f(x); \quad ({}_aI_x^{-n} f)(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = f^{(n)}(x). \qquad (3.21)$$

□

Note 3.1. The property given in (3.17) is called the semigroup property of fractional integration.

Proposition 3.2. *The following result holds:*

$$\int_a^b f(x)({}_aI_x^\alpha g)\mathrm{d}x = \int_a^b g(x)({}_xI_b^\alpha f)\mathrm{d}x. \qquad (3.22)$$

The result (3.22) can be established by interchanging the order of integration in the integral on the left of (3.22) and then by using the Dirichlet formula (3.8).

Remark 3.1. Stanislavsky (2004) derived a specific interpretation of fractional calculus. It was shown that there exists a relation between stable probability distribution and the fractional integral. The relation investigated shows that the parameter of the stable distribution coincides with the exponent of the fractional integral.

3.3.3 Illustrative Examples

Example 3.1. If $f(x) = (x-a)^{\beta-1}$, then find the value of ${}_aI_x^\alpha f(x)$.

Solution 3.1. We have

$$({}_aI_x^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}(t-a)^{\beta-1}\mathrm{d}t.$$

If we substitute $t = a + y(x-a)$ in the above integral, it reduces to

$$\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1},$$

where $\Re(\beta) > 0$. Thus,

$$({}_aI_x^\alpha f)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}, \tag{3.23}$$

provided $\alpha, \beta \in C$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$.

Example 3.2. It can be similarly shown that

$$({}_xI_b^\alpha g)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-x)^{\alpha+\beta-1}, \quad x < b, \tag{3.24}$$

where $g(x) = (b-x)^{\beta-1}, \alpha, \beta \in C; \min\{\Re(\alpha), \Re(\beta)\} > 0$.

Note 3.2. It may be noted that (3.23) and (3.24) give the Riemann–Liouville integrals of the power functions $f(x) = (x-a)^{\beta-1}$ and $g(x) = (b-x)^{\beta-1}, \min\{\Re(\alpha), (\Re(\beta)\} > 0$.

Exercises 3.2

3.2.1. Prove that

$$({}_aI_x^\alpha[x \pm c]^{\gamma-1})(x) = \frac{(a \pm c)^{\gamma-1}}{\Gamma(\alpha+1)}(x-a)^\alpha {}_2F_1\left(1, 1-\gamma; \alpha+1; \frac{a-x}{a \pm c}\right),$$

where $\Re(\alpha) > 0, \alpha, \gamma \in C, \left|\frac{a-x}{a\pm c}\right| < 1$.

3.2.2. Prove that

$$\begin{aligned}&\left({}_aI_x^\alpha(x-a)^{\beta-1}(b-x)^{\gamma-1}\right)(x)\\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\frac{(x-a)^{\alpha+\beta-1}}{(b-a)^{1-\gamma}} {}_2F_1(\beta, 1-\gamma; \alpha+\beta; \frac{a-x}{b-a}),\end{aligned}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \alpha, \beta, \gamma \in C, \left|\frac{a-x}{b-a}\right| < 1$.

3.2.3. Prove that

$$\left({}_aI_x^{\alpha}[\mathrm{e}^{\lambda x}]\right)(x) = \mathrm{e}^{\lambda a}(x-a)^{\alpha}E_{1,\alpha+1}(\lambda x - \lambda a).$$

where $x > a, \alpha, \lambda \in C, \Re(\alpha) > 0$ and $E_{1,\alpha+1}(.)$ is the Mittag-Leffler function.

3.2.4. Prove that

$$\left({}_aI_x^{\alpha}[\mathrm{e}^{\lambda x}(x-a)^{\beta-1}]\right)(x) = \mathrm{e}^{\lambda a}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}{}_1F_1(\beta;\alpha+\beta;\lambda x - \lambda a),$$

where $\alpha, \beta \in C, \min\{\Re(\alpha), \Re(\beta)\} > 0$.

3.2.5. Prove that

$$\begin{aligned}&\left({}_aI_x^{\alpha}([x-a]^{\beta-1}\ln(x-a)\right)(x)\\&= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}[\ln(x-a) + \psi(\beta) - \psi(\alpha+\beta)],\end{aligned}$$

where $\alpha, \beta \in C, \min\{\Re(\alpha), \Re(\beta)\} > 0$; and $\psi(.)$ is the logarithmic derivative of the gamma function.

3.2.6. Prove that

$$\left({}_aI_x^{\alpha}[(x-a)^{\frac{\nu}{2}}J_{\nu}[\lambda\sqrt{x-a}]\right)(x) = \left(\frac{2}{\lambda}\right)^{\alpha}(x-a)^{(\alpha+\nu)/2}J_{\alpha+\nu}(\lambda\sqrt{x-a}),$$

where $\alpha, \nu \in C, \Re(\alpha) > 0, \Re(\nu) > -1$.

3.2.7. Prove that

$$\left({}_aI_x^{\nu}((x-a)^{\beta-1}E_{\mu,\beta}[(x-a)]^{\mu}\right)(x) = (x-a)^{\nu+\beta-1}E_{\mu,\nu+\beta}[(x-a)^{\mu}],$$

where $\beta, \mu, \nu \in C, \Re(\nu) > 0$.

3.2.8. Prove that

$$\begin{aligned}&\left({}_0I_x^{\nu}[x^{\mu-1}\sin ax]\right)(x)\\&= \frac{x^{\mu+\nu-1}}{2i}\frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}\left[{}_1F_1(\mu;\mu+\nu;iax) - {}_1F_1(\mu;\mu+\nu;-iax)\right],\end{aligned}$$

where $\beta, \nu \in C, a > 0, \min\{\Re(\nu), \Re(\mu)\} > 0$.

3.2.9. Prove that

$$(I_{b-}^n g)(x) = \int_x^b \mathrm{d}t_1 \int_{t_1}^b \mathrm{d}t_2 \cdots \int_{t_{n-1}}^b g(t_n)\mathrm{d}t_n = \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} g(t)\mathrm{d}t,\ n \in \mathrm{N}.$$

3.2.10. Prove that Riemann–Liouville fractional integrals ${}_aI_x^\alpha$ and ${}_xI_b^\alpha$ with $\Re(\alpha) > 0$ are bounded in $L_1[a,b]$. That is

$$||{}_aI_x^\alpha h||_1 \leq \frac{(b-a)^{\Re(\alpha)}}{|\Gamma(\alpha)|\Re(\alpha)}||h||_1, ||{}_xI_b^\alpha h||_1 \leq \frac{(b-a)^{\Re(\alpha)}}{|\Gamma(\alpha)|\Re(\alpha)}||h||_1, \tag{3.25}$$

where $\alpha \in C, \Re(\alpha) > 0$.

3.2.11. Prove that the Riemann–Liouville fractional integral I_{0+}^α of the H-function exists and the following result holds:

$$\left(I_{0+}^\alpha t^{\rho-1} H_{p,q}^{m,n}\left[t^\sigma\Big|{}_{(b_q, B_q)}^{(a_p, A_p)}\right]\right)(x) = x^{\rho+\alpha-1} H_{p+1,q+1}^{m,n+1}\left[x^\sigma\Big|{}_{(b_q, B_q), (1-\rho-\alpha, \sigma)}^{(1-\rho, \sigma), (a_p, A_p)}\right],$$

provided $\alpha \in C, \Re(\alpha) > 0, a_i, b_j \in C, A_i, B_j > 0, i = 1, \ldots, p; j = 1, \ldots, q,$ $\rho \in C, \sigma > 0$. Further let

$$\alpha^* = \sum_{j=1}^n A_j - \sum_{j=n+1}^p A_j + \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_j > 0, \tag{3.26}$$

or

$$\alpha^* = 0, \gamma\mu + \Re(\delta) < -1; \sigma_{1\leq j\leq m}^{\min}\left[\frac{\Re(b_j)}{B_j}\right] + \Re(\rho) > 0, \tag{3.27}$$

and

$$\gamma\sigma < \Re(\rho), \quad \text{where the contour of integration is } L = L_{i\gamma\infty}. \tag{3.28}$$

3.4 Riemann–Liouville Fractional Derivatives

Notation 3.5. $\{\alpha\}$ means the fractional part of number $\alpha, 0 \leq \{\alpha\} < 1$.

Notation 3.6. $[\alpha]$ means the integral part of number α.

Note 3.3. We note that

$$\alpha = \{\alpha\} + [\alpha].$$

Notation 3.7. ${}_aD_x^\alpha\ \varphi;\ D_{a+}^\alpha\ \varphi$: Riemann–Liouville left-sided fractional derivative of the function $\varphi(x)$ of order α.

Notation 3.8. ${}_bD_x^\alpha\ \varphi,\ I_{b-}^\alpha\ \varphi$: Riemann–Liouville right-sided fractional derivative of the function $\varphi(x)$ of order α.

Definition 3.5. The left-sided Riemann–Liouville fractional derivative of order $\alpha \in C, \Re(\alpha) \geq 0$ of the function $\varphi(x)$ is defined by

$$\begin{aligned}({}_aD_x^{\alpha}\,\varphi)(x) &= (D_{a+}^{\alpha}\,\varphi)(x)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x \frac{\varphi(t)\mathrm{d}t}{(x-t)^{\alpha-n+1}}, \quad n = [\Re(\alpha)]+1;\ x > a, \end{aligned}\tag{3.29}$$

where $[\Re(\alpha)]$ means the integral part of $\Re(\alpha)$.

Definition 3.6. The right-sided Riemann–Liouville fractional derivative of order $\alpha \in C,\ \Re(\alpha) \geq 0$ of the function $\varphi(x)$is defined by

$$\begin{aligned}({}_xD_b^{\alpha}\,\varphi)(x) &= (D_{b-}^{\alpha}\,\varphi)(x)\\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^b \frac{\varphi(t)\mathrm{d}t}{(t-x)^{\alpha-n+1}}, \quad n = [\Re(\alpha)]+1;\ x < b. \end{aligned}\tag{3.30}$$

In short, one can express (3.29) in the form

$$({}_aD_x^{\alpha}\varphi)(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}({}_aI_x^{n-\alpha}\varphi)(x), \tag{3.31}$$

and (3.30) as

$$({}_xD_b^{\alpha}\,\varphi)(x) = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}({}_xI_b^{n-\alpha}\,\varphi)(x). \tag{3.32}$$

For $\alpha \in R^+$, the equations (3.29) and (3.30) take the forms

$$\begin{aligned}({}_aD_x^{\alpha}\,\varphi)(x) &= (D_{a+}^{\alpha}\,\varphi)(x)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x \frac{\varphi(t)\mathrm{d}t}{(x-t)^{\alpha-n+1}}, \quad n = [\alpha]+1;\ x < b \end{aligned}\tag{3.33}$$

and

$$\begin{aligned}{}_xD_b^{\alpha}\,\varphi)(x) &= (D_{b-}^{\alpha}\,\varphi)(x)\\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^b \frac{\varphi(t)\mathrm{d}t}{(t-x)^{\alpha-n+1}}, \quad n = [\alpha]+1;\ x < b. \end{aligned}\tag{3.34}$$

We shall also employ the notations

$$({}_aD_x^{\alpha})\,\varphi = ({}_aI_x^{-\alpha})\,\varphi = ({}_aI_x^{\alpha})^{-1}\varphi;\ \alpha \geq 0.$$

Similarly, we have

$$({}_xD_b^{\alpha})\varphi = ({}_xI_b^{-\alpha})\varphi = ({}_xI_b^{\alpha})^{-1}\varphi;\ \alpha \geq 0.$$

Remark 3.2. Geometric and physical interpretations of fractional integration and fractional differentiation were given by Podlubny (2002), also see Nigmatullin (1992).

Notation 3.9. $\Omega = [a, b], -\infty < a < b < \infty, \Omega$ may be a finite interval, a half line or a whole line.

Notation 3.10. $AC(\Omega)$, the space of absolutely continuous functions.

Notation 3.11. $AC^n(\Omega)$. If $n \in N$, the space of complex-valued functions $h(x)$ which have continuous derivatives up to order $n-1$ on $[a, b]$ with $h^{(n-1)}(x) \in AC[a, b]$ is denoted by $AC^n[a, b]$. That is

$$AC^n[a, b] = \left\{h : [a, b] \to C \text{ and } \left(D^{n-1}h\right)(x) \in AC[a, b]\right\}, \ D = \frac{\mathrm{d}}{\mathrm{d}x}, \quad (3.35)$$

where C is the set of complex numbers. It is evident that $AC^1[a, b] = AC[a, b]$.

We now present some properties of the operators defined by (3.29) and (3.30) (see Samko et al. (1993)).

Proposition 3.3. *Let $AC[a, b]$ be the space of absolutely continuous functions h on $[a, b]$. It is known [see Kolmogorov and Fomin 1984, p. 338] that $AC[a, b]$ coincides with the space of primitives of Lebesgue summable functions:*

$$h(x) \in AC[a, b] \Leftrightarrow h(x) = c + \int_a^x \varphi(t)\mathrm{d}t, \ \varphi(t) \in L(a, b). \quad (3.36)$$

Hence absolutely continuous function $h(x)$ has a summable derivative $h'(x) = \varphi(x)$ almost everywhere on $[a, b]$. Thus (3.36) gives

$$\varphi(t) = h'(t) \text{ and } c = h(a). \quad (3.37)$$

The following lemma can be established with the help of (3.36), which provides the characterization of the space $AC^n[a, b]$.

Lemma 3.1. *The space $AC^n[a, b]$ consists of those and only those functions $h(x)$, which are represented in the form*

$$h(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)\mathrm{d}t + \sum_{r=0}^{n-1} c_r(x-a)^r, \quad (3.38)$$

where $\varphi(x) \in L(a, b)$ and $c_r, r = 0, 1, \ldots, n-1$ are arbitrary constants. It follows from (3.38) that

$$\varphi(x) = h^{(n)}(x) \ \text{ and } c_r = \frac{h^{(r)}(a)}{r!}, \ r = 0, 1, \ldots, n-1. \quad (3.39)$$

The next theorem characterizes the conditions for the existence of the fractional derivatives in the space $AC^n[a,b]$, defined by (3.35)

Theorem 3.1. *If* $\alpha \in C, \Re(\alpha) \geq 0; n = |\Re(\alpha)| + 1,$ *and* $h(x) \in AC^n[a.b]$*, then the fractional differentiation operators* ${}_aD_x^\alpha h$ *and* ${}_xD_b^\alpha h$ *exist almost everywhere on* $[a,b]$ *and may be represented in the forms*

$$({}_aD_x^\alpha h)(x) = \sum_{r=o}^{n-1} \frac{h^{(r)}(a)}{\Gamma(1+r-\alpha)}(x-a)^{r-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{h^{(n)}(t)\mathrm{d}t}{(x-t)^{\alpha-n+1}}, \tag{3.40}$$

and

$$({}_xD_b^\alpha h)(x) = \sum_{r=o}^{n-1} \frac{(-1)^r h^{(r)}(b)}{\Gamma(1+r-\alpha)}(b-x)^{r-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{h^{(n)}(t)\mathrm{d}t}{(t-x)^{\alpha-n+1}}, \tag{3.41}$$

respectively.

To prove the first part of the theorem, we observe that since $h(x) \in AC^n$, consequently the representation (3.38) holds. Using this in the definition of the fractional derivative ${}_aD_x^\alpha h$ (3.29) and taking (3.39) into account, the result (3.40) follows. The second part can be proved similarly by using the Definition 3.6 and the representation for the function $f(x) \in AC^n[a,b]$ of the form (3.38):

$$f(x) = \frac{(-1)^n}{(n-1)!} \int_x^b (t-x)^{n-1}\theta(t)\mathrm{d}t + \sum_{r=0}^{n-1}(-1)^r \mathrm{e}_r (b-x)^r, \tag{3.42}$$

where

$$\theta(t) = f^{(n)}(t) \;\text{ and } \mathrm{e}_r = \frac{f^{(r)}(b)}{r!}. \tag{3.43}$$

Corollary 3.1. *If* $\alpha \in C, 0 \leq \Re(\alpha) < 1, \alpha \neq 0,$ *and* $h(x) \in AC[a,b]$*, then there holds the relations*

$$({}_aD_x^\alpha h)(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{h(a)}{(x-a)^\alpha} + \int_a^x \frac{h'(t)\mathrm{d}t}{(x-t)^\alpha} \right], \tag{3.44}$$

and

$$({}_xD_b^\alpha h)(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{h(b)}{(b-x)^\alpha} - \int_x^b \frac{h'(t)\mathrm{d}t}{(t-x)^\alpha} \right]. \tag{3.45}$$

Lemma 3.2. *If $\alpha \in C, \Re(\alpha) > 0$ and $h(x) \in L_p(a,b), 1 \leq p < \infty$, then the following formulae*

$$({}_aD_x^\alpha \; {}_aI_x^\alpha h)(x) = h(x) \text{ and } ({}_xD_b^\alpha \; {}_xI_b^\alpha h)(x) = h(x), \; \Re(\alpha) > 0 \tag{3.46}$$

hold almost everywhere on $[a,b]$.

Remark 3.3. The above assertion shows that the fractional differentiation is an operation inverse to fractional integration from the left.

Lemma 3.3. *If $\alpha, \beta \in C, \Re(\alpha) > \Re(\beta) > 0$, then for $h(x) \in L_p(a,b),\; 1 \leq p < \infty$, the composition relations*

$$({}_aD_x^\beta \; {}_aI_x^\alpha h)(x) = {}_aI_x^{\alpha-\beta} h(x) \quad \text{and} \quad ({}_xD_b^\beta \; {}_xI_b^\alpha h)(x) = {}_xI_b^{\alpha-\beta} h(x), \tag{3.47}$$

hold almost everywhere on $[a,b]$.

The first part in (3.47) readily follows from the results (3.40), Theorem 3.2 and Lemma 3.2. The second part can be proved similarly.

Notation 3.12. ${}_aI_x^\alpha(L_p)$: Space of functions.

Notation 3.13. ${}_xI_b^\alpha(L_p)$: Space of functions,

Definition 3.7. The space of functions ${}_aI_x^\alpha(L_p)$ is defined by

$${}_aI_x^\alpha(L_p) = \{h : h = {}_aI_x^\alpha \varphi; \varphi \in L_p(a,b)\}, \tag{3.48}$$

for $\alpha \in C, \Re(\alpha) > 0$ and$1 \leq p < \infty$.

Definition 3.8. The space of functions ${}_xI_b^\alpha(L_p)$ is defined by

$${}_xI_b^\alpha(L_p) = \{h : h = {}_xI_b^\alpha \varphi; \varphi \in L_p(a,b)\}, \tag{3.49}$$

for $\alpha \in C, \Re(\alpha) > 0$ and $1 \leq p \leq \infty$.

Theorem 3.2. *Let $\alpha \in C, \Re(\alpha) > 0, n = |\Re(\alpha)| + 1$ and let $h_{n-\alpha}(x) = ({}_aI_x^{n-\alpha} h)(x)$ be the fractional integral of order $n - \alpha$, defined by (3.15). Then the following results hold:*

(i) If $h(x) \in {}_aI_x^\alpha(L_p),\; 1 \leq p < \infty$, then

$$\left({}_aI_x^\alpha \, {}_aD_x^\alpha h\right)(x) = h(x). \tag{3.50}$$

(ii) If $h(x) \in L_1[a.b]$ and $h_{n-\alpha}(x) \in AC^n[a.b]$, then the formula

$$\left({}_aI_x^\alpha \, {}_aD_x^\alpha h\right)(x) = h(x) - \sum_{j=1}^{n} \frac{h_{n-\alpha}^{(n-j)}(a)}{\Gamma(n-j+1)} (x-a)^{n-j}, \tag{3.51}$$

holds almost everywhere on $[a,b]$.

3.4.1 Illustrative Examples

Example 3.3. Prove that

$$({}_0D_x^{\alpha}[t^{\gamma}])(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \alpha \geq 0, \gamma \in C, \Re(\gamma) > -1, x > 0, \quad (3.52)$$

Solution 3.2. We have

$$\begin{aligned} ({}_0D_x^{\alpha}[t^{\gamma}])(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x t^{\gamma}(x-t)^{n-\alpha-1}\mathrm{d}t \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+n+1-\alpha)} (\gamma-\alpha+1)_n x^{\gamma-\alpha} \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \end{aligned} \quad (3.53)$$

for $\gamma \in C, \Re(\gamma) > -1$.

Note 3.4. It is interesting to observe that for $\gamma = 0$, (3.53) yields

$$({}_0D_x^{\alpha}1)(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}; \alpha \neq 1, 2, \ldots, \quad (3.54)$$

which is a surprising result and indicates that the fractional derivative of a constant is, in general, not equal to zero. Thus it is not difficult to show that

$$({}_aD_x^{\alpha}1)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \text{ and } ({}_xD_b^{\alpha}1)(x) = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)}; 0 < \Re(\alpha) < 1. \quad (3.55)$$

Example 3.4. Prove that

$$({}_0I_x^{\nu}[\ln t])(x) = \frac{x^{\nu}}{\Gamma(\nu+1)}[\ln x - \gamma - \psi(\nu+1)],$$

where γ is the Euler's constant and $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Solution 3.3. We have

$$({}_0I_x^{\nu}[\ln t])(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} \ln t \, \mathrm{d}t.$$

If we use the substitution $t = xu$, then

$$\begin{aligned} ({}_0I_x^{\nu}[\ln t])(x) &= \frac{1}{\Gamma(\nu)} \int_0^1 x^{\nu}(1-u)^{\nu-1}(\ln x + \ln u)\mathrm{d}u \\ &= \frac{x^{\nu}\ln x}{\Gamma(\nu+1)} + \frac{x^{\nu}}{\Gamma(\nu)} \int_0^1 (1-u)^{\nu-1} \ln u \mathrm{d}u. \end{aligned}$$

We know that

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\ln t\mathrm{d}t = B(\alpha,\beta)[\psi(\alpha)-\psi(\alpha+\beta)], \tag{3.56}$$

where $\alpha, \beta \in C, \Re(\alpha) > 0, \Re(\beta) > 0$. Applying the formula (3.56) for $\alpha = 1$ and noting that $\psi(1) = -\gamma$, we see that

$$({}_0I_x^{\alpha}[\ln t])(x) = \frac{x^{\nu}}{\Gamma(\nu+1)}[\ln x - \gamma - \psi(\nu+1)].$$

Similarly, we can prove the result in the next example.

Example 3.5. Prove that

$$({}_0D_x^{\alpha}[\ln t])(x) = \frac{x^{-\nu}}{\Gamma(1-\nu)}[\ln x - \gamma - \psi(-\nu+1)].$$

Example 3.6.

$$({}_0D_x^{\alpha}[\mathrm{e}^{at}])(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}{}_1F_1(1;1-\alpha;ax).$$

Solution 3.4. We have

$$\begin{aligned}({}_0D_x^{\alpha}[\exp(at)])(x) &= \sum_{r=0}^{\infty}\frac{a^r}{r!}{}_0D_x^{\alpha}(x^r)\\ &= \sum_{r=0}^{\infty}\frac{a^r}{r!}\frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}x^{r-\alpha}\\ &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}{}_1F_1\ (1;1-\alpha;ax).\end{aligned}$$

Remark 3.4. One can unify the definitions of Riemann-Liouville fractional integral defined by (3.15) and Riemann–Liouville fractional derivative defined by (3.29) of arbitrary order $\alpha, \alpha \in C, \Re(\alpha) \neq 0, n \in N$, in the form

$$({}_aD_x^{\alpha}f)(x) = \begin{cases}\frac{1}{\Gamma(-\alpha)}\int_a^x (x-t)^{-\alpha-1}f(t)\mathrm{d}t, \ \Re(\alpha) < 0\\ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n({}_aI_x^{n-\alpha}f)(x), \ \Re(\alpha) > 0; n-1 \leq \Re(\alpha) < n,\end{cases} \tag{3.57}$$

which is called differintegral of f of order α. This process is also called fractional integro–differentiation. (Butzer and Westphal 2000)

Exercises 3.3

3.3.1. Prove that

$$\left({}_0D_x^\alpha[x^p \exp(ax)]\right)(x) = \frac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)}\,{}_1F_1(p+1; p-\alpha+1; ax),$$

where $\alpha, p \in C, \Re(p) > -1$.

3.3.2. Prove that

$$J_\nu(z) = \pi^{-1/2}2^{1-\nu}z^{-\nu}{}_0D_z^{-\nu+(1/2)}(\sin z).$$

3.3.3. Prove that

$$\psi(x) = -\gamma + \ln z - \Gamma(x)z^{1-x}{}_0D_z^{1-x}(\ln z),$$

where $\psi(x)$ is the logarithmic derivative of the gamma function and γ is the Euler's constant.

3.3.4. Prove that

$$\gamma(a,z) = \Gamma(a)\mathrm{e}^{-z}{}_0D_z^{-a}(\exp z),$$

where $\gamma(a,z)$ is the incomplete gamma function.

3.3.5. Prove that

$$\left({}_0D_x^\nu[x^{\mu/2}J_\mu(x^{\frac{1}{2}})]\right)(x) = 2^{-\nu}x^{\frac{1}{2}(\mu-\nu)}J_{\mu-\nu}(x^{\frac{1}{2}}).$$

where $\mu \in C, \Re(\mu) > -1$.

3.3.6. Prove that

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)}z^{1-c}{}_0D_z^{b-c}[z^{b-1}(1-z)^{-a}].$$

3.3.7. Establish the result

$$({}_0D_x^\nu[x^\lambda{}_2F_1(a,b;c;x)])(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)}x^{\lambda-\nu}{}_3F_2(\lambda+1,a,b;c;\lambda-\nu+1;x),$$

where $\lambda, \nu, a, b, c \in C, \Re(\lambda) > -1, \Re(\lambda-\nu) > -1$ and $c \neq 0, -1, -2, \ldots;$ and $|x| < 1$.

3.3.8. Prove that

$$({}_aD_x^\alpha\ {}_aI_x^\alpha h)(x) = h(x).$$

3.3.9. Prove that
$$({}_aD_x^{\beta}\ {}_aI_x^{\alpha}h)(x) = {}_aI_x^{\alpha-\beta}h(x),$$
where $\alpha, \beta \in C$, $\min\{\Re(\alpha), \Re(\beta)\} > 0$, $h(x) \in L(a, b)$.

3.5 The Weyl Integral

Notation 3.14. ${}_xW_{\infty}^{\alpha}, {}_xI_{\infty}^{\alpha}, I_{-}^{\alpha}$ Weyl integral of order α.

Definition 3.9. The Weyl integral of $f(x)$ of order α, denoted by ${}_xW_{\infty}^{\alpha}$, is defined by

$$\begin{aligned}({}_xW_{\infty}^{\alpha}f)(x) &= ({}_xI_{\infty}^{\alpha}f)(x) = (I_{-}^{\alpha}f)(x)\\ &= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}f(t)\mathrm{d}t,\ -\infty < x < \infty \end{aligned} \tag{3.58}$$

where $\alpha \in C, \Re(\alpha) > 0$.

Notation 3.15. ${}_xD_{\infty}^{\alpha},\ {}_xD_{-}^{\alpha}$: Weyl fractional derivative.

Definition 3.10. The Weyl fractional derivative of $f(x)$ of order α, denoted by ${}_xD_{\infty}^{\alpha}$, is defined by

$$\begin{aligned}({}_xD_{\infty}^{\alpha}f)(x) &= (D_{-}^{\alpha}f)(x) = (-1)^m\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\left({}_xW_{\infty}^{m-\alpha}f(x)\right)\\ &= (-1)^m\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\frac{1}{\Gamma(m-\alpha)}\int_x^{\infty}\frac{f(t)\mathrm{d}t}{(t-x)^{1+\alpha-m}}, -\infty < x < \infty\end{aligned} \tag{3.59}$$

where $m-1 \le \alpha < m;\ \ m \in N, \alpha \in C$.

3.5.1 Basic Properties of Weyl Integrals

Proposition 3.4. *The following result holds.*

$$\int_0^{\infty}\varphi(x)\left({}_0I_x^{\alpha}\psi(x)\right)\mathrm{d}x = \int_0^{\infty}\psi(x)\left({}_xW_{\infty}^{\alpha}\varphi(x)\right)\mathrm{d}x. \tag{3.60}$$

Equation (3.60) is called the formula for fractional integration by parts. It is also called the Parseval equality. Equation (3.60) can be established by interchanging the order of integration.

Proposition 3.5. *Weyl fractional integrals obey the semigroup property. That is*

$$\left({}_xW_\infty^\alpha \ {}_xW_\infty^\beta f\right)(x) = ({}_xW_\infty^{\alpha+\beta} f)(x) = \left({}_xW_\infty^\beta \ {}_xW_\infty^\alpha f\right)(x). \tag{3.61}$$

Proof 3.3. We have

$$\left({}_xW_\infty^\alpha \ {}_xW_\infty^\beta f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty \mathrm{d}t\,(t-x)^{\alpha-1} \times \frac{1}{\Gamma(\beta)}\int_t^\infty (u-t)^{\beta-1} f(u)\mathrm{d}u.$$

Using the modified form of Dirichlet formula (3.8), namely

$$\int_x^a \mathrm{d}t\,(t-x)^{\alpha-1}\int_t^a (u-t)^{\beta-1} f(u)\mathrm{d}u = B(\alpha,\beta)\int_t^a (u-t)^{\alpha+\beta-1}\mathrm{d}u \tag{3.62}$$

and letting $a \to \infty$, (3.62) yields the desired result

$$\left({}_xW_\infty^\alpha \ {}_xW_\infty^\beta f\right)(x) = ({}_xW_\infty^{\alpha+\beta} f)(x). \tag{3.63}$$

The second part of Eq. (3.61) can be similarly proved. □

3.5.2 Illustrative Examples

Example 3.7. Prove that

$$\left({}_xW_\infty^\alpha[\mathrm{e}^{-\lambda x}]\right)(x) = \frac{\mathrm{e}^{-\lambda x}}{\lambda^\alpha}$$

where $\Re(\alpha) > 0$.

Solution 3.5. We have

$$({}_xW_\infty^\alpha[\mathrm{e}^{-\lambda x}])(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}\mathrm{e}^{-\lambda t}\mathrm{d}t, \lambda > 0,$$
$$= \frac{\mathrm{e}^{-\lambda x}}{\Gamma(\alpha)}\int_0^\infty u^{\alpha-1}\mathrm{e}^{-\lambda u}\mathrm{d}u = \frac{\mathrm{e}^{-\lambda x}}{\lambda^\alpha}.$$

Example 3.8. Find the value of $({}_xD_\infty^\alpha[\mathrm{e}^{-\lambda x}])(x), \lambda > 0$.

Solution 3.6. We have

$$({}_xD_\infty^\alpha[\mathrm{e}^{-\lambda x}])(x) = (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m {}_xW_\infty^{m-\alpha}\mathrm{e}^{-\lambda x}$$
$$= (-1)^m \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \lambda^{-(m-\alpha)}\mathrm{e}^{-\lambda x} = \lambda^\alpha \mathrm{e}^{-\lambda x}.$$

Exercises 3.4

3.4.1. Prove that

$$({}_xW_\infty^\nu[x^{-\lambda}\exp(a/x)])(x) = \frac{\Gamma(\lambda-\nu)}{\Gamma(\lambda)} x^{\nu-\lambda}\Phi(\lambda-\nu,\lambda;a/x),$$

where $\lambda, \nu \in C, 0 < \Re(\nu) < \Re(\lambda)$.

3.4.2. Prove that

$$({}_xW_\infty^\nu[x^{\nu-1}\exp(-ax)])(x) = \pi^{-\frac{1}{2}}(x/a)^{\nu-\frac{1}{2}}\exp(-ax/2)K_{\nu-\frac{1}{2}}(ax/2),$$

where $\Re(ax) > 0, \nu \in C, \Re(\nu) > 0$.

3.4.3. Prove that

$$({}_xW_\infty^\alpha[x^{-\alpha-\gamma}E_{\beta,\gamma}^\delta(ax^{-\beta})])(x) = x^{-\gamma}E_{\beta,\alpha+\gamma}^\delta(ax^{-\beta}),$$

where $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0;\ \ \alpha, \beta, \gamma \in C, a \in R$.

3.4.4. Prove that the Riemann–Liouville fractional integral I_-^α of the H-function exists and the following relation holds:

$$\left(I_-^\alpha t^{\rho-1} H_{p,q}^{m,n}\left[t^\sigma \middle| \begin{matrix}(a_p, A_p)\\ (b_q, B_q)\end{matrix}\right]\right)(x) = x^{\rho+\alpha-1}$$
$$\times H_{p+1,q+1}^{m+1,n}\left[x^\sigma \middle| \begin{matrix}(a_p, A_p), (1-\rho,\sigma)\\ (1-\rho-\alpha,\sigma), (b_q, B_q)\end{matrix}\right],$$

provided $\alpha \in C, \Re(\alpha) > 0$ and further the constants $a_i, b_j \in C, A_i, B_j > 0$ $i = 1, \ldots, p;\ j = 1, \ldots, q, \rho \in C, \sigma > 0$ satisfy

$$\sigma \max_{1\le j\le n}\left[\frac{\Re(a_j)-1}{A_j}\right] + \Re(\rho) + \Re(\alpha) < 1,$$

and $1 + \gamma\sigma > \Re(\rho) + \Re(\alpha)$; the contour of integration being $L = L_{i\gamma\infty}$.

3.6 Laplace Transform

In this section, we derive the Laplace transforms of fractional integrals and fractional derivatives which are applicable in certain problems associated with fractional reaction, fractional diffusion fractional reaction–diffusion, etc.

3.6.1 Laplace Transform of Fractional Integrals

We have

$$({}_0I_x^\nu f)(x) = I_{0+}^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\mathrm{d}t, \tag{3.64}$$

where $\nu \in C, \Re(\nu) > 0$.

Application of the convolution theorem of the Laplace transform to (3.64) gives

$$\begin{aligned} L\{{}_0I_x^\nu f; s\} &= L\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}; s\right\} L\{f(t); s\} \\ &= s^{-\nu} F(s), \end{aligned} \tag{3.65}$$

where $s, \nu \in C, \Re(s) > 0, \Re(\nu) > 0$.

3.6.2 Laplace Transform of Fractional Derivatives

Let $n \in N$, then by the theory of the Laplace transform, we know that

$$L\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n} f; s\right\} = s^n F(s) - \sum_{r=0}^{n-1} s^{n-r-1} f^{(r)}(0_+) \tag{3.66}$$

$$= s^n F(s) - \sum_{r=0}^{n-1} s^r f^{(n-r-1)}(0_+), \tag{3.67}$$

where $s \in C, \Re(s) > 0$ and $F(s)$ is the Laplace transform of $f(t)$.

By virtue of the definition of the Riemann–Liouville fractional derivative, we find that

$$
\begin{aligned}
L\left[{}_0D_x^{\alpha} f; s\right] &= L\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\, {}_0I_x^{n-\alpha} f; s\right\} \\
&= s^n L\left[{}_0I_x^{n-\alpha} f; s\right] - \sum_{r=0}^{n-1} s^r \frac{\mathrm{d}^{n-r-1}}{\mathrm{d}x^{n-r-1}}\, {}_0I_x^{n-\alpha} f(0_+) \\
&= s^{\alpha} F(s) - \sum_{r=0}^{n-1} s^r \frac{\mathrm{d}^{\alpha-r-1}}{\mathrm{d}x^{\alpha-r-1}} f(0_+) \qquad (3.68) \\
&= s^{\alpha} F(s) - \sum_{r=1}^{n} s^{r-1} \frac{\mathrm{d}^{\alpha-r}}{\mathrm{d}x^{\alpha-r}} f(0_+), \qquad (3.69) \\
&= s^{\alpha} F(s) - \sum_{r=1}^{n} s^{r-1} D^{\alpha-r} f(0_+), \left(D = \frac{\mathrm{d}}{\mathrm{d}x}\right), \quad n-1 < \alpha \le n, \qquad (3.70)
\end{aligned}
$$

where $\Re(s) > 0$.

3.6.3 Laplace Transform of Caputo Derivative

Notation 3.16. ${}_a^C D_x^{\alpha} f$: Caputo fractional derivative of $f(t)$.

Definition 3.11. The Caputo fractional derivative of a casual function $f(t)$ (that is $f(t) = 0$ for $t < 0$) with $\alpha > 0$ was defined by Caputo (1969) in connection with certain boundary value problems arising in the theory of viscoelasticity and the hereditary solid mechanics in the form

$$
\begin{aligned}
({}_a^C D_x^{\alpha} f)(x) &= {}_aI_x^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = {}_aD_x^{-(n-\alpha)} f^{(n)}(x) \qquad (3.71) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\mathrm{d}t, n-1 < \alpha < n \qquad (3.72) \\
&= \frac{\mathrm{d}^n f}{\mathrm{d}x^n}, \text{ if } \alpha = n, n \in N. \qquad (3.73)
\end{aligned}
$$

From the Eqs. (3.65), (3.67) and (3.71), it follows that

$$
L\left\{{}_0^C D_x^{\alpha} f; s\right\} = s^{-(n-\alpha)} L\{f^{(n)}(t)\}. \qquad (3.74)
$$

On using (3.66) and (3.73), we see that

$$L\left\{{}_0^C D_x^\alpha f; s\right\} = s^{-(n-\alpha)}\left[s^n F(s) - \sum_{r=0}^{n-1} s^{n-r-1} f^{(r)}(0_+)\right]$$
$$= s^\alpha F(s) - \sum_{r=0}^{n-1} s^{\alpha-r-1} f^{(r)}(0_+), n-1 < \alpha \le n, \tag{3.75}$$

where $\alpha, s \in C, \Re(s) > 0, \Re(\alpha) > 0$.

Note 3.5. From (3.71), it can be seen that

$${}_0^C D_x^\alpha A = 0, \tag{3.76}$$

where A is a constant, and whereas the Riemann–Liouville derivative

$${}_0D_x^\alpha A = \frac{At^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \ne 1, 2, \ldots, \tag{3.77}$$

which is a surprising result.

Remark 3.5. In a recent paper, Freed and Diethelm (2007) have extended the Fung's elastic law to one that is appropriate for the viscoelastic representation of soft biological tissues, and whose kinetics are of fractional order.

3.7 Mellin Transforms

Notation 3.17. $M_p(0,\infty)$, : a subspace of $L_p(0,\infty)$.

Definition of the subspace $M_p(0,\infty)$: $M_p(0,\infty)$ denotes the class of all functions $f(x)$ of $L_p(0,\infty)$, with $p > 2$, which are inverse Mellin transforms of functions of $L_q(-\infty,\infty)$; $q = \frac{p}{p-1}$.

Theorem 3.3. *The following result holds true.*

$$M\left({}_0I_x^\alpha f\right)(s) = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)} f^*(s+\alpha), \tag{3.78}$$

where $s, \alpha \in C, \Re(\alpha) > 0$ *and* $\Re(\alpha+s) < 1$.

Proof 3.4. We have

$$M\left({}_0I_x^\alpha f\right)(s) = \int_0^\infty z^{s-1} \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} f(t)\mathrm{d}t\mathrm{d}z \tag{3.79}$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty f(t)\mathrm{d}t \int_t^\infty z^{s-1}(z-t)^{\alpha-1}\mathrm{d}z.$$

Setting $z = t/u$, the z-integral becomes

$$t^{\alpha+s-1}\int_0^1 u^{-\alpha-s}(1-u)^{\alpha-1}\mathrm{d}u = t^{\alpha+s-1}B(\alpha, 1-\alpha-s), \tag{3.80}$$

where $\Re(\alpha) > 0, \Re(\alpha+s) < 1$. The result (3.78) now follows from (3.80). □

Similarly, we can establish the following result:

Theorem 3.4. *The following result holds true:*

$$M\left({}_xI_\infty^\alpha f\right)(s) = \frac{\Gamma(s)}{\Gamma(s+\alpha)}M\left(t^\alpha f(t); s\right) \tag{3.81}$$

$$= \frac{\Gamma(s)}{\Gamma(s+\alpha)}f^*(s+\alpha), \tag{3.82}$$

where $s, \alpha \in C, \Re(\alpha) > 0, \Re(s) > 0.$

3.7.1 Mellin Transform of the nth Derivative

Theorem 3.5. *If* $n \in N$, *then*

$$M\left\{f^{(n)}(t); s\right\} = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)}M\left\{f(t); s-n\right\}, \tag{3.83}$$

where $s \in C, \Re(s-n) > 0$

Equation (3.83) can be proved by integrating by parts and using the definition of the Mellin transform.

3.7.2 Illustrative Examples

Example 3.9. Find the Mellin transform of the Riemann–Liouville fractional derivative ${}_0D_t^\alpha$.

Solution 3.7. We have

$${}_0D_t^\alpha f = ({}_0D_t^n\, {}_0D_t^{\alpha-n})f = ({}_0D_t^n\, {}_0I_t^{n-\alpha})f. \tag{3.84}$$

Therefore,

$$M\left({}_0D_t^{\alpha} f\right)(s) = \frac{(-1)^n \Gamma(s)}{\Gamma(s-n)} M\left({}_0I_t^{n-\alpha} f\right)(s-n),\ n-1 \le \Re(\alpha) < n \quad (3.85)$$

$$= \frac{(-1)^n \Gamma(s)\Gamma(1-(s-\alpha))}{\Gamma(s-n)\Gamma(1-s+n)} M\{f(t); s-\alpha\}, \quad (3.86)$$

where $\alpha, s \in C, \Re(s) > 0,\ \Re(s) < 1 + \Re(\alpha)$.

Example 3.10. In a similar manner, we can prove

$$M\left({}_0D_t^{\alpha} f\right)(s) = \frac{(-1)^n \Gamma(s) \sin[\pi(s-n)]}{\Gamma(s-\alpha)\sin[\pi(s-\alpha)]} M\{f(t); s-\alpha\}, \quad (3.87)$$

where $\alpha, s \in C, \Re(s) > 0, \Re(\alpha - s) > -1$.

Exercises 3.6

3.6.1. Find the Mellin transform of the Caputo derivative.

3.8 Kober Operators

Kober operators are the generalization of Riemann-Liouville and Weyl operators. These operators have been used by many authors in deriving the solution of single, dual, and triple integral equations possessing special functions of mathematical physics, as their kernels. These operators $(I_{(\alpha,\eta)} f)(x)$, are also called Erdélyi–Kober operators.

3.8.1 Erdélyi–Kober Operators

These operators are applicable in deriving the solution of certain integral equations involving special functions of mathematical physics which possess a Mellin–Barnes type integral representation. In this connection, refer to the works of Fox (1961, 1963, 1965, 1971), Saxena (1966, 1967, 1967a), Narain (1965, 1967), Nasim (1983), Habibullah (1977), and others. For further details see the survey paper entitled "Operators of fractional integration and their applications" by Srivastva and Saxena (2001).

Notation 3.18. $I[f(x)],\ I[\alpha, \eta;\ f(x)], E_{0,x}^{\alpha,\eta} f, I_x^{\eta,\alpha} f, (I_{\eta,\alpha}^{+} f)(x)$: Erdélyi–Kober fractional integral of the first kind.

Notation 3.19. R[f(x)], $R[\alpha,\zeta; f(x)]$, $K_{x,\infty}^{\alpha,\zeta} f$, $K_x^{\zeta,\alpha} f$, $(K_{\zeta,\alpha}^{-} f)(x)$, $(K(\alpha,\zeta f))(x)$: Erdélyi–Kober fractional integral of the second kind.

Definition 3.12.

$$I[f(x)] = I[\alpha,\eta; f(x)] = E_{0,x}^{\alpha,\eta} f = I_x^{\eta,\alpha} f = (I_{\eta,\alpha}^{+} f)(x) = (I(\alpha,\eta) f)(x)$$
$$= \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x t^{\eta}(x-t)^{\alpha-1} f(t)\mathrm{d}t, \alpha,\eta \in C; \Re(\alpha) > 0, \tag{3.88}$$

Definition 3.13.

$$R[f(x)] = R[\alpha,\zeta; f(x)] = K_{x,\infty}^{\alpha,\zeta} f = K_x^{\zeta,\alpha} f = (K_{\zeta,\alpha}^{-} f)(x) = (K(\alpha,\zeta) f)(x)$$
$$= \frac{x^{\zeta}}{\Gamma(\alpha)} \int_x^{\infty} t^{-\zeta-\alpha}(t-x)^{\alpha-1} f(t)\mathrm{d}t, \alpha,\zeta \in C; \Re(\alpha) > 0. \tag{3.89}$$

Equations (3.88) and (3.89) exist under the following set of conditions :

$$f \in L_p(0,\infty), \Re(\alpha) > 0, \Re(\eta) > -\frac{1}{q}, \Re(\zeta) > -\frac{1}{p}, \frac{1}{p} + \frac{1}{q} = 1, p \geq 1.$$

When $\eta = 0$, (3.88) reduces to Riemann-Liouville operator. That is

$$I_x^{0,\alpha} f = x^{-\alpha} {}_0 I_x^{\alpha} f. \tag{3.90}$$

For $\zeta = 0$, (3.89) yields the Weyl operator of the function $t^{-\alpha} f(t)$. That is

$$K_x^{0,\alpha} f = {}_x W_{\infty}^{\alpha} t^{-\alpha} f(t). \tag{3.91}$$

Theorem 3.6. *(Kober 1940) If $\alpha,\eta,s \in C, \Re(\alpha) > 0, \Re(\eta - s) > -1, f \in L_p(0,\infty)$, $1 \leq p \leq 2$ (or $f \in M_p(0,\infty)$, a subspace of $L_p(0,\infty)$ and $p > 2$), $\Re(\eta) > -\frac{1}{q}; \frac{1}{p} + \frac{1}{q} = 1$, then there holds the formula*

$$M\{I(\alpha,\eta) f\}(s) = \frac{\Gamma(1+\eta-s)}{\Gamma(\alpha+\eta+1-s)} M\{f(x); s\}. \tag{3.92}$$

The proof of (3.92) can be developed on similar lines to that of Theorem 3.3. In a similar manner, we can establish

Theorem 3.7. *(Kober 1940) If $\alpha,s,\zeta \in C, \Re(\alpha) > 0, \Re(\zeta + s) > 0, f \in L_p(0,\infty), 1 \leq p \leq 2$ (or $f \in M_p(0,\infty)$, a subspace of $L_p(0,\infty)$ and $p > 2$), $\Re(\zeta) > -\frac{1}{p}; \frac{1}{p} + \frac{1}{q} = 1$, then there holds the formula*

$$M\{R(\alpha,\zeta) f\}(s) = \frac{\Gamma(\zeta+s)}{\Gamma(\alpha+\zeta+s)} M\{f(x); s\}. \tag{3.93}$$

Semigroup property of the Erdélyi–Kober operators has been given in the form of the following theorem, which can be proved in the same way:

Theorem 3.8. *If* $\alpha, \eta \in C, \Re(\alpha) > 0, \Re(\eta) > \max\{-\frac{1}{p}, -\frac{1}{q}\}$; $f \in L_p(0,\infty), g \in L_q(0,\infty), 1 \leq p \leq 2$ *(or* $f \in M_p(0,\infty)$, *a subspace of* $L_p(0,\infty)$ *and* $p > 2$*),* $\frac{1}{p} + \frac{1}{q} = 1$, *then there holds the formula*

$$\int_0^\infty g(x)\,(I(\alpha,\eta;f))\,(x)\mathrm{d}x = \int_0^\infty f(x)\,(R(\alpha,\eta;g))\,(x)\mathrm{d}x. \tag{3.94}$$

Remark 3.6. Operators more general than the operators defined by (3.88) and (3.89) are defined by Galué et al. (2000) in the form

$$(I_{0_+}^{\alpha,0,\eta} f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_a^x t^\eta (x-t)^{\alpha-1} f(t)\mathrm{d}t, \alpha, \eta \in C; \Re(\alpha) > 0. \tag{3.95}$$

Exercises 3.7

3.7.1. For the Erdélyi–Kober operators defined by

$$I_{\eta,\alpha}^+ f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^x (x^2-t^2)^{\alpha-1} t^{2\eta+1} f(t)\mathrm{d}t,$$

where $\Re(\alpha) > 0$, establish the following results (Sneddon 1975):

(i) $I_{\eta,\alpha}^+ x^{2\beta} f(x) = x^{2\beta} I_{\eta+\beta,\alpha}^+ f(x)$.

(ii) $I_{\eta,\alpha}^+ I_{\eta+\alpha,\beta}^+ = I_{\eta,\alpha+\beta}^+ = I_{\eta+\alpha.\beta}^+ I_{\eta,\alpha}^+$.

(iii) $(I_{\eta,\alpha}^+)^{-1} = I_{\eta+\alpha,-\alpha}^+$

Remark 3.7. The results of Exercise 3.7.1 also hold for the operator, defined by

$$K_{\eta,\alpha}^- f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^\infty (t^2-x^2)^{\alpha-1} t^{-2\alpha-2\eta+1} f(t)dt,$$

where $\Re(\alpha) > 0$.

3.7.2. Prove that the Erdélyi–Kober fractional integral $I_{\eta,\alpha}^+$ of the H-function exists and the following result holds:

$$\left(I_{\eta,\alpha}^+ t^{\rho-1} H_{p,q}^{m,n} \left[t^\sigma \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] \right)(x)$$
$$= x^{\rho-1} H_{p+1,q+1}^{m,n+1} \left[x^\sigma \middle| \begin{matrix} (1-\rho-\eta,\sigma), (a_p, A_p) \\ (b_q, B_q), (1-\rho-\alpha-\eta,\sigma) \end{matrix} \right],$$

provided $\alpha, \eta \in C, \Re(\alpha) > 0$, and further the constants $a_i, b_j \in C, A_i, B_j > 0$, $i = 1, \ldots, p; j = 1, \ldots, q, \rho \in C, \sigma > 0$ satisfy

$$\sigma \min_{1 \le j \le m} \left[\frac{\Re(b_j)}{B_j} \right] + \Re(\rho) + \min[0, \Re(\eta)] > 0.$$

and $\gamma\sigma < -\Re(\rho) - \min[0, \Re(\eta)]$. (Here the contour of integration is $L = L_{i\gamma\infty}$.

3.7.3. Prove that the Erdélyi–Kober fractional integral $K^{-}_{\eta,\alpha}$ of the H-function exists and the following relation holds:

$$\left(K^{-}_{\eta,\alpha} t^{\rho-1} H^{m,n}_{p,q} \left[t^{\sigma} \middle| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] \right)(x)$$
$$= x^{\rho-1} H^{m+1,n}_{p+1,q+1} \left[x^{\sigma} \middle| \begin{matrix} (a_p, A_p), (1-\rho+\alpha+\eta, \sigma) \\ (1-\rho+\eta, \sigma), (b_q, B_q) \end{matrix} \right],$$

provided $\alpha, \eta \in C, \Re(\alpha) > 0$, and further the constants $a_i, b_j \in C, A_i, B_j > 0$, $i = 1, \ldots, p; j = 1, \ldots, q, \rho \in C, \sigma > 0$ satisfy

$$\sigma \max_{1 \le j \le n} \left[\frac{\Re(a_j) - 1}{A_j} \right] + \Re(\rho) < 1 + \Re(\eta)$$

and $1 - \gamma\sigma > \Re(\rho) - \Re(\eta)$.

3.9 Generalized Kober Operators

Notation 3.20. $I[\alpha, \beta, \gamma : m, k, \eta, a : f(x)], I[f(x)]$

Notation 3.21. $J[\alpha, \beta, \gamma : m, k, \delta, a : f(x)], K[f(x)]$

Notation 3.22. $R[f(x)], R \begin{bmatrix} \alpha, \beta, \gamma : \\ \sigma, \rho, a : \end{bmatrix} f(x) \Big]$

Notation 3.23. $K[f(x)], K \begin{bmatrix} \alpha, \beta, \gamma : \\ \zeta, \rho, a : \end{bmatrix} f(x) \Big]$

Let $\alpha, \beta, \gamma, k, \eta, \zeta, \sigma, \rho \in C, x \in R_+$.

Definition 3.14.

$$I[f(x)] = I[\alpha, \beta, \gamma : m, k, \eta, a : f(x)];$$
$$= \frac{k x^{-\eta-1}}{\Gamma(1-\alpha)} \int_0^x {}_2F_1 \left(\alpha, \beta + m; \gamma; \frac{a t^k}{x^k} \right) t^{\eta} f(t) \mathrm{d}t, \qquad (3.96)$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function.

Definition 3.15.

$$
\begin{aligned}
K[f(x)] &= K[\alpha,\beta,\gamma : m,k,\zeta,a : f(x)] \\
&= \frac{k x^{\zeta}}{\Gamma(1-\alpha)} \int_x^{\infty} {}_2F_1\left(\alpha,\beta+m;\gamma;\frac{a x^k}{t^k}\right) t^{-\zeta-1} f(t)\mathrm{d}t. \qquad (3.97)
\end{aligned}
$$

Operators defined by (3.96) and (3.97) exist under the following conditions:

(i) $p \geq 1, q < \infty, \frac{1}{p} + \frac{1}{q} = 1, |\arg(1-a)| < \pi, k > 0.$
(ii) $\Re(1-\alpha) > -m, \Re(\eta) > -1/q, \Re(\zeta) > -1/p, \Re(\gamma-\alpha-\beta-m) > -1, m \in N_0; \gamma \neq 0,-1,-2,\ldots$
(iii) $f \in L_p(0,\infty).$

The equations (3.96) and (3.97) are introduced by Kalla and Saxena (1969).

Remark 3.8. It is interesting to note that for $\gamma = \beta, a = k = 1$, the equations (3.96) and (3.97) reduce to the generalized Kober operators introduced and studied by Saxena (1967b).

Definition 3.16.

$$
\begin{aligned}
R[f(x)] &= R\left[\begin{matrix}\alpha,\beta,\gamma : \\ \sigma,\rho,a :\end{matrix} f(x)\right] \\
&= \frac{x^{-\sigma-\rho}}{\Gamma(\rho)} \int_0^x t^{\sigma}(x-t)^{\rho-1} {}_2F_1\left[\alpha,\beta;\gamma;a\left(1-\frac{t}{x}\right)\right] f(t)\mathrm{d}t. \qquad (3.98)
\end{aligned}
$$

Definition 3.17.

$$
\begin{aligned}
K[f(x)] &= K\left[\begin{matrix}\alpha,\beta,\gamma : \\ \zeta,\rho,a :\end{matrix} f(x)\right] \\
&= \frac{x^{\zeta}}{\Gamma(\rho)} \int_x^{\infty} t^{-\zeta-\rho}(t-x)^{\rho-1} {}_2F_1\left[\alpha,\beta;\gamma;a\left(1-\frac{x}{t}\right)\right] f(t)\mathrm{d}t. \qquad (3.99)
\end{aligned}
$$

The conditions of the validity of the operators (3.98) and (3.99) are given below:

(i) $p \geq 1, q < \infty, \frac{1}{p} + \frac{1}{q} = 1, |\arg(1-a)| < \pi$
(ii) $\Re(\sigma) > -1/q, \Re(\zeta) > -1/p, \Re(\gamma-\alpha-\beta) > 0, \Re(\rho) > 0; \gamma \neq 0,-1,-2,\ldots$
(iii) $f \in L_p(0,\infty).$

Remark 3.9. The operators defined by (3.98) and (3.99) are given by Saxena and Kumbhat (1973). For multidimensional generalized Kober operators associated with Gauss hypergeometric function, which provides an elegant multivariate analogue of the operators (3.98) and (3.99), see Saxena et al. (1990).

When a is replaced by a/α and $\alpha \to \infty$, the operators defined by (3.98) and (3.99) reduce to the following operators associated with confluent hypergeometric functions:

Definition 3.18.

$$R[f(x)] = R\left[\begin{matrix}\beta,\gamma: \\ \sigma,\rho,a:\end{matrix} f(x)\right] = \lim_{\alpha\to\infty} R\left[\begin{matrix}\alpha,\beta,\gamma: \\ \sigma,\rho,a/\alpha:\end{matrix} f(x)\right]$$
$$= \frac{x^{-\sigma-\rho}}{\Gamma(\rho)}\int_0^x t^{\sigma}(x-t)^{\rho-1}\,\Phi\left[\beta;\gamma;a\left(1-\frac{t}{x}\right)\right] f(t)\mathrm{d}t. \tag{3.100}$$

Definition 3.19.

$$K[f(x)] = K\left[\begin{matrix}\beta,\gamma: \\ \zeta,\rho,a:\end{matrix} f(x)\right] = \lim_{\alpha\to\infty} K\left[\begin{matrix}\alpha,\beta,\gamma: \\ \zeta,\rho,a/\alpha:\end{matrix} f(x)\right]$$
$$= \frac{x^{\zeta}}{\Gamma(\rho)}\int_x^{\infty} t^{-\zeta-\rho}(t-x)^{\rho-1}\,\Phi\left[\beta;\gamma;a\left(1-\frac{x}{t}\right)\right] f(t)\mathrm{d}t, \tag{3.101}$$

where $\Re(\rho) > 0, \Re(\zeta) > 0$; and $\Phi(\beta,\gamma;z)$ is the confluent hypergeometric function (Erdélyi et al. 1953, p. 248).

Many interesting and useful properties of the operators defined by (3.98) and (3.99) are investigated by Saxena and Kumbhat (1975), which deal with relations of these operators with well-known integral transforms, such as Laplace, Mellin, and Hankel transforms. Equation (3.98) was first considered by Love (1967).

Remark 3.10. In the special case, $\sigma = 0$, when α is replaced by $\alpha+\beta, \gamma$ by α and β by $-\eta$, then (3.98) reduces to the operator (3.102) considered by Saigo (1978). Similarly, (3.99) reduces to another operator (3.104) introduced by Saigo (1978).

3.10 Saigo Operators

An interesting extension of both the Riemann–Liouville and Erdélyi–Kober fractional integration operators was introduced by Saigo (1978) in terms of Gauss's hypergeometric function. In a series of papers, Saigo (1978, 1979, 1980,1981), Saigo et al. (1992, 1992a), Saigo and Raina (1991), Srivastava and Saigo (1987), Saigo and Saxena (1998), and others obtained several interesting properties of these operators and then applied in many problems. In this section, we present definitions and certain important properties of Saigo operators. Following Saigo (1978), we define the following generalized fractional calculus operators associated with Gauss hypergeometric function in the kernel.

Notation 3.24. $I_{0+}^{\alpha,\beta,\gamma}$: Left-sided generalized fractional integral operator.

Notation 3.25. $I_{-}^{\alpha,\beta,\gamma}$: Right-sided generalized fractional integral operator.

Notation 3.26. $D_{0_+}^{\alpha,\beta,\gamma}$: Left-sided generalized fractional derivative operator.

Notation 3.27. $D_{-}^{\alpha,\beta,\gamma}$: Right-sided generalized fractional derivative operator.

Let $\alpha, \beta, \eta \in C$, and let $x \in \Re_+$ the generalized fractional integral and generalized fractional derivative of a function $f(x)$ on $\Re_+$ are defined in the following forms:

Definition 3.20.

$$(I_{0_+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)\mathrm{d}t, \Re(\alpha) > 0 \tag{3.102}$$

$$= \frac{\mathrm{d}^n}{\mathrm{d}x^n}(I_{0_+}^{\alpha+n,\beta-n,\eta-n} f)(x), \Re(\alpha) \leq 0; n = [\Re(-\alpha)] + 1. \tag{3.103}$$

Definition 3.21.

$$(I_{-}^{\alpha,\beta,\eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)\mathrm{d}t, \Re(\alpha) > 0 \tag{3.104}$$

$$= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(I_{-}^{\alpha+n,\beta-n,\eta} f)(x), \Re(\alpha) \leq 0; n = [\Re(-\alpha)] + 1. \tag{3.105}$$

Definition 3.22.

$$\begin{aligned}(D_{0_+}^{\alpha,\beta,\gamma} f)(x) &= (I_{0_+}^{-\alpha,-\beta,\alpha+\eta} f)(x) \\ &= \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^n (I_{0_+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f)(x), \Re(\alpha) > 0; n = [\Re(\alpha)] + 1.\end{aligned} \tag{3.106}$$

Definition 3.23.

$$\begin{aligned}(D_{-}^{\alpha,\beta,\gamma} f)(x) &= (I_{-}^{-\alpha,-\beta,\alpha+\eta} f)(x) \\ &= \left(-\frac{\mathrm{d}}{\mathrm{dx}}\right)^n (I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta} f)(x), \Re(\alpha) > 0; n = [\Re(\alpha)] + 1.\end{aligned} \tag{3.107}$$

For $\beta = -\alpha$, the operators defined by (3.102), (3.104), (3.106) and (3.107) reduce to the classical Riemann–Liouville fractional calculus operators for $\Re(\alpha) > 0$, namely the Riemann–Liouville operator $I_{0_+}^{\alpha}$, defined in Sect. 3.2 by the equation (3.15), the Weyl operator I_{-}^{α}, defined in Sect. 3.4 by the equation (3.58) and the fractional derivative operators $D_{0_+}^{\alpha}$ and D_{-}^{α}, defined below by the Eqs. (3.109) and (3.111) respectively.

Notation 3.28. D^{α}_{0+} : Riemann–Liouville left-sided fractional derivative of order α.

Notation 3.29. D^{α}_{-} : Riemann–Liouville right-sided fractional derivative of order α.

Definition 3.24.

$$(D^{\alpha,-\alpha,\eta}_{0+} f)(x) = \left(D^{\alpha}_{0+} f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\alpha)]+1} \left(I^{1-\alpha+[\Re(\alpha)]}_{0+} f\right)(x), x > 0 \tag{3.108}$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+[\Re(\alpha)])} \int_0^x \frac{f(t)}{(x-t)^{\alpha-[\Re(\alpha)]}} \mathrm{d}t, x > 0. \tag{3.109}$$

Definition 3.25.

$$(D^{\alpha,-\alpha,\eta}_{-} f)(x) = \left(D^{\alpha}_{-} f\right)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\alpha)]+1} \left(I^{1-\alpha+[\Re(\alpha)]}_{-} f\right)(x), x > 0 \tag{3.110}$$

$$= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\Re(\alpha)]+1} \frac{1}{\Gamma(1-\alpha+[\Re(\alpha)])} \int_x^\infty \frac{f(t)}{(t-x)^{\alpha-[\Re(\alpha)]}} \mathrm{d}t, x>0, \tag{3.111}$$

where the symbol $[\varsigma]$ means the integral part of a real positive number ς that is the largest integer not exceeding ς. In particular for real $\alpha > 0$, D^{α}_{0+} and D^{α}_{-} take the interesting forms

$$\begin{aligned}(D^{\alpha,-\alpha,\eta}_{0+} f)(x) &= \left(D^{\alpha}_{0+} f\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1} \left(I^{1-\{\alpha\}}_{0+} f\right)(x),\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}} \mathrm{d}t, \ x > 0,\end{aligned} \tag{3.112}$$

and

$$\begin{aligned}(D^{\alpha,-\alpha,\eta}_{-} f)(x) &= (D^{\alpha}_{-} f)(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1} \left(I^{1-\{\alpha\}}_{-} f\right)(x)\\ &= \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_x^\infty \frac{f(t)}{(t-x)^{\{\alpha\}}} \mathrm{d}t, \ x > 0,\end{aligned} \tag{3.113}$$

where $\{\varsigma\}$ denotes the fractional part of ς, that is $\{\varsigma\} = \varsigma - [\varsigma]$.

If we set $\beta = 0$, then the operators defined by (3.102) and (3.104) yield the Erdélyi–Kober operators, defined by (3.88) and (3.89) respectively.

3.10.1 Relations Among the Operators

We note that the relation connecting the operators (3.102) and (3.104) is given by

$$\left(I_{-}^{\alpha,\beta,\eta} f\left[\frac{1}{t}\right]\right)(x) = x^{-\beta-1}\left(I_{0+}^{\alpha,\beta,\eta}\left[t^{\beta-1} f(t)\right]\right)\left(\frac{1}{x}\right). \tag{3.114}$$

To prove the result (3.114), we observe that if we start from its left hand side then by a simple change of variable, we obtain the desired result.

When $\beta = -\alpha$, in (3.114), it gives the relation between the operators (3.111) and (3.58) given by Kilbas (2005):

$$\begin{aligned}\left(I_{-}^{\alpha,-\alpha,\eta} f\left[\frac{1}{t}\right]\right)(x) &= \left(I_{-}^{\alpha} f\left[\frac{1}{t}\right]\right)(x) = \left(W_{x,\infty}^{\alpha} f\left[\frac{1}{t}\right]\right)(x)\\ &= x^{\alpha-1}\left(I_{0+}^{\alpha,-\alpha,\eta}\left[t^{-\alpha-1} f(t)\right]\right)\left(\frac{1}{x}\right) = x^{\alpha-1}\left(I_{0+}^{\alpha}\left[t^{-\alpha-1} f(t)\right]\right)\left(\frac{1}{x}\right).\end{aligned} \tag{3.115}$$

On the other hand, for $\beta = 0$, we obtain the relation between the operators (3.88) and (3.89) as

$$\begin{aligned}\left(I_{-}^{\alpha,0,\eta} f\left[\frac{1}{t}\right]\right)(x) &= \left(K_{\eta,\alpha}^{-} f\left[\frac{1}{t}\right]\right)(x) = x^{-1}\left(I_{0+}^{\alpha,0,\eta}\left[t^{-1} f(t)\right]\right)\left(\frac{1}{x}\right)\\ &= x^{-1}\left(I_{\eta,\alpha}^{+}\left[t^{-1} f(t)\right]\right)\left(\frac{1}{x}\right).\end{aligned} \tag{3.116}$$

Note 3.6. We observe that the operators (3.106) and (3.107) are inverse to the operators (3.102) and (3.104):

$$D_{0+}^{\alpha,\beta,\eta} = (I_{0+}^{\alpha,\beta,\eta})^{-1} \text{ and } D_{-}^{\alpha,\beta,\eta} = (I_{-}^{\alpha,\beta,\eta})^{-1}. \tag{3.117}$$

3.10.2 Power Function Formulae

By making use of the following integral

$$\int_0^t x^{\rho-1}(t-x)^{c-1}{}_2F_1\left(a,b;c;1-\frac{x}{t}\right)\mathrm{d}x = \frac{\Gamma(c)\Gamma(\rho)\Gamma(\rho+c-a-b)}{\Gamma(\rho+c-a)\Gamma(\rho+c-b)}t^{\rho+c-1}, \tag{3.118}$$

where $\rho, a, b, c \in C, \Re(\rho) > 0, \Re(c) > 0, \Re(\rho + c - a - b) > 0$ and

$$\int_t^{\infty} x^{\rho-1}(x-t)^{c-1} {}_2F_1(a, b; c; 1 - \frac{t}{x})\mathrm{d}x = \frac{\Gamma(c)\Gamma(1-\rho-c)\Gamma(1-\rho-a-b)}{\Gamma(1-\rho-a)\Gamma(1-\rho-b)} t^{\rho+c-1}, \tag{3.119}$$

where $\rho, a, b, c \in C; \Re(c) > 0, \Re(\rho + c) < 1, \Re(\rho + a + b) < 1$, we obtain the following power function formulae for the operators $\left(I_{0+}^{\alpha,\beta,\eta}\right)$ and $\left(I_{-}^{\alpha,\beta,\eta}\right)$:

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\lambda}\right)(x) = \frac{\Gamma(1+\lambda)\Gamma(1+\lambda+\eta-\beta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)} x^{\lambda-\beta}, \tag{3.120}$$

where α, β, η and $\lambda \in C, \Re(\alpha) > 0$ and $\Re(\lambda) > \max\{0, \Re(\beta - \eta)\} - 1$;

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\lambda}\right)(x) = \frac{\Gamma(\beta-\lambda)\Gamma(\eta-\lambda)}{\Gamma(-\lambda)\Gamma(\alpha+\beta+\eta-\lambda)} x^{\lambda-\beta}, \tag{3.121}$$

where α, β, η, and $\lambda \in C$, $\Re(\alpha) > 0$, $\Re(\lambda) < \min\{\Re(\beta), \Re(\eta)\}$, or if $\Re(\alpha) \le 0$; $0 < \Re(\alpha) + n \le 1$ and $\Re(\lambda) < \min\{\Re(\beta) - n, \Re(\eta)\}$, where n is a positive integer.

For $\beta = -\alpha$, (3.120) and (3.121) give rise to the formulae

$$\left(I_{0+}^{\alpha} t^{\lambda}\right)(x) = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+\alpha)} x^{\lambda+\alpha}, \tag{3.122}$$

where $\alpha, \lambda \in C, \Re(\alpha) > 0, \Re(\lambda) > -1$; and

$$\left(I_{-}^{\alpha} t^{-\lambda}\right)(x) = \frac{\Gamma(\lambda-\alpha)}{\Gamma(\lambda)} x^{\alpha-\lambda}, \tag{3.123}$$

where $\alpha, \lambda \in C, \Re(\lambda) > \Re(\alpha) > 0$.

Similarly, for $\beta = 0$, we obtain

$$\left(I_{\eta,\alpha}^{+} t^{\lambda}\right)(x) = \frac{\Gamma(1+\lambda+\eta)}{\Gamma(1+\alpha+\lambda+\eta)} x^{\lambda}, \tag{3.124}$$

where $\alpha, \lambda, \eta \in C, \Re(\lambda + \eta) > -1$ and

$$\left(K_{\eta,\alpha}^{-} t^{\lambda}\right)(x) = \frac{\Gamma(\eta-\lambda)}{\Gamma(\alpha+\eta-\lambda)} x^{\lambda}, \tag{3.125}$$

where $\alpha, \lambda, \eta \in C, \Re(\alpha) > 0, \Re(\eta) > \Re(\lambda)$.

The discussion in the next two sections is based on the work of Saigo et al. (1992).

3.10.3 Mellin Transform of Saigo Operators

Theorem 3.9. *If* $\alpha, \beta, \eta \in C, \Re(\alpha) > 0$*, and* $\Re(s) < 1 + \min[0, \Re(\eta - \beta)]$*, then the following formula holds for* $f(x) \in L_p(0,\infty)$ *with* $1 \leq p \leq 2$ *or* $f(x) \in M_p(0,\infty)$ *with* $p > 2$*:*

$$M\left\{x^\beta I_{0+}^{\alpha,\beta,\eta} f; s\right\} = \frac{\Gamma(1-s)\Gamma(\eta-\beta+1-s)}{\Gamma(1-s-\beta)\,\Gamma(\alpha+\eta+1-s)} M\{f(x); s\}, \quad (3.126)$$

where $M_p(0,\infty)$*is defined in Sect. 3.6.*

Theorem 3.10. *If* $\alpha, \beta, \eta \in C, \Re(\alpha) > 0$*, and* $\Re(s) > -\min[\Re(\beta), \Re(\eta)]$*, then the following formula holds for* $f(x) \in L_p(0,\infty)$ *with* $1 \leq p \leq 2$ *or* $f(x) \in M_p(0,\infty)$ *with* $p > 2$*:*

$$M\left\{x^\beta I_{-}^{\alpha,\beta,\eta} f; s\right\} = \frac{\Gamma(\beta+s)\Gamma(\eta+s)}{\Gamma(s)\,\Gamma(\alpha+\beta+\eta+s)} M\{f(x); s\}. \quad (3.127)$$

3.10.4 Representation of Saigo Operators

A representation of Erdélyi–Kober operators (3.88) and (3.89) in terms of the Laplace transform operator L and its inverse L^{-1} was given by Fox (1971, 1972). Certain relations connecting L and L^{-1} operators, and fractional integration operators of Saxena (1967) were derived by Kumbhat and Saxena (1975) generalizing the results of Fox (1971, 1972). In this section we present certain representations of the Saigo operators by L and L^{-1}.

Theorem 3.11. *Let* $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\eta - \beta) > 0$ *and* $\Re(\eta) < 0$*. If a function* $f(x)$ *satisfies the following conditions:*

(i) $f(x) \in L(0,\infty)$
(ii) $y^{-\frac{1}{2}} f(y) \in L(0,\infty)$*, where* $f(y)$ *is of bounded variation near the point* $y = x$
(iii) $M\{f(x); s\} = F(s) \in L\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$
(iv) $y^{\beta-\frac{1}{2}} I_{0+}^{\alpha,\beta,\eta} f \in L(0,\infty)$ *and* $y^\beta I_{0+}^{\alpha,\beta,\eta} f$ *is of bounded variation near the point* $y = x$*, then there holds the relation*

$$I_{0+}^{\alpha,\beta,\eta} f = x^{-\alpha-\beta-\eta} L^{-1}\left[t^{-\alpha-\eta} L\left\{x^\beta L^{-1}\left[t^\eta L\left\{x^{\eta-\beta} f(x)\right\}\right]\right\}\right]. \quad (3.128)$$

Remark 3.11. For $\beta = 0$, (3.128) reduces to a result given by Fox (1972, p. 198).

Theorem 3.12. *Let* $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\beta) > 0$ *and* $\Re(\eta) > 0$*. If a function* $f(x)$ *satisfies the following conditions:*

(i) $f(x) \in L(0,\infty)$
(ii) $y^{-\frac{1}{2}} f(y) \in L(0,\infty)$*, where* $f(y)$ *is of bounded variation near the point* $y = x$
(iii) $M\{f(x);\} = F(s) \in L\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$
(iv) $y^{\beta-\frac{1}{2}} I_{-}^{\alpha,\beta,\eta} f \in L(0,\infty)$ *and* $y^{\beta} I_{-}^{\alpha,\beta,\eta} f$ *is of bounded variation near the point* $y = x$*, then there holds the relation*

$$I_{-}^{\alpha,\beta,\eta} f = x^{-\alpha-2\beta-\eta+1} L^{-1}\left[t^{-\alpha-\eta} L\left\{x^{\beta} L^{-1}\left[t^{\eta} L\left\{x^{\eta-1} f\left(\frac{1}{x}\right)\right\}\right]\right\}\right]_{x=\frac{1}{x}}. \tag{3.129}$$

Remark 3.12. For $\beta = 0$, (3.129) reduces to a result given by Fox (1972, p. 199).

Exercises 3.9

3.9.1. Let $\alpha^* > 0$ or $\alpha^* = 0$ and $\gamma\mu + \Re(\delta) < -1$. Further, let $\alpha, \beta, \eta \in C$, $\Re(\alpha) > 0, \Re(\beta) \neq \Re(\eta); \rho \in C$ and $\kappa > 0$ satisfy the conditions

$$\Re(\rho) + \kappa \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] + \min[0, \Re(\eta - \beta)] > 0.$$

for $\alpha^* > 0$ or $\alpha^* = 0, \mu \geq 0$, and

$$\Re(\rho) + \kappa \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu}\right] + \min[0, \Re(\eta - \beta)] > 0,$$

for $\alpha^* = 0$ and $\mu < 0$. Then show that the generalized fractional integration $I_{0+}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n}\left[t^{\kappa}\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right]\right)(x)$$
$$= x^{\rho-\beta-1} H_{p+2,q+2}^{m,n+2}\left[x^{\kappa}\left|\begin{matrix}(1-\rho,\kappa),(1+\beta-\eta-\rho,\kappa),(a_p,A_p)\\(b_q,B_q),(1+\beta-\rho,\kappa),(1-\rho-\alpha-\eta,\kappa)\end{matrix}\right.\right], \tag{3.130}$$

where μ, δ, α^* and γ are defined by (1.17), (1.18), (3.26), and (3.28) respectively.

3.9.2. Let either $\alpha^* > 0$ or $\alpha^* = 0$ and $\gamma\mu + \Re(\delta) < -1$. Further let $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\beta) \neq \Re(\eta); \rho \in C$ and $\kappa > 0$ satisfy the conditions

$$\Re(\rho) + \kappa \max_{1\le i\le n}\left[\frac{\Re(a_i) - 1}{A_i}\right] < 1 + \min[\Re(\beta), \Re(\eta)],$$

for $\alpha^* > 0$ or $\alpha^* = 0, \mu \leq 0$, and

$$\Re(\rho) + \kappa \max_{1 \leq i \leq n} \left[\frac{\Re(a_i) - 1}{A_i}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] < 1 + \min[\Re(\beta), \Re(\eta)],$$

for $\alpha^* = 0$ and $\mu > 0$. Then show that the generalized fractional integration $I_{-}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n} \left[t^{\kappa} \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] \right)(x)$$
$$= x^{\rho-\beta-1} H_{p+2,q+2}^{m+2,n} \left[x^{\kappa} \left| \begin{matrix} (a_p, A_p), (1-\rho, \kappa), (1+\alpha+\beta+\eta-\rho, \kappa) \\ (1-\rho+\beta, \kappa), (1-\rho+\eta, \kappa), (b_q, B_q) \end{matrix} \right. \right], \tag{3.131}$$

where α^* is defined in (3.26).

Note 3.7. In Exercise 3.9.1, left-sided generalized fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of the H-function is considered, whereas Exercise 3.9.2 gives the right-sided generalized fractional integral $I_{-}^{\alpha,\beta,\eta}$ of the H-function.

3.9.3. Let $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\alpha + \beta + \eta) \neq 0, \rho \in C, \kappa > 0$. Let $\alpha^* > 0$ or $\alpha^* = 0$, and $\gamma\mu + \Re(\delta) < -1$ satisfy the conditions

$$\Re(\rho) + \kappa \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{B_j} \right] + \min[0, \Re(\alpha + \beta + \eta)] > 0.$$

for $\alpha^* > 0$, or $\alpha^* = 0, \mu \geq 0$, and

$$\Re(\rho) + \kappa \min_{1 \leq j \leq m} \left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta) + \frac{1}{2}}{\mu} \right] + \min[0, \Re(\alpha + \beta + \eta)] > 0,$$

for $\alpha^* > 0$ and $\mu < 0$. Then show that the generalized fractional differentiation $D_{0+}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n} \left[t^{\kappa} \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] \right)(x)$$
$$= x^{\rho+\beta-1} H_{p+2,q+2}^{m,n+2} \left[x^{\kappa} \left| \begin{matrix} (1-\rho, \kappa), (1-\rho-\eta-\alpha-\beta, \kappa), (a_p, A_p) \\ (b_q, B_q), (1-\rho-\beta, \kappa), (1-\rho-\eta, \kappa) \end{matrix} \right. \right], \tag{3.132}$$

where α^* is defined in (3.26). Hence or otherwise show that the Riemann-Liouville fractional derivative D_{0+}^{α} of the H-function exists and the following result holds:

$$\left(D_{0+}^{\alpha} t^{\rho-1} H_{p,q}^{m,n}\left[t^{\sigma}\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right]\right)(x)$$

$$= x^{\rho-\alpha-1} H_{p+1,q+1}^{m,n+1}\left[x^{\sigma}\left|\begin{matrix}(1-\rho,\sigma),(a_p, A_p)\\(b_q, B_q),(1-\rho+\alpha,\sigma)\end{matrix}\right.\right], \qquad (3.133)$$

provided $\alpha \in C, \Re(\alpha) > 0$, and further $\alpha.\beta, \eta \in C, \Re(\alpha) > 0, \Re(\alpha+\beta+\eta) \neq 0$, $\rho \in C, \sigma > 0$; either $\alpha^* > 0$ or $\alpha^* = 0$, and $\gamma\mu + \Re(\delta) < -1$ satisfy the conditions

$$\Re(\rho) + \sigma \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right] > 0$$

for $\alpha^* > 0$ and $\alpha^* = 0, \mu \geq 0$, and

$$\Re(\rho) + \sigma \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] > 0.$$

for $\alpha^* = 0, \mu < 0, \alpha^*$ is defined in (3.26). (Kilbas and Saigo 1998)

3.9.4. Let either $\alpha^* > 0$ or $\alpha^* = 0$ and $\gamma\mu + \Re(\delta) < -1$. Further let $\alpha, \beta, \eta \in C$, $\Re(\alpha) > 0, \rho \in C; \Re(\alpha+\beta+\eta)+[\Re(\alpha)]+1 \neq 0$, and $\kappa > 0$ satisfy the conditions

$$\Re(\rho) + \max[\Re(\beta), [\Re(\alpha)]+1, -\Re(\alpha+\eta)] + \kappa \max_{1\le i\le n}\left[\frac{\Re(a_i)-1}{A_i}\right] < 1,$$

for $\alpha^* > 0$ or $\alpha^* = 0, \mu \leq 0$, and

$$\Re(\rho)+\max[\Re(\beta), [\Re(\alpha)]+1, -\Re(\alpha+\eta)]+\kappa \max_{1\le i\le n}\left[\frac{\Re(a_i)-1}{A_i}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] < 1,$$

for $\alpha^* = 0$ and $\mu > 0$. Then show that the generalized fractional differentiation $D_{-}^{\alpha,\beta,\eta}$ of the H-function exists and there holds the formula

$$\left(D_{-}^{\alpha,\beta,\eta} t^{\rho-1} H_{p,q}^{m,n}\left[t^{\kappa}\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right]\right)(x) = (-1)^{[\Re(\alpha)]+1} x^{\rho+\beta-1} H_{p+2,q+2}^{m+2,n}$$

$$\times\left[x^{\kappa}\left|\begin{matrix}(a_p, A_p),(1-\rho,\kappa),(1-\rho-\beta+\eta,\kappa)\\(1-\rho-\beta,\kappa),(1-\rho+\alpha+\eta,\kappa),(b_q, B_q)\end{matrix}\right.\right]. \qquad (3.134)$$

Hence or otherwise show that the Riemann–Liouville fractional derivative D_{-}^{α} of the H-function exists and the following relation holds:

$$\left(D_{-}^{\alpha} t^{\rho-1} H_{p,q}^{m,n}\left[t^{\sigma}\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right]\right)(x)$$

$$= (-1)^{[\Re(\alpha)]+1} x^{\rho-\alpha-1} H_{p+1,q+1}^{m+1,n}\left[x^{\sigma}\left|\begin{matrix}(a_p, A_p), (1-\rho,\sigma)\\(1-\rho+\alpha,\sigma), (b_q, B_q)\end{matrix}\right.\right], \quad (3.135)$$

provided $\alpha, \beta, \eta \in C, \Re(\alpha) > 0, \Re(\alpha+\beta+\eta) \neq 0, \rho \in C, \sigma > 0$; further $\alpha^* > 0$ or $\alpha^* = 0$, and $\gamma\mu + \Re(\delta) < -1$ satisfy the conditions

$$\Re(\rho) + \sigma \max_{1\leq j\leq n}\left[\frac{\Re(a_j)-1}{A_j}\right] - \{\Re(\alpha)\} > 0,$$

for $\alpha^* > 0$ or $\alpha^* = 0, \mu \leq 0$; while

$$\Re(\rho) + \sigma \max_{1\leq j\leq n}\left[\frac{\Re(a_j)-1}{A_j}, \frac{\Re(\delta)+\frac{1}{2}}{\mu}\right] - \{\Re(\alpha)\} > 0.$$

for $\alpha^* = 0$ and $\mu > 0$ where $\{\Re(\alpha)\}$ is the fractional part of $\Re(\alpha)$.

(Kilbas and Saigo 1999)

Note 3.8. In Exercise 3.9.3, we consider the left-sided generalized fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ of the H-function, whereas Exercise 3.9.4 provides the right-sided generalized fractional derivative $D_{-}^{\alpha,\beta,\eta}$ of the H-function.

Note 3.9. It is observed that the result of Exercise 3.9.1 also holds for the generalized fractional integro–differentiation $I_{0+}^{\alpha,\beta,\eta}$ of the H-function defined by (3.103). Similarly the result of Exercise 3.9.2 also gives the generalized fractional integro–differentiation $I_{-}^{\alpha,\beta,\eta}$ of the H-function defined by (3.105).

Remark 3.13. Certain properties of the Riemann–Liouville fractional calculus operators associated with generalized Mittag-Leffler function are obtained by Saxena and Saigo (2005). Saigo-Maeda operators of fractional calculus associated with Appell function F_3 (Saigo-Maeda 1998), which are the generalizations of Saigo fractional Calculus operators, are studied by Saxena and Saigo (2001), which provide the extensions of the theorems given in this section (Exercises 3.9.1–3.9.4). For further results on Saigo-Maeda fractional calculus operators, refer to the papers by Saxena et al. (2002) and Kiryakova (2006).

3.9.5. With the help of the following chain rules for the Saigo operators

$$(I_{0+}^{\alpha,\beta,\eta} I_{0+}^{\gamma,\delta,\alpha+\eta} f)(x) = (I_{0+}^{\alpha+\gamma,\beta+\delta,\eta} f)(x),$$

and

$$(I_{-}^{\alpha,\beta,\eta} I_{-}^{\gamma,\delta,\alpha+\eta} f)(x) = (I_{-}^{\alpha+\gamma,\beta+\delta,\eta} f)(x),$$

derive the inverses

$$\left(I_{0_+}^{\alpha,\beta\eta}\right)^{-1} = I_{0_+}^{-\alpha,-\beta,\alpha+\eta}, \tag{3.136}$$

and

$$\left(I_{-}^{\alpha,\beta\eta}\right)^{-1} = I_{-}^{-\alpha,-\beta,\alpha+\eta}. \tag{3.137}$$

3.9.6. Establish the following property of Saigo operators called "integration by parts"

$$\int_0^\infty f(x)\left(I_{0_+}^{\alpha,\beta,\eta} g)\right)(x)\mathrm{d}x = \int_0^\infty g(x)\left(I_{-}^{\alpha,\beta,\eta} f\right)(x)\mathrm{d}x. \tag{3.138}$$

3.9.7. Show that

$$\left(I_{0_+}^{\alpha,\beta,\eta} x^{\sigma-1}(a+bx)^c\right)(x) = a^c \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)} \times {}_3F_2\left(1, \eta-\beta+1, -c; 1-\beta, \alpha+\eta+1; -\frac{bx}{a}\right).$$

Also give the conditions of validity of this result.

3.11 Multiple Erdélyi–Kober Operators

Fractional integration operators associated with the H-functions are studied by Saxena et al. (1974), Kalla (1969), Kalla and Kiryakova (1990), Srivastava and Buschman (1973). A detailed and comprehensive account of fractional integration operators and their applications studied by various authors during the last four decades can be found in the paper of Srivastava and Saxena (2001). The discussion in this section is based on the work of Galué et al. (1993).

Notation 3.30. $I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)}$: Multiple Erdélyi–Kober operator of Riemann–Liouville type.

Notation 3.31. C_α: Space of continuous functions.

Notation 3.32. $K_{(\varepsilon_k),(\xi_k),n}^{(\tau_k),(\alpha_k)} f(x)$: Multiple Erdélyi–Kober operator of Weyl type.

Notation 3.33. $C_{\alpha^*}^*$: Space of continuous functions.

Definition 3.26. Space of functions C_α is defined as

$$C_\alpha = \{f(x) = x^p f^\sim(x) : p > \alpha, f^\sim(x) \in C[0,\infty)\}$$
$$\text{with } \alpha =^{\max}_{1\le k\le m} [-\beta(\gamma_k+1)] \tag{3.139}$$

Definition 3.27. Space of functions $C^*_{\alpha^*}$ is defined as

$$C^*_{\alpha^*} = \{f(x) = x^q g(x); q < \alpha^*, g \in C(0,\infty); |g| \leq A_g\} \quad \text{with } \alpha^* = \min_{1\leq k\leq m} (\beta\tau_k) \tag{3.140}$$

Definition 3.28. A multiple Erdélyi–Kober operator of Riemann–Liouville type is defined in the form

$$I[f(x)] = I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x)$$
$$= \begin{cases} \int_0^1 H^{m,0}_{m,m}\left[u \left| \begin{matrix} (\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\lambda_k},\frac{1}{\lambda_k})_1^m \end{matrix} \right. \right] f(xu)\mathrm{d}u \\ \quad \text{if } \sum_1^m \delta_k > 0. \\ f(x), \quad \text{if } \delta_k = 0 \text{ and } \lambda_k = \beta_k, k = 1,\ldots,m, \end{cases} \tag{3.141}$$

where $m \in Z^+, \beta_k > 0, \delta_k \geq 0$, and $\gamma_k, k = 1,\ldots,m$ are real numbers. Furthermore

$$\sum_{k=1}^m \frac{1}{\lambda_k} \geq \sum_{k=1}^m \frac{1}{\beta_k},$$

and $f(x) \in C_\alpha$, where C_α is defined by (3.139), and

$$\alpha \geq \max_{1\leq k\leq m} [-\lambda_k(\gamma_k + 1)].$$

The definition (3.141) can be rewritten in the familiar form :

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x) = \frac{1}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\left| \begin{matrix} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\lambda_k},\frac{1}{\lambda_k}\right)_1^m \end{matrix} \right. \right] f(t)\mathrm{d}t. \tag{3.142}$$

Remark 3.14. It is interesting to note that for $\lambda_k = \beta_k, k = 1,2,\ldots,m$, we obtain the operator defined by Kalla and Kiryakova (1990). If, however, we set $m = 1$ and $\beta_k = \lambda_k, k = 1,\ldots,m$, we obtain a slight variant form of Erdélyi–Kober operator defined in (3.88). The following properties of this operator holds.

3.11.1 A Mellin Transform

$$M\left\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x); s\right\} = \prod_{k=1}^m \frac{\Gamma(\gamma_k + 1 - \frac{s}{\lambda_k})}{\Gamma(\gamma_k + \delta_k + 1 - \frac{s}{\lambda_k})} M\{f(x); s\}, \tag{3.143}$$

where
$$\sum_{k=1}^{m}\frac{1}{\lambda_k}-\sum_{k=1}^{m}\frac{1}{\beta_k}>0 \text{ and } \Re(s)<\min_{1\le k\le m}[\lambda_k(1+\gamma_k)].$$

3.11.2 Properties of the Operators

Some basic properties of the operator defined by (3.141) are given below:

$$I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)}x^{\rho}=\prod_{k=1}^{m}\frac{\Gamma(\gamma_k+1+\frac{\rho}{\lambda_k})}{\Gamma(\gamma_k+\delta_k+1+\frac{\rho}{\lambda_k})}x^{\rho}, \tag{3.144}$$

where $\Re(\rho)+\max_{1\le k\le m}[\lambda_k(1+\gamma_k)]>0$.

$$I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)}I_{(\varepsilon_k),(\xi_k),n}^{(\tau_k),(\alpha_k)}f(x)=I_{((\beta_k)_1^m,(\varepsilon_k)_1^n),((\lambda_k)_1^m,(\xi_k)_1^n),m+n}^{((\gamma_k)_1^m,(\tau_k)_1^n),((\delta_k)_1^m,(\alpha_k)_1^n)}f(x), \tag{3.145}$$

where
$$\sum_{k=1}^{n}\frac{1}{\xi_k}-\sum_{k=1}^{n}\frac{1}{\varepsilon_k}>0,\ A=\sum_{k=1}^{m}\frac{1}{\lambda_k}-\sum_{k=1}^{m}\frac{1}{\beta_k}>0,$$
$\max_{1\le k\le m}[1-\lambda_k(\gamma_k+1)]<0<\max_{1\le k\le n}[\xi_k(\tau_k+1)-1]$, and
$$\left|\arg\left(\frac{1}{x}\right)\right|<(\pi A/2).$$

The inverse of the operator $I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)}$ is given by

$$\left(I_{(\beta_k),(\lambda_k),m}^{(\gamma_k),(\delta_k)}\right)^{-1}f(x)=I_{(\lambda_k),(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)}f(x). \tag{3.146}$$

The results (3.143) and (3.146) are useful in deriving the solutions of a certain class of integral equations.

Definition 3.29. Another multiple Erdélyi–Kober fractional integral operator of Weyl type is defined by

$$K[f(x)]=K_{(\varepsilon_k),(\xi_k)n}^{(\tau_k),(\alpha_k)}f(x)=\int_x^{\infty}H_{n,n}^{n,0}\left[\frac{1}{u}\left|\begin{matrix}(\tau_k+\alpha_k+\frac{1}{\varepsilon_k},\frac{1}{\varepsilon_k})_1^n\\(\tau_k+\frac{1}{\xi_k},\frac{1}{\xi_k})_1^n\end{matrix}\right.\right]f(xu)\mathrm{d}u, \tag{3.147}$$

if $\sum_1^n\alpha_k>0$, and $f(x)$, if $\alpha_k=0$ and $\varepsilon_k=\xi_k, k=1,\ldots,n$, where

$$\sum_{k=1}^{n}\frac{1}{\xi_k}-\sum_{k=1}^{n}\frac{1}{\varepsilon_k}>0,$$

$n \in N, \varepsilon_k > 0, \xi_k > 0, \alpha_k \geq 0$ and $\tau_k, k = 1, \ldots, n$ are real numbers, $f(x) \in C^*_{\alpha^*}$, where $C^*_{\alpha^*}$ is defined by (3.140) and

$$\alpha^* \leq \min_{1 \leq k \leq n} (\xi_k \tau_k).$$

The definition (3.147) can easily be put in the familiar form :

$$K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} f(x) = \frac{1}{x} \int_x^\infty H^{n,0}_{n,n} \left[\frac{x}{t} \middle| \begin{matrix} (\tau_k + \alpha_k + \frac{1}{\varepsilon_k}, \frac{1}{\varepsilon_k})_1^n \\ (\tau_k + \frac{1}{\xi_k}, \frac{1}{\xi_k})_1^n \end{matrix} \right] f(t) \mathrm{d}t, \qquad (3.148)$$

provided that

$$\sum_{k=1}^n \alpha_k > 0.$$

Remark 3.15. It is interesting to note that for $\varepsilon_k = \xi_k, k = 1, 2, \ldots, n$, we obtain the operator defined by Kalla and Kiryakova (1990). If, however, we set $n = 1$ and $\varepsilon_k = \xi_k, k = 1, \ldots, n$, we obtain a slight variation of the Erdélyi–Kober operator of Weyl type defined in (3.89). The following properties of this operator holds.

3.11.3 Mellin Transform of a Generalized Operator

It can be easily seen with the help of the Mellin transform of the H-function given by the equation (2.8) that

$$M\left\{K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} f(x); s\right\} = \prod_{k=1}^n \frac{\Gamma(\tau_k + \frac{s}{\xi_k})}{\Gamma(\tau_k + \alpha_k + \frac{s}{\varepsilon_k})} M\{f(x); s\}, \qquad (3.149)$$

where

$$\sum_{k=1}^n \frac{1}{\xi_k} - \sum_{k=1}^n \frac{1}{\varepsilon_k} > 0 \text{ and } \max_{1 \leq k \leq n} (-\xi_k \tau_k) < \Re(s).$$

The power function formula for the operator $K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n}$ is given by

$$K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} x^\rho = \prod_{k=1}^m \frac{\Gamma(\tau_k - \frac{\rho}{\xi_k})}{\Gamma(\tau_k + \alpha_k - \frac{\rho}{\varepsilon_k})} x^\rho, \qquad (3.150)$$

where $\Re(\rho) < \min_{1 \leq k \leq m} [\tau_k \xi_k]$. Further

$$K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} K^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x) = K^{((\tau_k)_1^n,(\gamma_k)_1^m),((\alpha_k)_1^n,(\delta_k)_1^m)}_{((\varepsilon_k)_1^n,(\beta_k)_1^m),((\xi_k)_1^n,(\lambda_k)_1^m),m+n} f(x), \qquad (3.151)$$

where

$$B = \sum_{k=1}^{n} \frac{1}{\xi_k} - \sum_{k=1}^{n} \frac{1}{\varepsilon_k} > 0, \sum_{k=1}^{m} \frac{1}{\lambda_k} - \sum_{k=1}^{m} \frac{1}{\beta_k} > 0,$$
$$\max_{1 \le k \le m} (-\lambda_k \gamma_k - 1) < 0 < \min_{1 \le k \le n} (\xi_k \tau_k + 1),$$

and

$$|\arg x| < \frac{1}{2}\pi B.$$

Finally, the inverse of the operator $K_{(\varepsilon_k),(\xi_k),n}^{(\tau_k),(\alpha_k)}$ is given by

$$\left(K_{(\varepsilon_k),(\xi_k),n}^{(\tau_k),(\alpha_k)}\right)^{-1} f(x) = K_{(\xi_k),(\varepsilon_k),n}^{(\tau_k+\alpha_k),(-\alpha_k)} f(x). \tag{3.152}$$

Remark 3.16. Solutions of certain dual integral equations involving general H-functions have been developed by Galué et al. (1993) by the application of the operators (3.141) and (3.147). It is interesting to observe that the results given earlier by Kalla and Kiryakova (1990) for the multiple Erdélyi–Kober and Weyl operators follow easily from the results of this section.

Remark 3.17. Representations of fractional integration operators of multiple Riemann–Liouville and Weyl type defined by (3.141) and (3.147), in terms of the Laplace and inverse Laplace transforms, are recently obtained by Saxena et al. (2006). Integral formulae for the H-function generalized fractional integration operators discussed in this section are derived by Saxena et al. (2004a, 2007). Integral formulas for the generalized Erdélyi–Kober operator of Weyl type, defined by the equation (3.147), are recently evaluated by Saxena et al. (2005).

Chapter 4
Applications in Statistics

4.1 Introduction

Special functions are used in almost all areas of statistics. Statistical densities are basically elementary special functions or product of such functions. Hence, the theory of special functions is directly applicable to statistical distribution theory. While studying generalized densities, structural properties of densities, Bayesian inference, distributions of test statistics, characterization of densities and related studies of probability theory, stochastic processes and time series problems, and special functions and generalized special functions in the categories of Meijer's G-functions and H-functions come in naturally.

When looking at multivariate and matrix-variate distributions, the theory of special functions of matrix argument is directly applicable. Functions of matrix argument in the categories of matrix variable gamma, type-1 beta and type-2 beta, are the most commonly used special functions in current statistical literature.

In this chapter, a brief introduction to the applications of H-functions in statistical distribution theory will be given. Problems which fit directly into the definition of an H-function are dealt with in this chapter. With the knowledge of the basic materials discussed in this chapter, the reader will be able to tackle more complicated situations of applications of special functions in statistics. Only the real variable case is discussed in this chapter.

4.2 General Structures

General structures in statistical literature where H-function will be applicable are many. The simplest of the structures are products and ratios of statistically independently distributed positive real scalar random variables. A real scalar random variable x is said to have a generalized gamma density when the density is of the form

$$f(x) = \begin{cases} \frac{\beta a^{\frac{\alpha}{\beta}}}{\Gamma\left(\frac{\alpha}{\beta}\right)} x^{\alpha-1} \mathrm{e}^{-a x^{\beta}}, & x > 0, a > 0, \alpha > 0, \beta > 0 \\ 0, \text{elsewhere.} \end{cases} \tag{4.1}$$

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Note 4.1. Usually, in statistical problems, the parameters are real; hence, we will assume that the parameters $a, \alpha,\ and\ \beta$ are real.

Let

$$u = x_1 x_2 \cdots x_k, \tag{4.2}$$

where x_j has the density in (4.1) with the parameters $a_j > 0, \alpha_j > 0, \beta_j > 0, j = 1, 2, \cdots, k$ and let $x_1, \cdots, x_k$ be statistically independently distributed. Note that for $\beta_j = 1$ in (4.1), one has the standard gamma density. Hence, if y_1 has the density in (4.1) with $\beta_j = 1$, then a density of the structure in (4.1) can be created by considering $x_j = y_j^{\beta_j}, j = 1, \cdots, k$. Hence,

$$u^* = y_1^{\beta_1} \cdots y_k^{\beta_k}, \tag{4.3}$$

and u in (4.2) can be studied by using the same procedures. If one is interested in deriving the exact density of (4.2), then one of the methods, and possibly the easiest way, is to compute the Mellin transform of the density of u. If the unknown density of u is denoted by $g(u)$, one can evaluate the Mellin transform of $g(u)$, without knowing $g(u)$, by making use of the independence properties of $x_1, \cdots, x_k$. In the standard terminology in statistical literature, let E denote the mathematical expectation, then $E(x^h)$, when x has the density in (4.1), is given by

$$E(x^h) = \frac{\Gamma\left(\frac{\alpha+h}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\beta}\right) a^{\frac{h}{\beta}}}, \quad \text{for } \Re(\alpha + h) > 0, \tag{4.4}$$

where $\Re(\cdot)$ denotes the real part of $(\cdot)$. Thus, when α and h are real, this expected value or the hth moment of x can exist for some negative value of h also such that $\alpha + h > 0$. Due to statistical independence,

$$\begin{aligned} E(u^h) &= [E(x_1^h)][E(x_2^h)] \cdots [E(x_k^h)] \\ &= \prod_{j=1}^{k} \frac{\Gamma\left(\frac{\alpha_j+h}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right) a_j^{\frac{h}{\beta_j}}}, \quad \Re(\alpha_j + h) > 0, \quad j = 1, \ldots, k. \end{aligned} \tag{4.5}$$

But, with h replaced by $s - 1$, one has the Mellin transform of $g(u)$. That is,

$$\begin{aligned} E(u^{s-1}) &= \int_0^\infty u^{s-1} g(u) \mathrm{d}u \\ &= \prod_{j=1}^{k} \frac{\Gamma\left(\frac{\alpha_{j-1}}{\beta_j} + \frac{s}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)} \frac{a_j^{\frac{1}{\beta_j}}}{a_j^{\frac{s}{\beta_j}}}, \quad \Re(\alpha_j + s - 1) > 0, \quad j = 1, \ldots, k. \end{aligned} \tag{4.6}$$

Then, the unknown density $g(u)$ of u is available from the inverse Mellin transform. That is,

$$\begin{aligned} g(u) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[E(u^{s-1})\right]u^{-s}\mathrm{d}s,\ \ i=\sqrt{-1},\ c>-\alpha_j+1,\ j=1,\cdots,k \\ &= \left\{\prod_{j=1}^{k}\frac{a_j^{\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)}\right\}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\prod_{j=1}^{k}\Gamma\left(\frac{\alpha_j-1}{\beta_j}+\frac{s}{\beta_j}\right)\right\}\left[\left(\prod_{j=1}^{k}a_j^{\frac{1}{\beta_j}}\right)u\right]^{-s}\mathrm{d}s \\ &= \left\{\prod_{j=1}^{k}\frac{a_j^{\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)}\right\}H_{0,k}^{k,0}\left[a_1^{\frac{1}{\beta_1}}\cdots a_k^{\frac{1}{\beta_k}}u\Big|_{\left(\frac{\alpha_j-1}{\beta_j},\frac{1}{\beta_j}\right),j=1,\ldots,k}\right],\ \ 0<u<\infty, \end{aligned} \tag{4.7}$$

and 0 elsewhere, is the density of u.

Note that for $\beta_j = 1,\ \ j = 1,\cdots,k$, the H-function in (4.7) reduces to a Meijer's G-function $G_{0,k}^{k,0}(\cdot)$. Further, for special values of k, one can evaluate (4.7) in terms of elementary special functions.

Note 4.2. Since statistical densities, in general, can be written in terms of elementary special functions and the H-function is a very generalized special function, one can represent almost all densities, in current use, in terms of H-functions.

Note 4.3. Special cases of the gamma density in (4.1) include the following:

(a) Weibull density ($\beta = \alpha$);
(b) chisquare density ($\beta = 1, a = \frac{1}{2}, \alpha = \frac{m}{2}, m = 1, 2, \cdots$);
(c) standard gamma density ($\beta = 1$);
(d) exponential density ($\beta = 1, \alpha = 1$);
(e) folded Gaussian ($\beta = 2, \alpha = 1$);
(f) chi density ($\beta = 2, \alpha = 1, n = 1, 2, \cdots$);
(g) Helley's density ($\beta = 1, \alpha = 1, a = \frac{mg}{KT}$);
(h) Helmert's density ($\beta = 2, a = \frac{n}{2\sigma^2}, \alpha = n - 1 > 0$);
(i) Maxwell–Boltzmann density ($\beta = 2, \alpha = 3$);
(j) Rayleigh density ($\beta = 2, \alpha = 2$).

Note 4.4. When x in (4.1) is replaced by $|x|, -\infty < x < \infty$, we obtain more generalized densities. The most important special cases will then be the Gaussian ($\beta = 2, \alpha = 1$) and the Laplace density ($\beta = 1, \alpha = 1$).

4.2.1 Product of Type-1 Beta Random Variables

A real scalar random variable is said to have a real type-1 beta distribution, if the density is of the following form:

$$f_1(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1, \alpha > 0, \beta > 0 \\ 0, \text{elsewhere,} \end{cases} \tag{4.8}$$

where the parameters α and β are assumed to be real. The following discussion holds even when α and β are complex quantities. In that case, the condition becomes $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ where $\Re(\cdot)$ means the real part of $(\cdot)$. The hth moment of x, when x has the density in (4.8), is given by

$$E(x^h) = \frac{\Gamma(\alpha+h)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+h)}, \Re(\alpha+h) > 0. \tag{4.9}$$

When α and h are real, the moments can exist for some negative values of h also such that $\alpha + h > 0$. The Mellin transform of $f_1(x)$ is obtained from (4.9), by replacing h by $s-1$ for some complex s.

Consider a set of real scalar random variables $x_1, \cdots, x_k$, mutually independently distributed, where x_j has the density in (4.8) with the parameters (α_j, β_j), $j = 1, \cdots, k$ and consider the product

$$u_1 = x_1 x_2 \cdots x_k. \tag{4.10}$$

Then, the Mellin transform of the density $g_1(u)$ of u_1 is obtained from the property of statistical independence and is given by,

$$\begin{aligned}
\int_0^\infty u^{s-1} g_1(u)\mathrm{d}u &= E(u_1^{s-1}) = [E(x_1^{s-1})]\cdots[E(x_k^{s-1})] \\
&= \prod_{j=1}^k \frac{\Gamma(\alpha_j+s-1)}{\Gamma(\alpha_j)}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j+\beta_j+s-1)} \\
&= \left[\prod_{j=1}^k \frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)}\right]\left[\prod_{j=1}^k \frac{\Gamma(\alpha_j+s-1)}{\Gamma(\alpha_j+\beta_j+s-1)}\right]. \qquad (4.11)
\end{aligned}$$

Then, the unknown density $g_1(u)$ is available by taking the inverse Mellin transform of (4.11). This can be written in terms of a Meijer's G-function of the type $G_{k,k}^{k,0}(\cdot)$. We can consider more general structures in the same category. For example, consider the structure

$$u_2 = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_k^{\gamma_k}, \gamma_j > 0, j = 1, \ldots, k \tag{4.12}$$

where $x_1, \ldots, x_k$ are mutually independently distributed as in (4.10). Then, observing that

$$\begin{aligned}
E(u_2^{s-1}) &= E(x_1^{\gamma_1(s-1)})E(x_2^{\gamma_2(s-1)})E(x_k^{\gamma_k(s-1)}) \qquad (4.13) \\
&= \left\{\prod_{j=1}^k \frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)}\right\}\left\{\prod_{j=1}^k \frac{\Gamma(\alpha_j-\gamma_j+\gamma_j s)}{\Gamma(\alpha_j+\beta_j-\gamma_j+\gamma_j s)}\right\}, \qquad (4.14) \\
&\Re(\alpha_j-\gamma_j+\gamma_j s) > 0, j = 1, \cdots, k,
\end{aligned}$$

the density $g_2(u_2)$ of u_2 is available by taking the inverse Mellin transform, that is,

$$\begin{aligned} g_2(u_2) &= \left\{\prod_{j=1}^{k}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)}\right\}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\prod_{j=1}^{k}\frac{\Gamma(\alpha_j-\gamma_j+\gamma_j s)}{\Gamma(\alpha_j+\beta_j-\gamma_j+\gamma_j s)}u_2^{-s}\mathrm{d}s \\ &= \left\{\prod_{j=1}^{k}\frac{\Gamma(\alpha_j+\beta_j)}{\Gamma(\alpha_j)}\right\}H_{k,k}^{k,0}\left[u_2\Big|_{(\alpha_j-\gamma_j,\gamma_j),j=1,\cdots,k}^{(\alpha_j+\beta_j-\gamma_j,\gamma_j),j=1,\cdots,k}\right], 0<u_2<1. \end{aligned} \tag{4.15}$$

Observe that when $\gamma_j = 1,\ j = 1,\cdots,k$, the H-function reduces to the G-function. The case in (4.15) is slightly different from x_j having a generalized type-1 beta density and then considering the product $x_1\cdots x_k$. Suppose x_j has a generalized type-1 beta density given by

$$f_2(x) = \begin{cases} \frac{\gamma a^{\frac{\alpha}{\gamma}}}{B\left(\frac{\alpha}{\gamma},\beta\right)}x^{\alpha-1}(1-ax^{\gamma})^{\beta-1}, 0<x<a^{-\frac{1}{\gamma}}, \alpha>0, \beta>0, \gamma>0, a>0, \\ \quad 1-ax^{\gamma}>0, \\ 0, \text{elsewhere}, \end{cases} \tag{4.16}$$

where $B(\cdot,\cdot)$ is a beta function

$$B\left(\frac{\alpha}{\gamma},\beta\right) = \frac{\Gamma\left(\frac{\alpha}{\gamma}\right)\Gamma(\beta)}{\Gamma\left(\frac{\alpha}{\gamma}+\beta\right)}, \alpha>0, \beta>0, \gamma>0.$$

If x follows the density in (4.16), then the $(s-1)$th moment of x is given by,

$$E(x^{s-1}) = \int_0^{a^{-\frac{1}{\gamma}}} x^{s-1}f_2(x)\mathrm{d}x = \frac{\Gamma\left(\frac{\alpha+s-1}{\gamma}\right)}{a^{\frac{s-1}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}\right)}\frac{\Gamma\left(\frac{\alpha}{\gamma}+\beta\right)}{\Gamma\left(\frac{\alpha+s-1}{\gamma}+\beta\right)}. \tag{4.17}$$

Let, x_j have the density in (4.16) with parameters $(a_j,\alpha_j,\beta_j,\gamma_j),\ j = 1,\cdots,k$ and let $x_1,\cdots,x_k$ be independently distributed. Then, if

$$u_3 = x_1\cdots x_k, \tag{4.18}$$

then

$$E(u_3^{s-1}) = \prod_{j=1}^{k}\left\{\frac{1}{a_j^{\frac{s-1}{\gamma_j}}}\frac{\Gamma\left(\frac{\alpha_j+s-1}{\gamma_j}\right)}{\Gamma\left(\frac{\alpha_j}{\gamma_j}\right)}\frac{\Gamma\left(\frac{\alpha_j}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j+s-1}{\gamma_j}+\beta_j\right)}\right\}. \tag{4.19}$$

The density of u_3, denoted by $g_3(u_3)$, is available from the inverse Mellin transform in (4.19). That is,

$$g_3(u_3)=\left\{\prod_{j=1}^{k}\frac{a_j^{\frac{1}{\gamma_j}}\Gamma\left(\frac{\alpha_j}{\gamma_j}+\beta_j\right)}{\Gamma\left(\frac{\alpha_j}{\gamma_j}\right)}\right\}H_{k,k}^{k,0}\left[a_1^{\frac{1}{\gamma_1}}\cdots a_k^{\frac{1}{\gamma_k}}u_3\Big|_{\left(\frac{\alpha_j-1}{\gamma_j},\frac{1}{\gamma_j}\right),j=1,\ldots,k}^{\left(\frac{\alpha_j-1}{\gamma_j}+\beta_j,\frac{1}{\gamma_j}\right),j=1,\cdots,k}\right]$$

$$0<a_1^{\frac{1}{\gamma_1}}\cdots a_k^{\frac{1}{\gamma_k}}u_3<1. \tag{4.20}$$

Note that (4.20) is different from (4.15).

4.2.2 Real Scalar Type-2 Beta Structure

A real scalar random variable x is said to have a type-2 beta density, if x has the density

$$f_3(x)=\begin{cases}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1+x)^{-(\alpha+\beta)}, 0<x<\infty, \alpha>0, \beta>0\\ 0, \text{elsewhere.}\end{cases} \tag{4.21}$$

Then, the Mellin transform of $f_3(x)$ is given by,

$$\int_0^\infty x^{s-1}f_3(x)\mathrm{d}x=E(x^{s-1})=\frac{\Gamma(\alpha+s-1)}{\Gamma(\alpha)}\frac{\Gamma(\beta-s+1)}{\Gamma(\beta)}$$
$$\text{for } \Re(\alpha+s-1)>0, \Re(\beta-s+1)>0. \tag{4.22}$$

This is obtained from the normalizing constant in (4.21) by observing that $\alpha+\beta=(\alpha+s-1)+(\beta-s+1)$. As in the previous cases, consider

$$u_4=x_1^{\gamma_1}\cdots x_k^{\gamma_k}, \tag{4.23}$$

where $\gamma_1>0,\cdots,\gamma_k>0$, with x_j having the density in (4.21) with the parameters (α_j,β_j), $j=1,\cdots,k$ and $x_1,\cdots,x_k$ are independently distributed. Then, as in the previous situations, the Mellin transform of the density $g_4(u_4)$ of u_4 is given by

$$\int_0^\infty u_4^{s-1}g_4(u_4)\mathrm{d}u_4=E(u_4^{s-1})=\prod_{j=1}^{k}\frac{\Gamma(\alpha_j+\gamma_j s-\gamma_j)}{\Gamma(\alpha_j)}\frac{\Gamma(\beta_j-\gamma_j s+\gamma_j)}{\Gamma(\beta_j)}, \tag{4.24}$$

for $\Re(\alpha_j-\gamma_j+\gamma_j s)>0,\ \Re(\beta_j+\gamma_j-\gamma_j s)>0,\ j=1,\cdots,k.$

Then, by taking the inverse Mellin transform in (4.24), one has the density,

$$g_4(u_4) = \left\{\prod_{j=1}^{k} \frac{1}{\Gamma(\alpha_j)\Gamma(\beta_j)}\right\} H_{k,k}^{k,k}\left[u_4\Big|_{(\alpha_j-\gamma_j,\gamma_j),\ j=1,\cdots,k}^{(1-\beta_j-\gamma_j,\gamma_j),\ j=1,\cdots,k}\right],\ 0 < u_4 < \infty. \tag{4.25}$$

As illustrated before, the density of a product of generalized type-2 beta random variables will be different from (4.25). A generalized type-2 beta density has the form

$$f_4(x) = \begin{cases} \frac{\gamma\ a^{\frac{\alpha}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}+\beta\right)}{\Gamma(\frac{\alpha}{\gamma})\Gamma(\beta)} x^{\alpha-1}(1+ax^{\gamma})^{-(\alpha+\beta)}, 0 < x < \infty, \alpha > 0, \\ \quad \beta > 0, a > 0, \gamma > 0, \\ 0,\ \text{elsewhere.} \end{cases} \tag{4.26}$$

Thus, for all such special cases mentioned in Notes 4.2 and 4.3, the procedure discussed in this section is applicable. Observing that negative moments of the form $E(x^{-h}), h > 0$ are available from $E(x^h)$ with h replaced by $-h$ if $E(x^{-h})$ exists.

4.2.3 A More General Structure

We can consider more general structures. Let,

$$w = \frac{x_1x_2\cdots x_r}{x_{r+1}\cdots x_k}, \tag{4.27}$$

where $x_1, \cdots, x_k$ are mutually independently distributed real random variables having the density in (4.1) with x_j having parameters $a_j,\ \alpha_j,\ \beta_j,\ j = 1, \cdots, k$. Then,

$$E(w^h) = E(x_1^h)E(x_2^h)\cdots E(x_r^h)E(x_{r+1}^{-h})\cdots E(x_k^{-h}), \tag{4.28}$$

provided the right side in (4.28) exists. Then, from (4.4) we have,

$$E(w^{s-1}) = \left\{\prod_{j=1}^{r} \frac{\Gamma\left(\frac{\alpha_j+h}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)\ a_j^{\frac{h}{\beta_j}}}\right\} \left\{\prod_{j=r+1}^{k} \frac{\Gamma\left(\frac{\alpha_j-h}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)\ a_j^{-\frac{h}{\beta_j}}}\right\},\ h = s-1 \tag{4.29}$$

$$= \left\{\prod_{j=1}^{r} \frac{a_j^{\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)}\right\} \left\{\prod_{j=r+1}^{k} \frac{1}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)\ a_j^{\frac{1}{\beta_j}}}\right\}$$
$$\times \left\{\prod_{j=1}^{r} \Gamma\left(\frac{\alpha_j-1}{\beta_j}+\frac{s}{\beta_j}\right)\frac{1}{a_j^{\frac{s}{\beta_j}}}\right\} \left\{\prod_{j=r+1}^{k} \Gamma\left(\frac{\alpha_j+1}{\beta_j}-\frac{s}{\beta_j}\right) a_j^{\frac{s}{\beta_j}}\right\}, \tag{4.30}$$

for $\alpha_j + s - 1 > 0,\ j = 1, \cdots, r,\ \alpha_j - s + 1 > 0,\ j = r+1, \cdots, k$. Hence, the density of w, denoted by $g^*(w)$, is available from the inverse Mellin transform. That is,

$$g^*(w) = c^* \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ \prod_{j=1}^{r} \Gamma\left(\frac{\alpha_j - 1}{\beta_j} + \frac{s}{\beta_j}\right) \right\} \left\{ \prod_{j=r+1}^{k} \Gamma\left(\frac{\alpha_j + 1}{\beta_j} - \frac{s}{\beta_j}\right) \right\}$$
$$\times \left[\frac{\prod_{j=1}^{r} a_j^{\frac{1}{\beta_j}}}{\prod_{j=r+1}^{k} a_j^{\frac{1}{\beta_j}}} w \right]^{-s} \mathrm{d}s,\ i = \sqrt{-1},\ \max_{j=1,\cdots,r}(1-\alpha_j) < c < \min_{j=r+1,\cdots,k}(\alpha_j + 1)$$
$$= H_{k-r,r}^{r,k-r}\left[\delta u \Big|_{\left(\frac{\alpha_j - 1}{\beta_j}, \frac{1}{\beta_j}\right),\ j=1,\cdots,r}^{(1-\frac{\alpha_j+1}{\beta_j}, \frac{1}{\beta_j}),\ j=r+1,\cdots,k} \right],\ 0 < u < \infty \tag{4.31}$$

where $$\delta = \frac{\prod_{j=1}^{r} a_j^{\frac{1}{\beta_j}}}{\prod_{j=r+1}^{k} a_j^{\frac{1}{\beta_j}}},\ c^* = \frac{\delta}{[\prod_{j=1}^{k} \Gamma\left(\frac{\alpha_j}{\beta_j}\right)]}. \tag{4.32}$$

Remark 4.1. In statistical applications, sometimes, the variables $x_1, \cdots, x_k$ are independently and identically distributed. In this case, the parameters in (4.28)–(4.32) will be such that $a_j = a,\ \beta_j = \beta,\ \alpha_j = \alpha$ for some $a,\ \beta,\ \alpha$ and for $j = 1, \cdots, k$.

Remark 4.2. In (4.27), we took x_j's belonging to the generalized gamma density in (4.1). But, we could have considered w consisting of x_j's belonging to (4.1), (4.8), (4.12), (4.21), (4.23), and (4.26) or mixed cases provided $E(w^{s-1})$ exists. Then, the density of such a general structure will be available by taking the inverse Mellin transform of $E(w^{s-1})$. The density of w can be written in terms of an H-function. More of such cases are contained in the pathway model to be discussed in the next section.

Exercises 4.1

4.1.1. If x is a real scalar variable having a generalized gamma density then evaluate the Laplace transform of the density of $\frac{1}{x}$ and show that this Laplace transform can be written as a H-function. [Hint: Evaluate the integral

$$c \int_0^\infty \mathrm{e}^{-\frac{p}{x}} x^{\gamma-1} \mathrm{e}^{-bx^\rho} \mathrm{d}x,$$

where c is the normalizing constant and p is the Laplace parameter.]

4.1.2. Show that

$$c \int_0^\infty x^{-\gamma-1} \mathrm{e}^{-px - bx^{-\rho}} \mathrm{d}x,$$

also leads to the same result as in Exercise 4.1.1.

4.1.3. Show that the Laplace transform of $\frac{1}{x}$ in a generalized type-2 beta density, that is

$$c\int_0^\infty \mathrm{e}^{-\frac{p}{x}}x^{\gamma-1}[1+a(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}}\mathrm{d}x,$$

for $a>0,\delta>0,\alpha>1,\gamma>-1,\frac{1}{\alpha-1}-\frac{\gamma+1}{\delta}>0$, is an H-function, where c is a normalizing constant in the density.

4.1.4. Evaluate the integral

$$c\int_0^\infty \mathrm{e}^{-px}x^{-\gamma-1}[1+a(\alpha-1)x^{-\delta}]^{-\frac{1}{\alpha-1}}\mathrm{d}x,$$

for $\alpha>1,a>0,\delta>0$ and write down the conditions for the existence of the integral. Interpret it as a Laplace transform.

4.1.5. Let x_1 and x_2 be independently distributed type-1 beta random variables with the parameters (α_1,β_1), *and* (α_2,β_2), respectively. Let $u=x_1^{\gamma_1}x_2^{\gamma_2}$. Give the conditions under which u is distributed as a power of a type-1 beta random variable.

4.3 A Pathway Model

A general density that was introduced by Mathai (2005) is a matrix-variate pathway density. The scalar version of the pathway density in the real case is the following:

$$f_x(x)=c\,|x|^\gamma[1-a(1-\alpha)|x|^\delta]^{\frac{\eta}{1-\alpha}},\ \delta>0,\ \eta>0,\ a>0,\ 1-a(1-\alpha)|x|^\delta>0, \tag{4.33}$$

and $f_x(x)=0$ elsewhere, where c is the normalizing constant. When $\alpha<1$ the range of x is

$$-\frac{1}{[a(1-\alpha)]^{\frac{1}{\delta}}}<x<\frac{1}{[a(1-\alpha)]^{\frac{1}{\delta}}}. \tag{4.34}$$

As α moves toward 1, the range becomes larger and larger, and eventually $-\infty<x<\infty$ when $\alpha\to1$. Thus, for $\alpha<1$, (4.33) remains as a generalized type-1 beta family of densities. When $\alpha>1$, we can write $1-\alpha=-(\alpha-1),\ \alpha>1$, and then $1-a(1-\alpha)|x|^\delta=1+a(\alpha-1)|x|^\delta$, $-\infty<x<\infty$; then, the density in (4.33) becomes a generalized type-2 beta family of densities. When $\alpha\to1$, either from the left or from the right,

$$\lim_{\alpha\to1}[1-a(1-\alpha)|x|^\delta]^{\frac{\eta}{1-\alpha}}=\mathrm{e}^{-a\eta|x|^\delta}. \tag{4.35}$$

In this case, (4.33) becomes a generalized version of the density in (4.1). Thus, the model in (4.33) switches into three different families of densities, represented by three different functional forms, namely the generalized type-1 beta, type-2 beta, and gamma families. Then, α becomes a pathway parameter. As can be expected, c in (4.33) will be different for the three cases $\alpha < 1, \alpha > 1$, and $\alpha \to 1$, and the respective densities are the following:

$$f_1(x) = c_1\,|x|^\gamma[1 - a(1-\alpha)|x|^\delta]^{\frac{\eta}{1-\alpha}},\ \alpha < 1,\ a > 0,\ \delta > 0,\ \eta > 0, \tag{4.36}$$
$$-\frac{1}{[a(1-\alpha)]^{\frac{1}{\delta}}} < x < \frac{1}{[a(1-\alpha)]^{\frac{1}{\delta}}},\ \text{and}\ f_1(x) = 0,\ \text{elsewhere},$$

$$f_2(x) = c_2\,|x|^\gamma[1 + a(\alpha-1)|x|^\delta]^{-\frac{\eta}{\alpha-1}},\ a > 0,\ \delta > 0,\ \eta > 0,\ \alpha > 1,\ -\infty < x < \infty, \tag{4.37}$$

$$f_3(x) = c_3\,|x|^\gamma \mathrm{e}^{-a\eta|x|^\delta},\ a > 0,\ \eta > 0,\ \delta > 0,\ -\infty < x < \infty, \tag{4.38}$$

where the conditions on γ will be available from the normalizing constants c_1, c_2, *and* c_3, and these constants are evaluated with the help of type-1 beta integral, type-2 beta integral, and gamma integral, respectively, and they are the following:

$$c_1 = \frac{\delta\,[a(1-\alpha)]^{\frac{\gamma+1}{\delta}}\,\Gamma\left(\frac{\gamma+1}{\delta} + \frac{\eta}{1-\alpha} + 1\right)}{2\Gamma\left(\frac{\gamma+1}{\delta}\right)\Gamma\left(\frac{\eta}{1-\alpha} + 1\right)}, \alpha < 1, \gamma > -1, a > 0, \eta > 0, \delta > 0, \tag{4.39}$$

$$c_2 = \frac{\delta\,[a(\alpha-1)]^{\frac{\gamma+1}{\delta}}\ \Gamma\left(\frac{\eta}{\alpha-1}\right)}{2\,\Gamma\left(\frac{\gamma+1}{\delta}\right)\Gamma\left(\frac{\eta}{\alpha-1} - \frac{\gamma+1}{\delta}\right)}, \alpha > 1, \gamma > -1, \frac{\eta}{\alpha-1} - \frac{\gamma+1}{\delta} > 0,\ \delta > 0, \eta > 0, a > 0, \tag{4.40}$$

$$c_3 = \frac{\delta\ (a\eta)^{\frac{\gamma+1}{\delta}}}{2\,\Gamma\left(\frac{\gamma+1}{\delta}\right)},\ \delta > 0, a > 0, \gamma > -1, \eta > 0. \tag{4.41}$$

4.3.1 Independent Variables Obeying a Pathway Model

Consider k-independent real scalar variables, distributed according to the pathway density in (4.33) with different parameters. Let, $u = x_1x_2\cdots x_k$. We can compute the density of u by following the procedure in Sect. 4.1. To this end, let us look at the $(s-1)$th moment of x in (4.33). This will have three different forms depending upon the cases $\alpha < 1$, $\alpha > 1$, and $\alpha \to 1$, and these are available from (4.39), (4.40), and (4.41), respectively. That is,

$$E(|x|^{s-1}) = \frac{1}{[a(1-\alpha)]^{\frac{s-1}{\delta}}} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{\gamma+1}{\delta}\right)} \frac{\Gamma\left(\frac{\gamma+1}{\delta}+\frac{\eta}{1-\alpha}+1\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+1+\frac{\gamma+s}{\delta}\right)},$$

$$\text{for}\, \alpha < 1,\ a > 0,\ \eta > 0,\ \gamma+s > 0,\ \delta > 0,\ \gamma+1 > 0, \quad (4.42)$$

$$= \frac{1}{[a(\alpha-1)]^{\frac{s-1}{\delta}}} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{\gamma+1}{\delta}\right)} \frac{\Gamma\left(\frac{\eta}{\alpha-1}-\frac{\gamma+1}{\delta}\right)}{\Gamma\left(\frac{\eta}{\alpha-1}-\frac{\gamma+s}{\delta}\right)}$$

$$\text{for}\ \alpha > 1,\ a > 0,\ \eta > 0,\ \gamma+s > 0,$$

$$\frac{\eta}{\alpha-1}-\frac{\gamma+s}{\delta} > 0,\ \frac{\eta}{\alpha-1}-\frac{\gamma+1}{\delta} > 0,\ \gamma+1 > 0, \quad (4.43)$$

$$= \frac{1}{(a\eta)^{\frac{s-1}{\delta}}} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{\gamma+1}{\delta}\right)} \quad \text{for}\, a>0,\ \eta>0, \gamma+s>0, \gamma+1 > 0. \quad (4.44)$$

The density of $|u| = |x_1 \cdots x_k| = |x_1| \cdots |x_k|$ is available by inverting

$$E|u|^{s-1} = E|x_1|^{s-1} E|x_2|^{s-1} \cdots E|x_k|^{s-1}.$$

Let the densities of $|u|$ for $\alpha < 1,\ \alpha > 1$ and $\alpha \to 1$ be denoted by $g_1(|u|), g_2(|u|)$, and $g_3(|u|)$, respectively. Then,

$$g_1(|u|) = \left\{\prod_{j=1}^{k} \frac{[a_j(1-\alpha)]^{\frac{1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} \Gamma\left(\frac{\gamma_j+1}{\delta_j}+\frac{\eta_j}{1-\alpha}+1\right)\right\}$$

$$\times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left\{\prod_{j=1}^{k} \Gamma\left(\frac{\gamma_j+s}{\delta_j}\right) \frac{1}{\Gamma\left(\frac{\gamma_j+s}{\delta_j}+\frac{\eta_j}{1-\alpha}+1\right)}\right\}$$

$$\times \frac{|u|^{-s}}{[\prod_{j=1}^{k} a_j(1-\alpha)]^{\frac{s}{\delta_j}}} \mathrm{ds}$$

$$= \left\{\prod_{j=1}^{k} \frac{[a_j(1-\alpha)]^{\frac{1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} \Gamma\left(\frac{\gamma_j+1}{\delta_j}+\frac{\eta_j}{1-\alpha}+1\right)\right\}$$

$$\times H_{k,k}^{k,0}\left[\left[\prod_{j=1}^{k} a_j^{\frac{1}{\delta_j}} (1-\alpha)^{\frac{1}{\delta_j}}\right] |u| \Big|_{\left(\frac{\gamma_j}{\delta_j}, \frac{1}{\delta_j}\right), j=1,\cdots,k}^{\left(\frac{\gamma_j}{\delta_j}+\frac{\eta_j}{1-\alpha}+1, \frac{1}{\delta_j}\right), j=1,\cdots,k}\right] \quad (4.45)$$

$$\text{for}\quad -\frac{1}{(\prod_{j=1}^{k} a_j^{\frac{1}{\delta_j}} (1-\alpha)^{\frac{1}{\delta_j}})} < u < \frac{1}{(\prod_{j=1}^{k} a_j^{\frac{1}{\delta_j}} (1-\alpha)^{\frac{1}{\delta_j}})},$$

$$\alpha < 1, a_j > 0, \delta_j > 0, \gamma_j+1 > 0, \eta_j > 0, j = 1, \cdots, k,$$

and 0 elsewhere.

$$g_2(|u|) = \left\{ \prod_{j=1}^{k} \frac{[a_j(\alpha-1)]^{\frac{1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} \frac{1}{\Gamma\left(\frac{\eta_j}{\alpha-1} - \frac{\gamma_j+1}{\delta_j}\right)} \right\}$$
$$\times H_{k,k}^{k,k}\left[\left(\prod_{j=1}^{k} a_j^{\frac{1}{\delta_j}} (\alpha-1)^{\frac{1}{\delta_j}} \right) |u| \Big|_{\left(\frac{\gamma_j}{\delta_j}, \frac{1}{\delta_j}\right), j=1,\cdots,k}^{(1-\frac{\eta_j}{\alpha-1}+\frac{\gamma_j}{\delta_j}, \frac{1}{\delta_j}), j=1,\cdots,k} \right], \quad (4.46)$$
$$-\infty < u < \infty, \alpha > 1, a_j > 0, \delta_j > 0, \gamma_j + 1 > 0, \eta_j > 0, j = 1, \cdots, k.$$

$$g_3(|u|) = \left\{ \prod_{j=1}^{k} \frac{(a_j\eta_j)^{\frac{1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} \right\} H_{0,k}^{k,0}\left[(\prod_{j=1}^{k} (a_j\eta_j)^{\frac{1}{\delta_j}})|u| \Big|_{\left(\frac{\gamma_j}{\delta_j}, \frac{1}{\delta_j}\right), j=1,\ldots,k} \right],$$
$$-\infty < u < \infty, a_j > 0, \eta_j > 0, \gamma_j + 1 > 0, \delta_j > 0, j = 1, \cdots, k. \quad (4.47)$$

Remark 4.3. When $\delta_j = 1, j = 1, \cdots, k$ or when $\frac{1}{\delta_j} = m_j, m_j = 1, 2, \cdots$, the H-functions in (4.45)–(4.47) become Meijer's G-functions. When $\frac{1}{\delta_j} = m_j, m_j = 1, 2, \cdots$, one can expand $\Gamma(m_j s)$ and $\Gamma(m_j\gamma_j + \frac{\eta_j}{1-\alpha} + 1 + m_j s)$ in (4.45), $\Gamma(m_j s)$ and $\Gamma\left(\frac{\eta_j}{\alpha-1} - m_j(\gamma + s)\right)$ in (4.46), and $\Gamma(m_j s)$ in (4.47) by using the multiplication formula for gamma functions. Then, the coefficients of s in all gammas become ± 1, thereby the H-functions reduce to G-functions.

Exercises 4.2

4.2.1. Let α be the pathway parameter in a real scalar version of the pathway model. By using Maple/Mathematica, draw the graphs of the model for varying values of α and for fixed values of the other parameters.

4.2.2. Show that

$$f(x) = cx^{\gamma-1}[1 + a_1(\alpha_1 - 1)x^{\delta_1}]^{-\frac{1}{\alpha_1-1}}[1 + a_2(\alpha_2 - 1)x^{-\delta_2}]^{-\frac{1}{\alpha_2-1}},$$

where $x > 0, \alpha_1 > 1, \alpha_2 > 1, a_1 > 0, a_2 > 0, \delta_1 > 0, \delta_2 > 0$ and $f(x) = 0$ for $x \le 0$ can create a statistical density. Then, evaluate the normalizing constant c.

4.2.3. In Exercise 4.2.2, let $\alpha_1 < 1$ and $\alpha_2 > 1$. Then, can $f(x)$ still form a density? If so, evaluate the normalizing constant c.

4.2.4. In Exercise 4.2.2, show that

$$\lim_{\alpha_1 \to 1} f(x), \lim_{\alpha_2 \to 1} f(x), \lim_{\alpha_1 \to 1, \alpha_2 \to 1} f(x),$$

can create statistical densities. Evaluate the normalizing constants in each case.

4.2.5. Consider the normalizing constant c in Exercise 4.2.2. Show that c goes to the normalizing constants in each case in Exercise 4.2.4 under the respective conditions.

4.4 A Versatile Integral

This section deals with a general class of integrals, the particular cases of which are connected to a large number of problems in different disciplines. Reaction rate probability integrals in the theory of nuclear reaction rates, Krätzel integrals in applied analysis, inverse Gaussian distribution, generalized type-1, type-2, and gamma families of distributions in statistical distribution theory, Tsallis statistics and superstatistics in statistical mechanics, and the general pathway model are all shown to be connected to the integral under consideration. Representations of the integral in terms of generalized special functions such as Meijer's G-function and Fox's H-function are also given.

Consider the following integral:

$$f(z_2|z_1) = \int_0^\infty x^{\gamma-1}[1+z_1^\delta(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}}[1+z_2^\rho(\beta-1)x^{-\rho}]^{-\frac{1}{\beta-1}} \tag{4.48}$$

$$\text{for } \alpha>1, \beta>1, z_1\geq 0, z_2\geq 0, \delta>0, \rho>0, \Re(\gamma+1)>0,$$

$$\Re\left(\frac{1}{\alpha-1}-\frac{\gamma+1}{\delta}\right)>0, \Re\left(\frac{1}{\beta-1}-\frac{1}{\rho}\right)>0$$

$$= \int_0^\infty \frac{1}{x} f_1(x) f_2\left(\frac{z_2}{x}\right)\mathrm{d}x, \tag{4.49}$$

where $\Re(\cdot)$ denotes the real part of $(\cdot)$.

$$f_1(x) = x^\gamma[1+z_1^\delta(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}},\ f_2(x) = [1+(\beta-1)x^\rho]^{-\frac{1}{\beta-1}}, \tag{4.50}$$

with Mellin transforms

$$M_{f_1}(s) = [\delta z_1^{\gamma+s}(\alpha-1)^{\frac{\gamma+s}{\delta}}]^{-1}\frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)\Gamma\left(\frac{1}{\alpha-1}-\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{1}{\alpha-1}\right)}, \tag{4.51}$$

$$\Re(\gamma+s)>0, \Re\left(\frac{1}{\alpha-1}-\frac{\gamma+s}{\delta}\right)>0,$$

and

$$M_{f_2}(s) = [\rho(\beta-1)^{\frac{s}{\rho}}]^{-1}\frac{\Gamma\left(\frac{s}{\rho}\right)\Gamma\left(\frac{1}{\beta-1}-\frac{s}{\rho}\right)}{\Gamma\left(\frac{1}{\beta-1}\right)} \tag{4.52}$$

$$\Re(s)>0, \Re\left(\frac{1}{\alpha-1}-\frac{s}{\rho}\right)>0.$$

Hence, the Mellin transform of $f(z_2|z_1)$, as a function of z_2, with parameter s is the following:

$$\begin{aligned} M_{f(z_2|z_1)}(s) &= M_{f_1}(s)M_{f_2}(s) \\ &= \frac{1}{\delta\, z_1^{\gamma+s}(\alpha-1)^{\frac{\gamma+s}{\delta}}} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)\Gamma\left(\frac{1}{\alpha-1}-\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{1}{\alpha-1}\right)} \\ &\times \frac{1}{\rho(\beta-1)^{\frac{s}{\rho}}} \frac{\Gamma\left(\frac{s}{\rho}\right)\Gamma\left(\frac{1}{\beta-1}-\frac{s}{\rho}\right)}{\Gamma\left(\frac{1}{\beta-1}\right)} \\ &\text{for } \Re(\gamma+s)>0, \Re\left(\frac{1}{\alpha-1}-\frac{\gamma+s}{\delta}\right)>0, \Re(s)>0, \\ &\Re\left(\frac{1}{\beta-1}-\frac{s}{\rho}\right)>0, z_1>0, z_2>0. \end{aligned} \tag{4.53}$$

Putting $y = \frac{1}{x}$ in (4.48), we have

$$f(z_1|z_2) = \int_0^\infty \frac{y^{-\gamma}}{y}[1+z_1^\delta(\alpha-1)y^{-\delta}]^{-\frac{1}{\alpha-1}}[1+z_2^\rho(\beta-1)y^\rho]^{-\frac{1}{\beta-1}}\mathrm{d}y. \tag{4.54}$$

Evaluating the Mellin transform of (4.54) with parameter s and treating it as a function of z_1, we have exactly the same expression in (4.53). Hence,

$$M_{f(z_2|z_1)}(s) = M_{f(z_1|z_2)}(s) = \text{right side in (4.53)}. \tag{4.55}$$

By taking the inverse Mellin transform of $M_{f(z_2|z_1)}(s)$, one can get the integral $f(z_2|z_1)$ as an H-function as follows:

Theorem 4.1.

$$f(z_2|z_1) = c^{-1}H_{2,2}^{2,2}\left[z_1z_2(\alpha-1)^{\frac{1}{\delta}}(\beta-1)^{\frac{1}{\rho}}\Big|_{(\frac{\gamma}{\delta},\frac{1}{\delta}),(0,\frac{1}{\rho})}^{(1-\frac{1}{\alpha-1}+\frac{\gamma}{\delta},\frac{1}{\delta}),(1-\frac{1}{\beta-1},\frac{1}{\rho})}\right] \tag{4.56}$$

where

$$c = \delta\rho z_1^\gamma(\alpha-1)^{\frac{\gamma}{\delta}},$$

and $H_{p,q}^{m,n}(\cdot)$ *is a* H*-function.*

The integral in (4.48) is connected to reaction rate probability integral in nuclear reaction rate theory in the nonresonant case, Tsallis statistics in nonextensive statistical mechanics, superstatistics in astrophysics, generalized type-2, type-1 beta, and gamma families of densities and the density of a product of two real positive random variables in statistical literature, Krätzel integrals in applied analysis, inverse Gaussian distribution in stochastic processes, and the like. Special cases include a wide range of functions appearing in different disciplines.

Observe that $f_1(x)$ and $f_2(x)$ in (4.50), multiplied by the appropriate normalizing constants, can produce statistical densities. Further, $f_1(x)$ and $f_2(x)$ are defined for $-\infty < \alpha < \infty, -\infty < \beta < \infty$. When $\alpha > 1$ and $z_1 > 0, \delta > 0$, $f_1(x)$ multiplied by the normalizing constant stays in the generalized type-2 beta family. When $\alpha < 1$, writing $\alpha - 1 = -(1-\alpha), \alpha < 1$, the function $f_1(x)$ switches into a generalized type-1 beta family and when $\alpha \to 1$,

$$\lim_{\alpha\to 1} f_1(x) = \mathrm{e}^{-z_1^\delta x^\delta}, \tag{4.57}$$

and hence $f_1(x)$ goes into a generalized gamma family. Similar is the behavior of $f_2(x)$ when β ranges from $-\infty$ to ∞. Thus, the parameters α and β create pathways to switch into different functional forms or different families of functions. Hence, we will call α and β pathway parameters in this case. Let us look into some interesting special cases. Take the special case $\beta \to 1$,

$$f_1(z_2|z_1) = \int_0^\infty x^{\gamma-1}[1 + z_1^\delta(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}}\mathrm{e}^{-z_2^\rho x^{-\rho}}\mathrm{d}x \tag{4.58}$$

$\alpha > 1, z_1 > 0, z_2 > 0, \delta > 0, \rho > 0$. Put $y = \frac{1}{x}$

$$f_1(z_1|z_2) = \int_0^\infty y^{-\gamma-1}[1 + z_1^\delta(\alpha-1)y^{-\delta}]^{-\frac{1}{\alpha-1}}\mathrm{e}^{-z_2^\rho y^\rho}\mathrm{d}y \tag{4.59}$$

$\alpha > 1, z_1 > 0, z_2 > 0, \delta > 0, \rho > 0$. Let $\alpha \to 1$ in (1)

$$f_2(z_2|z_1) = \int_0^\infty x^{\gamma-1}\mathrm{e}^{-z_1^\delta x^\delta}[1 + z_2^\rho(\beta-1)x^{-\rho}]^{-\frac{1}{\beta-1}}\mathrm{d}x \tag{4.60}$$

$\beta > 1, z_1 > 0, z_2 > 0, \delta > 0, \rho > 0.$

$$f_2(z_1|z_2) = \int_0^\infty x^{-\gamma-1}\mathrm{e}^{-z_1^\delta x^{-\delta}}[1 + z_2^\rho(\beta-1)x^\rho]^{-\frac{1}{\beta-1}}\mathrm{d}x \tag{4.61}$$

$\beta > 1, z_1 > 0, z_2 > 0, \delta > 0, \rho > 0$. Take $\alpha \to 1, \beta \to 1$ in (1)

$$f_3(z_2|z_1) = \int_0^\infty x^{\gamma-1}\mathrm{e}^{-z_1^\delta x^\delta - z_2^\rho x^{-\rho}}\mathrm{d}x \tag{4.62}$$

$z_1 > 0, z_2 > 0, \delta > 0, \rho > 0.$

$$f_3(z_1|z_2) = \int_0^\infty x^{-\gamma-1}\mathrm{e}^{-z_1^\delta x^{-\delta} - z_2^\rho x^\rho}\mathrm{d}x \tag{4.63}$$

$z_1 > 0, z_2 > 0, \delta > 0, \rho > 0.$

4.4.1 Case of $\alpha < 1$ or $\beta < 1$

When $\alpha < 1$, writing $\alpha - 1 = -(1-\alpha)$, we can define the function

$$g_1(x) = x^\gamma[1 + z_1^\delta(\alpha-1)x^\delta]^{-\frac{1}{\alpha-1}} = x^\gamma[1 - z_1^\delta(1-\alpha)x^\delta]^{\frac{1}{1-\alpha}}, \alpha < 1, \tag{4.64}$$

for $[1 - z_1^{\delta}(1-\alpha)x^{\delta}] > 0, \alpha < 1 \Rightarrow x < \frac{1}{z_1(1-\alpha)^{\frac{1}{\delta}}}$ and $g_1(x) = 0$ elsewhere. In this case, the Mellin transform of $g_1(x)$ is the following:

$$h_1(s) = \int_0^{\infty} x^{s-1} g_1(x)\mathrm{d}x = \int_0^{\frac{1}{z_1(1-\alpha)^{\frac{1}{\delta}}}} x^{\gamma+s-1}[1 - z_1^{\delta}(1-\alpha)x^{\delta}]^{\frac{1}{1-\alpha}}\mathrm{d}x \tag{4.65}$$

$$= \frac{1}{\delta[z_1(1-\alpha)^{\frac{1}{\delta}}]^{\gamma+s}} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\Gamma\left(\frac{1}{1-\alpha}+1+\frac{\gamma+s}{\delta}\right)}, \Re(\gamma+s) > 0, \alpha < 1, \delta > 0. \tag{4.66}$$

Then, the Mellin transform of $f(z_2|z_1)$ for $\alpha < 1, \beta > 1$ is given by

$$M_{z_2|z_1}(s) = \frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\delta\rho z_2^s z_1^{\gamma+s}(\beta-1)^{\frac{s}{\rho}}(1-\alpha)^{\frac{\gamma+s}{\delta}}} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{\gamma+s}{\delta}+\frac{1}{1-\alpha}+1\right)} \frac{\Gamma\left(\frac{s}{\rho}\right)\Gamma\left(\frac{1}{\beta-1}-\frac{s}{\rho}\right)}{\Gamma\left(\frac{1}{\beta-1}\right)}, \tag{4.67}$$
$$\Re(\gamma+s) > 0, \Re(s) > 0, \Re\left(\frac{1}{\beta-1}-\frac{s}{\rho}\right) > 0.$$

Hence, the inverse Mellin transform for $\alpha < 1, \beta > 1$ is given in

Theorem 4.2. *For* $\alpha < 1, \beta > 1$

$$f(z_2|z_1) = \frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\delta\rho z_1^{\gamma}(1-\alpha)^{\frac{\gamma}{\delta}}\Gamma\left(\frac{1}{\beta-1}\right)} \times H_{2,2}^{2,1}\left[z_1 z_2(1-\alpha)^{\frac{1}{\delta}}(\beta-1)^{\frac{1}{\rho}}\Big|_{\left(0,\frac{1}{\rho}\right),\left(\frac{\gamma}{\delta},\frac{1}{\delta}\right)}^{\left(1-\frac{1}{\beta-1},\frac{1}{\rho}\right),\left(1+\frac{1}{1-\alpha}+\frac{\gamma}{\delta},\frac{1}{\delta}\right)}\right], \tag{4.68}$$

$$\lim_{\beta\to1} f(z_2|z_1) = \frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\rho\delta z_1^{\gamma}(1-\alpha)^{\frac{\gamma}{\delta}}} H_{1,2}^{2,0}\left[z_1 z_2(1-\alpha)^{\frac{1}{\delta}}\Big|_{\left(0,\frac{1}{\delta}\right),\left(\frac{\gamma}{\delta},\frac{1}{\delta}\right)}^{\left(1+\frac{1}{1-\alpha}+\frac{\gamma}{\delta},\frac{1}{\delta}\right)}\right], \tag{4.69}$$

$$\lim_{\alpha\to1} f(z_2|z_1) = \frac{1}{\rho\delta\Gamma\left(\frac{1}{\beta-1}\right)z_1^{\gamma}} H_{1,2}^{2,1}\left[z_1 z_2(\beta-1)^{\frac{1}{\rho}}\Big|_{\left(0,\frac{1}{\rho}\right),\left(\frac{\gamma}{\delta},\frac{1}{\delta}\right)}^{\left(1-\frac{1}{\beta-1},\frac{1}{\rho}\right)}\right], \tag{4.70}$$

$$\lim_{\alpha\to1,\beta\to1} f(z_2|z_1) = \frac{1}{\rho\delta z_1^{\gamma}} H_{0,2}^{2,0}\left[z_1 z_2\Big|_{\left(0,\frac{1}{\rho}\right),\left(\frac{\gamma}{\delta},\frac{1}{\delta}\right)}\right]. \tag{4.71}$$

In $f(z_2|z_1)$, if $\beta < 1$, we may write $\beta-1 = -(1-\beta)$, and if we assume $[1-z_2^{\rho}(1-\beta)x^{-\rho}]^{\frac{1}{1-\beta}} > 0 \Rightarrow x > z_2(1-\beta)^{\frac{1}{\rho}}$, then the corresponding integrals can also be evaluated as H-functions. But, if $\alpha < 1$ and $\beta < 1$, then from the conditions

$$1-z_1^{\delta}(1-\alpha)x^{\delta} > 0 \Rightarrow x < \frac{1}{z_1(1-\alpha)^{\frac{1}{\delta}}} \text{ and } 1-z_2^{\rho}(1-\beta)x^{-\rho} > 0 \Rightarrow x > z_2(1-\beta)^{\frac{1}{\rho}}$$

the resulting integral may be zero. Hence, except this case of $\alpha < 1$ and $\beta < 1$, all other cases of $\alpha > 1, \beta > 1; \alpha < 1, \beta > 1; \alpha > 1, \beta < 1$ can be given meaningful interpretations as H-functions. Further, all these situations can be connected to practical problems. A few such practical situations will be considered next.

Remark 4.4. In the integrals in (4.48), (4.58)–(4.63), the exponents of x are taken as $(\delta, -\rho)$ or $(-\delta, \rho)$ with $\delta > 0, \rho > 0$. The cases where the exponents of x are (δ, ρ), $(-\delta, -\rho)$ with $\delta > 0, \rho > 0$ are not considered so far. But, these cases can be done by using the convolution property

$$g(z_1) = \int_0^{\infty} x f_1(z_1 x) f_2(x)\mathrm{d}x. \tag{4.72}$$

Remark 4.5. The convolution integrals in (4.49) and (4.72) can be interpreted easily in terms of independently distributed real scalar positive random variables when f_1 and f_2 are densities. Let x_1 and x_2 be statistically independently distributed real scalar positive random variables with densities $f_1(x_1)$ and $f_2(x_2)$ respectively. Let $u = x_1 x_2$ and $v = \frac{x_1}{x_2}$. Then, the densities of u and v are respectively given by

$$g_u(u) = \int_x \frac{1}{x} f_1(x) f_2\left(\frac{u}{x}\right)\mathrm{d}x \tag{4.73}$$

and

$$g_v(v) = \int_x x f_1(vx) f_2(x)\mathrm{d}x. \tag{4.74}$$

These are the two convolution formulae in (4.49) and (4.72), respectively. The densities $g_u(u)$ and $g_v(v)$ are available from the inverse Mellin transforms also. That is, whenever the Mellin transforms exist and invertible,

$$E(u^{s-1}) = E(x_1^{s-1})E(x_2^{s-1}) = h_1(s), \text{ say} \tag{4.75}$$

$$E(v^{s-1}) = E(x_1^{s-1})E(x_2^{1-s}) = h_2(s), \text{ say.} \tag{4.76}$$

Then

$$g_u(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h_1(s)u^{-s}\mathrm{d}s, \tag{4.77}$$

and

$$g_v(v) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} h_2(s)v^{-s}\mathrm{d}s. \tag{4.78}$$

4.4.2 Some Practical Situations

(a). Krätzel Integral

For $\delta = 1, z_2^{\rho} = z, z_1 = 1$ in $f_3(z_2|z_1)$ gives the Krätzel integral

$$f_3(z_2|z_1) = \int_0^{\infty} x^{\gamma-1} \mathrm{e}^{-x-zx^{-\rho}} \mathrm{d}x, \tag{4.79}$$

which was studied in detail by Krätzel (1979). Hence, f_3 can be considered as generalization of Krätzel integral. An additional property that can be seen from Krätzel integral as f_3 is that it can be written as a H-function of the type $H_{0,2}^{2,0}(\cdot)$. Hence all the properties of H-function can now be made use of to study this integral further.

(b). Inverse Gaussian Density in Statistics

Inverse Gaussian density is a popular density, which is used in many disciplines including stochastic processes. One form of the density is the following (Mathai 1993c, page 33):

$$f(x) = c\ x^{-\frac{3}{2}} \mathrm{e}^{-\frac{\lambda}{2}\left(\frac{x}{\mu^2}+\frac{1}{x}\right)}, \mu \neq 0, x > 0, \lambda > 0, \tag{4.80}$$

where $c = \pi^{-\frac{1}{2}} \mathrm{e}^{\frac{\lambda}{|\mu|}}$. Comparing this with our case $f_3(z_1|z_2)$, we see that the inverse Gaussian density is the integrand in $f_3(z_1|z_2)$ for $\gamma = \frac{1}{2}, \rho = 1, z_2 = \frac{\lambda}{2}\left(\frac{1}{\mu^2}\right)$, $\delta = 1$, $z_1 = \frac{\lambda}{2}$. Hence, f_3 can be used directly to evaluate the moments or Mellin transform in inverse Gaussian density.

(c). Reaction Rate Probability Integral in Astrophysics

In a series of papers Haubold and Mathai studied modifications of Maxwell–Boltzmann theory of reaction rates, a summary is given in Mathai and Haubold (1988). The basic reaction rate probability integral that appears there is the following:

$$I_1 = \int_0^{\infty} x^{\gamma-1} \mathrm{e}^{-ax-zx^{-\frac{1}{2}}} \mathrm{d}x. \tag{4.81}$$

This is the case in the nonresonant case of nuclear reactions. Compare integral I_1 with $f_3(z_2|z_1)$. The reaction rate probability integral I_1 is $f_3(z_2|z_1)$ for $\delta = 1, \rho = \frac{1}{2}, z_2^{\frac{1}{2}} = z$. The basic integral I_1 is generalized in many different forms for various situations of resonant and nonresonant cases of reactions, depletion of high energy tail, cut off of the high energy tail, and so on. Dozens of published papers are there in this area.

(d). Tsallis Statistics and Superstatistics

It is estimated that on Tsallis statistics in nonextensive statistical mechanics, over 1200 papers were published during the period 1990 to 2007. Tsallis statistics is of the following form:

$$f_X(x) = c_1[1 + (\alpha - 1)x]^{-\frac{1}{\alpha-1}}. \tag{4.82}$$

Compare $f_X(x)$ with the integrand in (1). For $z_2 = 0, \delta = 1,\ and\, \gamma = 1$, the integrand in (4.48) agrees with Tsallis statistics $f_X(x)$ given above. The three different forms of Tsallis statistics are available from $f_X(x)$ for $\alpha > 1, \alpha < 1,\ and\, \alpha \to 1$. The starting paper in nonextensive statistical mechanics may be seen from Tsallis (1988). But, the integrand in (4.48) with $z_2 = 0, z_1 = 1, \alpha > 1$ is the superstatistics of Beck and Cohen, see for example Beck and Cohen (2003), Beck (2006). In statistical language, this superstatistics is the unconditional density in a generalized gamma case when the scale parameter has a prior density belonging to the same class of generalized gamma density.

(e). Pathway Model

Mathai (2005) considered a rectangular matrix-variate function in the real case from where one can obtain almost all matrix-variate densities in current use in statistical and other disciplines. The corresponding version, when the elements are in the complex domain, is given in Mathai and Provost (2006). For the real scalar case, the function is of the following form:

$$f(x) = c^*|x|^\gamma[1 - a(1-\alpha)|x|^\delta]^{\frac{\eta}{1-\alpha}}, \tag{4.83}$$

for $-\infty < x < \infty, a > 0, \eta > 0, \delta > 0$, and c^* is the normalizing constant. Here, $f(x)$ for $\alpha < 1$ stays in the generalized type-1 beta family when $[1 - a(1-\alpha)|x|^\delta]^{\frac{\eta}{1-\alpha}} > 0$. When $\alpha > 1$, the function switches into a generalized type-2 beta family and when $\alpha \to 1$, it goes into a generalized gamma family of functions. Here α behaves as a pathway parameter, and hence the model is called a pathway model. Observe that the integrand in (4.48) is a product of two such pathway functions so that the corresponding integral is more versatile than a pathway model. Thus, for $z_2 = 0$ in (4.48), the integrand produces the pathway model of Mathai (2005).

Exercises 4.3

4.3.1. By normalizing the integrals in (4.58) to (4.63), create statistical densities corresponding to the integrands in the six equations.

4.3.2. Evaluate the hth moments for the six densities in Exercise 4.3.1.

4.3.3. Write down the hth moments in Exercise 4.3.2 for $h = 1, 2$ and compute the variances of the corresponding random variables.

4.3.4. Using Stirling's approximation on the gammas in (4.67), derive the corresponding Mellin–Barnes representations in (4.69)–(4.71).

4.3.5. Evaluate the series form in (4.71) for $\frac{1}{\rho} = 2, \frac{1}{\delta} = 3$.

Chapter 5
Functions of Matrix Argument

5.1 Introduction

Particular cases of a H-function with matrix argument are available for real as well as for complex matrices. For the general H-function only a class of functions is available analogous to the scalar variable H-function. Real-valued scalar functions of matrix argument is developed when the argument matrix is a real symmetric positive definite matrix or for hermitian positive definite matrices. We consider only real matrices here.

We will use the standard notations to denote matrices. The transpose of a matrix $X = (x_{ij})$ will be denoted by X' and trace of X by $\text{tr}(X)$ = sum of the eigenvalues = sum of the leading diagonal elements. Determinant of X will be denoted by $|X|$, a null matrix by a big O and an identity matrix by $I = I_n$. A diagonal matrix will be written as $\text{diag}(\lambda_1, \ldots, \lambda_p)$ where $\lambda_1, \ldots, \lambda_p$ are the diagonal elements. $X > 0$ will mean the real symmetric matrix $X = X'$ is positive definite. Definiteness is defined only for symmetric matrices when real and hermitian matrices when complex, $X \geq 0$ (positive semidefinite), $X < 0$ (negative definite), $X \leq 0$ (negative semidefinite). A matrix which does not fall in the categories $X > 0, X \geq 0, X < 0, X \leq 0$ is called indefinite. $\int_X f(X)\mathrm{d}X$ means the integral over X. $\int_A^B f(X)\mathrm{d}X$ means the integral over $0 < A < X < B$, that is, $X = X' > 0, A = A' > 0, B = B' > 0, X - A > 0, B - X > 0$ and the integral is taken over all such X.

It is difficult to develop the theory of a real-valued scalar function of a general matrix X. Even for a square matrix A rational powers will create problems. For example even for an identity matrix, even a simple item such as a square root will create difficulties. If the existence of a matrix B such that $B^2 = A$ is taken as the square root of A then consider

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have then

$$A_1^2 = I_2, \quad A_2^2 = I_2, \quad A_3^2 = I_2, \quad A_4^2 = I_2.$$

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Thus A_1, A_2, A_3, A_4 are all candidates for the square root of a nice matrix like an identity matrix. But if we confine our discussion to the class of positive definite matrices, when real, and hermitian positive definite matrices, when complex, then A_1 is the only candidate for the square root of I_2. In this class of positive definiteness, several items can be defined uniquely. Hence the theory is developed when the matrices are positive definite when real.

5.2 Exponential Function of Matrix Argument

Hypergeometric functions, in the scalar case, are special cases of a H-function. For example

$$ {}_0F_0(\,;\,;\pm x) = \mathrm{e}^{\pm x}, \tag{5.1}$$

when x is scalar. The corresponding function of matrix argument is

$$ {}_0F_0(\,;\,;\pm X) = \mathrm{e}^{\pm \mathrm{tr}(X)}, \tag{5.2}$$

where X is a $p \times p$ positive definite matrix. For any type of integral operations on (5.2) we need to define differential elements and wedge product of differentials.

Definition 5.1. Wedge product of differentials. Wedge product or skew symmetric product of differential elements $\mathrm{d}x$ and $\mathrm{d}y$ will be denoted by $\mathrm{d}x \wedge \mathrm{d}y$, where $\wedge =$ wedge, and will be defined by the relation

$$\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x. \tag{5.3}$$

That is, if the order is changed then it is to be multiplied by (-1). This will then imply that

$$\mathrm{d}x \wedge \mathrm{d}x = 0,\ \mathrm{d}x \wedge \mathrm{d}x \wedge \mathrm{d}x = 0,\ \mathrm{d}y \wedge \mathrm{d}y = 0,$$

and so on. An interesting consequence is there when products of differentials are taken. If X is a $p \times q$ matrix, $X = (x_{ij})$ then the wedge product of differentials is the following:

Notation 5.1.

$$\mathrm{d}X = \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge \cdots \wedge \mathrm{d}x_{1q} \wedge \mathrm{d}x_{21} \wedge \cdots \wedge \mathrm{d}x_{pq}. \tag{5.4}$$

If $X = X'$ and $p \times p$ then there are only $\frac{p(p+1)}{2}$ free elements in X because $x_{ij} = x_{ji}$ for all i and j, and then

$$\mathrm{d}X = \mathrm{d}x_{11} \wedge \cdots \wedge \mathrm{d}x_{1p} \wedge \mathrm{d}x_{22} \wedge \cdots \wedge \mathrm{d}x_{2p} \wedge \cdots \wedge \mathrm{d}x_{pp}. \tag{5.5}$$

Thus

$$\int_X f(X)\mathrm{d}X = \int_{X=X'>0} f(X)\mathrm{d}X = \int_{X>0} f(X)\mathrm{d}X$$

will mean that the integral is taken over all $X > 0$.

$$\int_{O<X<I} f(X)\mathrm{d}X = \int_0^I f(X)\mathrm{d}X$$

will mean the integral over all $X = X' > 0$ such that $I - X > 0$. Now we are in a position to define an integral analogous to a gamma integral in the scalar case. Consider

$$\Gamma_p(\alpha) = \int_{X=X'>0} |X|^{\alpha-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X, \tag{5.6}$$

where X is $p\times p$ real symmetric and positive definite. For $p = 1$, (5.6) corresponds to the gamma integral. How can we evaluate (5.6)? This requires some matrix transformations and the associated Jacobians. For simplicity let us look at functions of two scalar variables x_1 and x_2. Let

$$y_1 = f_1(x_1, x_2) \text{ and } y_2 = f_2(x_1, x_2).$$

Then from basic calculus

$$\mathrm{d}y_1 = \frac{\partial f_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\mathrm{d}x_2 \text{ and } \mathrm{d}y_2 = \frac{\partial f_2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_2}{\partial x_2}\mathrm{d}x_2.$$

Now if we take the wedge product of the differentials we have

$$\begin{aligned}\mathrm{d}y_1 \wedge \mathrm{d}y_2 &= \left[\frac{\partial f_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_1}{\partial x_2}\mathrm{d}x_2\right] \wedge \left[\frac{\partial f_2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f_2}{\partial x_2}\mathrm{d}x_2\right] \\ &= \left[\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\right]\mathrm{d}x_1 \wedge \mathrm{d}x_2 + 0 + 0,\end{aligned}$$

by using the results $\mathrm{d}x_1 \wedge \mathrm{d}x_1 = 0$ and $\mathrm{d}x_2 \wedge \mathrm{d}x_1 = -\mathrm{d}x_1 \wedge \mathrm{d}x_2$. Then

$$\begin{aligned}\mathrm{d}y_1 \wedge \mathrm{d}y_2 &= \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \\ &\Rightarrow \mathrm{d}Y = J\ \mathrm{d}X,\end{aligned} \tag{5.7}$$

where $\mathrm{d}Y = \mathrm{d}y_1 \wedge \mathrm{d}y_2$, $\mathrm{d}X = \mathrm{d}x_1 \wedge \mathrm{d}x_2$ and J is the Jacobian or the determinant of the matrix of partial derivatives. In general, if we have a transformation of $x_1, \ldots, x_p$ going to $y_1, \ldots, y_p$ then

$$\mathrm{d}Y = \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \cdots \wedge \mathrm{d}y_p = J\ \mathrm{d}X,\ \mathrm{d}X = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p,$$

and

$$J = \left| \left(\frac{\partial y_i}{\partial x_j} \right) \right|, \tag{5.8}$$

where the (i, j)th element in the matrix is the partial derivative of y_i with respect to x_j.

Example 5.1. Evaluate the Jacobian in the linear transformation $Y = AX$ where X is $p \times 1$, Y is $p \times 1$, A is $p \times p$ nonsingular constant matrix and X is of distinct real scalar variables. Verify the result for

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Solution 5.1. When $|A| \neq 0$ the transformation is unique or one-to-one. $Y = AX \Rightarrow X = A^{-1}Y$ where A^{-1} is the unique inverse of A. The transformation is of the form

$$\begin{aligned} y_1 &= a_{11}x_1 + \cdots + a_{1p}x_p \\ \vdots & \qquad\quad \vdots \\ y_p &= a_{p1}x_1 + \cdots + a_{pp}x_p \end{aligned} \quad \Rightarrow \frac{\partial y_i}{\partial x_j} = a_{ij}$$

$$\Rightarrow \frac{\partial Y}{\partial X} = A \Rightarrow J = |A|.$$

That is,

$$\mathrm{d}Y = |A|\mathrm{d}X \text{ or } \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_p = |A|\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p.$$

When

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}, \; |A| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1.$$

Hence, in this case,

$$\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_p = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p.$$

This may be stated as a theorem.

Theorem 5.1.

$$Y = AX \Rightarrow \mathrm{d}Y = |A|\mathrm{d}X, \tag{5.9}$$

where X and Y are $p \times 1, |A| \neq 0, X$ is of distinct real scalar variables.

If X is a $p \times q$ matrix of distinct real scalar variables then the wedge product of the differentials in X, denoted by $\mathrm{d}X$, is given by

$$\mathrm{d}X = \mathrm{d}x_{11} \wedge \mathrm{d}x_{12} \wedge \cdots \wedge \mathrm{d}x_{pq}. \tag{5.10}$$

5.3 Jacobians of Matrix Transformations

We considered one linear transformation involving a vector of variables X going to a vector of variables Y through a nonsingular linear transformation $Y = AX, |A| \neq 0$ and we found the Jacobian to be $|A|$. Now we consider a few more elaborate linear transformations and some nonlinear transformations.

Let X be a $m \times n$ matrix of distinct real scalar variables and let A be a $m \times m$ nonsingular matrix of constants. Consider the transformation $Y = AX$. Let $X^{(1)}, \ldots, X^{(n)}$ be the columns of X. Then

$$\begin{aligned} Y &= AX = A(X^{(1)}, \ldots, X^{(n)}) = (AX^{(1)}, \ldots, AX^{(n)}) \\ &= (Y^{(1)}, \ldots, Y^{(n)}), \end{aligned}$$

where $Y^{(1)}, \ldots, Y^{(n)}$ are the columns of Y. Then we can look at the transformation as

$$\begin{bmatrix} Y^{(1)} \\ \vdots \\ Y^{(n)} \end{bmatrix} = \begin{bmatrix} AX^{(1)} \\ \vdots \\ AX^{(n)} \end{bmatrix}.$$

Then from Theorem 5.1,

$$\frac{\partial Y^{(i)}}{\partial X^{(i)}} = A,\ i = 1, \ldots, n,\ \frac{\partial Y^{(i)}}{\partial X^{(j)}} = O, i \neq j.$$

The matrix of partial derivatives is of the following form:

$$\begin{bmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & A \end{bmatrix} \Rightarrow \begin{vmatrix} A & O & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & A \end{vmatrix} = |A|^n.$$

Hence we have the following theorem:

Theorem 5.2. *Let X be a $m \times n$ matrix of distinct real scalar variables or functionally independent real variables. Let A be a $m \times m$ nonsingular constant matrix. Then*

$$Y = AX \Rightarrow \mathrm{d}Y = |A|^n \mathrm{d}X. \tag{5.11}$$

In Theorem 5.2 we had a premultiplication of X by a constant nonsingular matrix A. Now let us consider a postmultiplication. Let B be a $n \times n$ nonsingular constant matrix. Then what will be the Jacobian in the transformation $Y = XB$? This can be computed exactly the same way by considering the rows of X. Let $X_{(1)}, \ldots, X_{(m)}$ be the m rows of X. Then

$$Y = XB = \begin{bmatrix} X_{(1)} \\ \vdots \\ X_{(m)} \end{bmatrix} B = \begin{bmatrix} X_{(1)}B \\ \vdots \\ X_{(m)}B \end{bmatrix} = \begin{bmatrix} Y_{(1)} \\ \vdots \\ Y_{(m)} \end{bmatrix},$$

where $Y_{(1)}, \ldots, Y_{(m)}$ are the m rows of Y. Now we can look at the long string

$$\begin{bmatrix} Y'_{(1)} \\ \vdots \\ Y'_{(m)} \end{bmatrix} = \begin{bmatrix} B'X'_{(1)} \\ \vdots \\ B'X'_{(m)} \end{bmatrix},$$

and apply Theorem 5.1. The matrix of partial derivatives will be

$$\begin{bmatrix} B' & O & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & B' \end{bmatrix} \Rightarrow \begin{vmatrix} B' & O & \cdots & O \\ \vdots & \vdots & \cdots & \vdots \\ O & O & \cdots & B' \end{vmatrix} = |B'|^m = |B|^m.$$

Hence we have the following result:

Theorem 5.3. *Let X be as in Theorem 5.2 and let B be a nonsingular $n \times n$ constant matrix. Then*

$$Y = XB \Rightarrow \mathrm{d}Y = |B|^m \mathrm{d}X. \tag{5.12}$$

Combining Theorems 5.2 and 5.3 we have the following result:

Theorem 5.4. *Let X, A, B be as in Theorems 5.2 and 5.3. Then*

$$Y = AXB \Rightarrow \mathrm{d}Y = |A|^n |B|^m \mathrm{d}X. \tag{5.13}$$

Example 5.2. Let X be a $m \times n$ matrix of functionally independent real variables. Let M, A, B be constant matrices where M is $m \times n$, A is $m \times m$, B is $n \times n$ with $|A| \neq 0, |B| \neq 0$ and further, let $A = A' > 0, B = B' > 0$ (positive definite matrices). Consider the function

$$f(X) = c\, \mathrm{e}^{-\mathrm{tr}[A(X-M)B(X-M)']}, \tag{5.14}$$

where f is a real-valued scalar function of X, c is a scalar constant and $\mathrm{tr}(\cdot)$ denotes the trace of $(\cdot)$. Evaluate $\int_X f(X)\mathrm{d}X$.

Solution 5.2. We wish to evaluate the total integral of $f(X)$ over all such $m \times n$ matrices X. From the theory of matrices we know that a positive definite matrix A (definiteness is defined only for symmetric matrices when real and hermitian matrices when complex) can be written as $A = A_1A_1'$ with $|A_1| \neq 0$ where $A_1' =$ transpose of A_1. We also know that for any two matrices P and Q,

$$\mathrm{tr}(PQ) = \mathrm{tr}(QP),$$

whenever PQ and QP are defined, where PQ need not be equal to QP. By using these two results we can write

$$\begin{aligned}\mathrm{tr}[A(X-M)B(X-M)'] &= \mathrm{tr}[A_1A_1'(X-M)B_1B_1'(X-M)'] \\ &= \mathrm{tr}[A_1'(X-M)B_1B_1'(X-M)'A_1] \\ &= \mathrm{tr}(YY') = \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^2,\end{aligned}$$

where

$$Y = A_1'(X-M)B_1 \Rightarrow \mathrm{d}Y = |A_1|^n|B_1|^m\mathrm{d}(X-M) = |A|^{\frac{n}{2}}|B|^{\frac{m}{2}}\mathrm{d}X$$

by using Theorem 5.4. Note that

$$|A| = |A_1A_1'| = |A_1||A_1'| = |A_1|^2 = |A_1'|^2,\ \mathrm{d}(X-M) = \mathrm{d}X,$$

since M is a constant matrix. Also from the theory of matrices we know that for any matrix G, $\mathrm{tr}(GG') =$ sum of squares of all elements in G. Hence

$$\begin{aligned}\int_X f(X)\mathrm{d}X &= c\int_X \mathrm{e}^{-\mathrm{tr}[A(X-M)B(X-M)']} \\ &= c|A|^{-\frac{n}{2}}|B|^{-\frac{m}{2}}\int_Y \mathrm{e}^{-\mathrm{tr}(YY')}\mathrm{d}Y \\ &= c|A|^{-\frac{n}{2}}|B|^{-\frac{m}{2}}\prod_{i=1}^{m}\prod_{j=1}^{n}\int_{-\infty}^{\infty}\mathrm{e}^{-y_{ij}^2}\mathrm{d}y_{ij} \\ &= c|A|^{-\frac{n}{2}}|B|^{-\frac{m}{2}}\pi^{\frac{mn}{2}},\end{aligned}$$

since

$$\int_{-\infty}^{\infty}\mathrm{e}^{-t^2}\mathrm{d}t = \sqrt{\pi}.$$

Hence if $f(X)$ is a density function then

$$c = \frac{|A|^{\frac{n}{2}}|B|^{\frac{m}{2}}}{\pi^{\frac{mn}{2}}}, \tag{5.15}$$

then the total integral is 1 and c in that case is called a normalizing constant and with this c, $f(X)$ becomes a density because by definition $f(X) > 0$ for all X, when $c > 0$ since it is an exponential function. The density in (5.14) with c in (5.15) is called a real matrix-variate Gaussian density.

In Theorem 5.4 our matrix X was rectangular. If $m = n$ then X is a square matrix with m^2 real scalar variables. If X is symmetric then there are only $\frac{m(m+1)}{2}$ distinct elements in X because $x_{ij} = x_{ji}$ for all i and j. What will happen to the Jacobian if we have a transformation of the type $Y = AXA', |A| \neq 0, X = X'$? This result will be stated here without proof.

Theorem 5.5. *Let $X = X'$ be $p \times p$ and of functionally independent real variables except for the condition $X = X'$. Let A be a $p \times p$ nonsingular constant matrix. Then*

$$Y = AXA' \Rightarrow \mathrm{d}Y = |A|^{p+1}\mathrm{d}X. \tag{5.16}$$

One way of proving this result is to represent the nonsingular matrix A as a product of basic elementary matrices and then look at the transformations successively. For example, let $A = E_1 E_2 \cdots E_k$ where $E_1, \ldots, E_k$ are some basic elementary matrices. Then

$$Y = AXA' = E_1 \cdots E_k X E_k' \cdots E_1'.$$

Now look at the transformations

$$Y_1 = E_k X E_k',\ Y_2 = E_{k-1} Y_1 E_{k-1}', \ldots, Y_k = E_1 Y_{k-1} E_1',$$

and evaluate the Jacobians successively.

$$\mathrm{d}Y_1 = J_1 \mathrm{d}X,\ \mathrm{d}Y_2 = J_2 \mathrm{d}Y_1 = J_2 J_1 \mathrm{d}X,$$

and so on. For more details on this and for other Jacobians see Mathai (1997).

5.4 Jacobians in Nonlinear Transformations

For a $p \times p$ positive definite matrix X of functionally independent real scalar variables consider the integral

$$\Gamma_p(\alpha) = \int_{X=X'>0} |X|^{\alpha-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X. \tag{5.17}$$

For $p = 1$, obviously (5.17) is the integral representation for the gamma function $\Gamma(\alpha)$. Hence $\Gamma_p(\alpha)$ in (5.17) is a matrix-variate version of $\Gamma(\alpha)$. The integral in (5.17) can be evaluated by using a triangular decomposition of X as $X = TT'$ where T is a lower triangular matrix. This transformation $X = TT'$ is not one-to-one. There can be many values for t_{ij}'s for given x_{ij}'s. But if we assume that the diagonal elements in T are positive, that is, $t_{jj} > 0, j = 1, \ldots, p$ then the transformation can be shown to be one-to-one. Take a case of $p = 3$, write X, T, TT' explicitly and verify this fact. Taking the x-variables in the order $x_{11}, x_{12}, \ldots, x_{1p}, x_{22}, \ldots, x_{2p}, \cdots, x_{pp}$ and the t-variables in the order $t_{11}, t_{21}, \ldots, t_{p1}, t_{22}, t_{32}, \ldots, t_{pp}$ we can easily see that the matrix of partial derivatives is of a triangular format with the diagonal elements t_{11} appearing p times, t_{22} appearing $p-1$ times and so on and t_{pp} appearing only once and a 2 appearing a total of p times in the diagonal. Hence we have the following result:

Theorem 5.6. *Let $X = X'$ be a positive definite matrix of functionally independent real scalar variables except for the symmetry condition. Let T be a lower triangular matrix with distinct real elements with the diagonal elements $t_{jj} > 0, j = 1, \ldots, p$. Then*

$$X = TT' \Rightarrow \mathrm{d}X = 2^p \left\{ \prod_{j=1}^{p} t_{jj}^{p+1-j} \right\} \mathrm{d}T.$$

Then by applying this triangular decomposition of X into t_{ij}'s, observing that

$$|X| = |TT'| = \left(\prod_{j=1}^{p} t_{jj}^2 \right),$$

$$\mathrm{tr}(X) = \mathrm{tr}(TT') = t_{11}^2 + \left(t_{21}^2 + t_{22}^2\right) + \cdots + \left(t_{p1}^2 + \cdots + t_{pp}^2\right),$$

and integrating out one has the following result:

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}} \Gamma(\alpha)\Gamma\left(\alpha - \frac{1}{2}\right) \cdots \Gamma\left(\alpha - \frac{p-1}{2}\right), \Re(\alpha) > \frac{p-1}{2}. \quad (5.18)$$

Notation 5.2. $\Gamma_p(\alpha)$: **Real matrix-variate gamma function**.

Definition 5.2. Real matrix-variate gamma function is defined by (5.18).

The equation in (5.17) gives the integral representation for the real matrix-variate gamma function, where $\Re(\cdot)$ means the real part of $(\cdot)$.

In a similar fashion one can define a real matrix-variate beta function. To this end we can start with

$$\Gamma_p(\alpha)\Gamma_p(\beta) = [\int_{X>0} |X|^{\alpha - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X][\int_{Y>0} |Y|^{\beta - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(Y)} \mathrm{d}Y],$$

where X and Y are $p \times p$ positive definite matrices. Then

$$\Gamma_p(\alpha)\Gamma_p(\beta) = \int_{X>0} \int_{Y>0} |X|^{\alpha - \frac{p+1}{2}} |Y|^{\beta - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X+Y)} \mathrm{d}X \wedge \mathrm{d}Y.$$

Make the transformation $U = X + Y$. Then

$$Y = U - X \Rightarrow |Y| = |U - X| = |U||I - U^{-\frac{1}{2}}XU^{-\frac{1}{2}}|.$$

Then put $Z = U^{-\frac{1}{2}}XU^{-\frac{1}{2}}$ for fixed U and integrate out X to obtain

$$\Gamma_p(\alpha)\Gamma_p(\beta) = \Gamma_p(\alpha+\beta)\int_Z |Z|^{\alpha-\frac{p+1}{2}}|I - Z|^{\beta-\frac{p+1}{2}}\mathrm{d}Z. \tag{5.19}$$

Notation 5.3. $B_p(\alpha,\beta)$: **Real matrix-variate beta function**.

Definition 5.3. $B_p(\alpha,\beta)$ is defined as

$$B_p(\alpha,\beta) = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} = B_p(\beta,\alpha), \Re(\alpha) > \frac{p-1}{2}, \Re(\beta) > \frac{p-1}{2}. \tag{5.20}$$

One integral representation is given in (5.19). By changing $Z = I - W$ we can have one more representation. That is,

$$B_p(\alpha,\beta) = \int_{O<W<I} |W|^{\beta-\frac{p+1}{2}}|I - W|^{\alpha-\frac{p+1}{2}} \tag{5.21}$$

$$= \int_{O<Z<I} |Z|^{\alpha-\frac{p+1}{2}}|I - Z|^{\beta-\frac{p+1}{2}}. \tag{5.22}$$

These two representations are known as type-1 integral representations for a real matrix-variate beta.

Another nonlinear transformation that we need is about a nonsingular matrix going to its unique inverse. The result will be given here without proof.

Theorem 5.7. *Let X be a $p \times p$ nonsingular matrix of functionally independent real variables. Let $Y = X^{-1}$, the regular inverse of X. Then, ignoring the sign,*

$$\begin{aligned} Y = X^{-1} \Rightarrow \mathrm{d}Y &= |X|^{-2p}\mathrm{d}X \textit{ for a general } X \\ &= |X|^{-(p+1)}\mathrm{d}X \textit{ for } X = X' \\ &= |X|^{-(p-1)}\mathrm{d}X \textit{ for } X = -X'. \end{aligned} \tag{5.23}$$

This can be proved by making the following observations: For any θ,

$$XX^{-1} = I \Rightarrow \frac{\partial}{\partial\theta}(XX^{-1}) = \frac{\partial}{\partial\theta}(I) = O.$$

But

$$\frac{\partial}{\partial\theta}(XX^{-1}) = X[\frac{\partial}{\partial\theta}X^{-1}] + [\frac{\partial}{\partial\theta}X]X^{-1} = O.$$

Hence by taking differentials on both sides the matrices of differentials, denoted by $(\mathrm{d}X)$ and $(\mathrm{d}X^{-1})$ are connected by the relation

$$(\mathrm{d}X)X^{-1} + X(\mathrm{d}X^{-1}) = O \Rightarrow (\mathrm{d}X^{-1}) = -X^{-1}(\mathrm{d}X)X^{-1}. \tag{5.24}$$

Then by taking the wedge product of the differentials on both sides, keeping in mind that X^{-1} does not contain differentials and hence behaves like a constant when taking wedge product of differentials on both sides, we have the result in (5.23).

With the help of Theorem 5.7 one can have other representations for real matrix-variate beta function from the type-1 integral representations in (5.21) and (5.22). For the W or call it X in (5.21) consider the transformations

$$U = (I - X)^{-\frac{1}{2}} X (I - X)^{-\frac{1}{2}} \text{ and } V = U^{-1}.$$

Then the integral representations for $B_p(\alpha, \beta)$ reduce to the following:

$$\begin{aligned} B_p(\alpha, \beta) &= \int_{U=U'>0} |U|^{\alpha - \frac{p+1}{2}} |I + U|^{-(\alpha+\beta)} \mathrm{d}U \\ &= \int_{V=V'>0} |V|^{\beta - \frac{p+1}{2}} |I + V|^{-(\alpha+\beta)} \mathrm{d}V. \end{aligned} \tag{5.25}$$

The representations in (5.25) are called type-2 integral representations for a real matrix-variate beta function.

5.5 The Binomial Function

In the real scalar case, when we take the Laplace transform of a negative exponential function or a gamma function we obtain the binomial function. For example, for the scalar variable $x > 0$ and for the scalar parameter t

$$L_{f_1}(t) = \int_0^\infty \mathrm{e}^{-tx} f_1(x) \mathrm{d}x, \tag{5.26}$$

is the Laplace transform of $f_1(x)$ defined for $x > 0$. If we take the Laplace transform of the gamma type function

$$f_2(x) = \frac{x^{\alpha-1}\mathrm{e}^{-x}}{\Gamma(\alpha)}, x > 0,$$

we have

$$\begin{aligned} L_{f_2}(t) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{e}^{-tx} x^{\alpha-1} \mathrm{e}^{-x} \mathrm{d}x \\ &= (1+t)^{-\alpha} \text{ for } 1 + t > 0. \end{aligned}$$

This is the binomial function or ${}_1F_0$ hypergeometric function. The Laplace transform in the matrix-variate case, analogous to the multivariate Laplace transform, is defined as

$$L_f(T^*) = \int_{X=X'>0} \frac{|X|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(X)}}{\Gamma_p(\alpha)}\mathrm{e}^{-\mathrm{tr}(T^*X)}\mathrm{d}X, \tag{5.27}$$

where

$$T^* = (t^*_{ij}),\ t^*_{ij} = \frac{1}{2}t_{ij}, i \neq j,\ t^*_{jj} = t_{jj}, t_{ij} = t_{ji},$$

for all $i, j = 1, \ldots, p$. Then

$$L_f(T^*) = |I + T^*|^{-\alpha} = |T^*|^{-\alpha}|I + T^{*-1}|^{-\alpha}, \tag{5.28}$$

for $T^* = T^{*\prime} > 0$ and $I + T^* > 0$. Then the hypergeometric function ${}_1F_0$ with matrix argument U will be defined as

$${}_1F_0(\alpha;\ ;U) = |I - U|^{-\alpha} \text{ for } O < U < I. \tag{5.29}$$

Observe that $O < U < I$ implies that $U = U' > 0, I - U > 0$ which means that the eigenvalues of U are in the open interval $(0, 1)$. We can make one more observation on

$${}_0F_0(\ ;\ ;-X) = \mathrm{e}^{-\mathrm{tr}(X)},$$

and

$${}_1F_0(\alpha;\ ;-X) = |I + X|^{-\alpha},$$

that we obtained so far. Consider the integral of the following type:

$$\int_{X=X'>0} |X|^{\rho-\frac{p+1}{2}} f(X)\mathrm{d}X = \int_{X>0} |X|^{\rho-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X = \Gamma_p(\rho), \tag{5.30}$$

and

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}}|I + X|^{-\alpha}\mathrm{d}X = \frac{\Gamma_p(\rho)\Gamma_p(\alpha-\rho)}{\Gamma_p(\alpha)}, \tag{5.31}$$

for $\Re(\rho) > \frac{p-1}{2},\ \Re(\alpha - \rho) > \frac{p-1}{2}$. The integral in (5.31) is evaluated by using the type-2 integral representation for a beta function in (5.25).

Notation 5.4. $M_f(\rho)$: M-transform of f.

Definition 5.4. The generalized matrix transform or M-transform of a real-valued scalar function of the real $p \times p$ matrix $X = X > 0$ is defined as

$$M_f(\rho) = \int_{X=X'>0} |X|^{\rho-\frac{p+1}{2}} f(X)\mathrm{d}X, \tag{5.32}$$

whenever $M_f(\rho)$ exists, where f is a symmetric function in the sense $f(AB) = f(BA)$ for all matrices A and B where AB and BA are defined.

Thus a class of functions f will have the M-transform $M_f(\rho)$ for the arbitrary parameter ρ. For example, when f is the ${}_0F_0$ or ${}_1F_0$ we have the M-transforms given in (5.30) and (5.31).

5.6 Hypergeometric Function and M-transforms

Notation 5.5. ${}_rF_s(a_1,\ldots,a_r;b_1,\ldots b_s;-X)$: **Hypergeometric function of matrix argument** $-X$.

Definition 5.5. A hypergeometric function of matrix argument $-X$ with r upper and s lower parameters is defined as the class of symmetric functions f having the following M-transform:

$$M_f(\rho)=\frac{\left\{\prod_{j=1}^{s}\Gamma_p(b_j)\right\}}{\left\{\prod_{j=1}^{r}\Gamma_p(a_j)\right\}}\Gamma_p(\rho)\frac{\left\{\prod_{j=1}^{r}\Gamma_p(a_j-\rho)\right\}}{\left\{\prod_{j=1}^{s}\Gamma_p(b_j-\rho)\right\}} \tag{5.33}$$

whenever the gammas on the right exist, where ρ is a parameter, and $a_1,\ldots,a_r$ and $b_1,\ldots,b_s$ are the upper and lower parameters of the hypergeometric function, which will be written as

$$f={}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;-X).$$

In (5.33) it is assumed that f is a symmetric function in the sense $f(AB)=f(BA)$ for all A and B whenever AB and BA are defined. An implication of this condition is the following: Let Q be an orthonormal matrix such that $QQ'=I=Q'Q$ and $Q'XQ=\mathrm{diag}(\lambda_1,\ldots,\lambda_p)$ where $\lambda_1,\ldots,\lambda_p$ are the eigenvalues of X, where it is assumed that the eigenvalues are distinct, then

$$\begin{aligned} f(X)&=f(XI)=f(XQQ')=f(Q'XQ)\\ &=f(D),\ D=\mathrm{diag}(\lambda_1,\ldots,\lambda_p), \end{aligned} \tag{5.34}$$

or $f(X)$ becomes a function of the p eigenvalues only. Thus, under the condition of symmetry on $f(X)$, this function of the $\frac{p(p+1)}{2}$ real scalar variables in X becomes a function of p variables, namely the p eigenvalues of X, which by assumption are real, distinct and positive.

There are other definitions for a hypergeometric function of matrix argument. All definitions have the basic assumption that the function is symmetric in the above sense. One definition based on the Laplace and inverse Laplace pair gives

${}_rF_s(a_1,\dots,a_r;b_1,\dots,b_s\ :\ X)$ as that function satisfying the following pair of integral equations:

$$
\begin{aligned}
&{}_{r+1}F_s(a_1,\dots,a_r,c;b_1,\dots,b_s;-\Lambda^{-1})|\Lambda|^{-c}\\
&\quad=\frac{1}{\Gamma_p(c)}\int_{U=U'>0}\mathrm{e}^{-\mathrm{tr}(\Lambda U)}{}_rF_s(a_1,\dots,a_r;b_1,\dots,b_s;-U)|U|^{c-\frac{p+1}{2}}\mathrm{d}U
\end{aligned}
\tag{5.35}
$$

$$
\begin{aligned}
&{}_rF_{s+1}(a_1,\dots,a_r;b_1,\dots,b_s,c;-\Lambda)|\Lambda|^{c-\frac{p+1}{2}}\\
&\quad=\frac{\Gamma_p(c)}{(2\pi i)^{\frac{p(p+1)}{2}}}\int_{\Re(Z)=Z_0>0}\mathrm{e}^{\mathrm{tr}(\Lambda Z)}{}_rF_s(a_1,\dots,a_r;b_1,\dots,b_s;-Z^{-1})|Z|^{-c}\mathrm{d}Z.
\end{aligned}
\tag{5.36}
$$

Under certain conditions the function ${}_rF_s$ defined through (5.35) and (5.36) can be shown to be unique. From this definition also the explicit forms are available only for ${}_0F_0$ and ${}_1F_0$. Others will remain as the solutions of a pair of integral equations.

The third definition available is in terms of zonal polynomials, which are certain symmetric functions in the eigenvalues of $X=X'>0$. For zonal polynomials and their properties see Mathai, Provost and Hayakawa (1995). Here ${}_rF_s$ will be defined as the following series:

$$
\begin{aligned}
&{}_rF_s(a_1,\dots,a_r;b_1,\dots,b_s;X)\\
&\quad=\sum_{k=0}^{\infty}\sum_{K}\frac{(a_1)_K\cdots(a_r)_K}{(b_1)_K\cdots(b_s)_K}\frac{C_K(X)}{k!},
\end{aligned}
\tag{5.37}
$$

where $K=(k_1,\dots,k_p), k=k_1+\cdots+k_p$

$$
(a)_K=\prod_{j=1}^{p}\left(a-\frac{j-1}{2}\right)_{k_j},
$$

and $C_K(X)$ is the zonal polynomial of order k. In this definition, ${}_rF_s$ is available explicitly for all r and s but zonal polynomials of higher orders are extremely difficult to evaluate and hence the practical utility of (5.37) is limited. The uniqueness of ${}_rF_s$, defined through (5.37), can be established by showing that (5.37) satisfies the pair of integral equations (5.35) and (5.36). For more details on (5.35), (5.36) and (5.37) and some applications see Mathai (1997).

Observe that H-functions and Meijer's G-functions, in the scalar cases, are defined in terms of their Mellin–Barnes representations. If we want a series representation then we have to take into account all the poles of the integrands in the Mellin–Barnes representations. Obviously the poles can be of all sorts of higher orders and then the series representations will be quite complicated involving, gamma, psi and generalized zeta functions as well as logarithmic terms. For a general expansion for the G-function see Mathai (1993c). The same procedure can be followed to obtain a series expansion for a H-function. This will be more complicated. Hence if we wish to extend the definition in (5.37) to a H-function of matrix argument it is extremely difficult because the series form need not correspond to the same in the scalar variable case. For the very special case of simple poles for the integrand in a Meijer's G-function one can obtain a series form in terms of hypergeometric series in the scalar case. If the series form is replaced by (5.37), the series form in zonal polynomials, still the procedure will not be correct because in $\Gamma_p(\alpha + s)$ itself the alternate gammas produce poles of higher orders, namely the poles of $\Gamma(\alpha+s), \Gamma(\alpha+s-1), \ldots$ are of higher orders and similar is the case for the poles of $\Gamma(\alpha+s-\frac{1}{2}), \Gamma(\alpha+s-\frac{3}{2}), \ldots$ Hence the procedure of making use of (5.37) is also not suitable for extending the definition to matrix variable case for a H-function. Therefore looking for a class of functions by using M-transforms may be the most convenient way of extending the definition to a matrix-variate H-function.

The above considerations lead to one important question. Is there a unique function which can be called the multivariate version of a given univariate function? The answer is obviously a big "no". There can be infinitely many multivariate functions, where the marginal functions yield your specified univariate functions. We can construct many examples.

Example 5.3. **Nonuniqueness of multivariate analogues**. Show that the following two bivariate functions

$$\text{(i)}\quad f_1(x, y, \rho) = \frac{1}{\pi\sqrt{1-\rho^2}}\mathrm{e}^{-\frac{(x^2-2\rho xy+y^2)}{1-\rho^2}},$$

for $1-\rho^2 > 0, -\infty < x < \infty, -\infty < y < \infty$ where ρ is a constant and

$$\text{(ii)}\quad f_2(x, y) = \alpha_1 f_1(x, y, \rho_1) + \cdots + \alpha_k f_k(x, y, \rho_k),$$

for $0 < \alpha_i < 1, 1-\rho_i^2 > 0, i = 1, \ldots, k, \alpha_1 + \cdots + \alpha_k = 1$ yield the same marginal functions

$$f(x) = \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}}, \text{ and } g(y) = \frac{\mathrm{e}^{-y^2}}{\sqrt{\pi}},$$

for $-\infty < x < \infty, -\infty < y < \infty$.

Solution 5.3. Let us consider the marginal function of x from $f_1(x, y, \rho)$ by integrating out y. Consider the exponent, excluding -1.

$$\begin{aligned}\frac{1}{1-\rho^2}[x^2 - 2\rho xy + y^2] &= x^2 + \left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2\\ &= x^2 + u^2,\ u = \frac{y-\rho x}{\sqrt{1-\rho^2}}\\ \mathrm{d}y &= \sqrt{1-\rho^2}\mathrm{d}u \text{ for fixed } x.\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} \frac{1}{\pi\sqrt{1-\rho^2}} e^{-\frac{1}{1-\rho^2}(x^2-2\rho xy+y^2)} dy$$
$$= \int_{-\infty}^{\infty} \frac{1}{\pi\sqrt{1-\rho^2}} e^{-[x^2+\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2]} dy$$
$$= \frac{e^{-x^2}}{\pi} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{e^{-x^2}}{\sqrt{\pi}}.$$

Similarly

$$\int_{-\infty}^{\infty} f_1(x, y, \rho) dx = \frac{e^{-y^2}}{\sqrt{\pi}}.$$

Thus for the given function

$$f(x) = \frac{e^{-x^2}}{\sqrt{\pi}}, \quad -\infty < x < \infty,$$

one can take $f_1(x, y, \rho)$ for all ρ such that $1 - \rho^2 > 0$ as a bivariate analogue.

Now, look at the process above. When we integrate out y from $f_2(x, y, \rho)$ we obtain

$$\alpha_1 \frac{e^{-x^2}}{\sqrt{\pi}} + \cdots + \alpha_k \frac{e^{-x^2}}{\sqrt{\pi}} = \frac{e^{-x^2}}{\sqrt{\pi}},$$

since $\alpha_1 + \cdots + \alpha_k = 1$. Thus all the classes of functions defined by f_2 can also be considered as bivariate extensions of the univariate function $f(x)$.

This example shows that for a given univariate function there is nothing called a unique bivariate or multivariate analogue. There will be several classes of functions which can all be legitimately called the multivariate analogues. Hence looking for a unique multivariate analogue for a given univariate H-function is a meaningless attempt. Looking for a nonempty class of matrix variable functions, where when the matrix is 1×1 or a scalar quantity the functions reduce to the one variable H-function, is the proper procedure. Keeping this in mind, the following classes of functions are defined as G and H-functions of matrix argument.

5.7 Meijer's G-Function of Matrix Argument

Let $f_1(X)$ be a symmetric function in the sense $f(AB) = f(BA)$ for all matrices A and B whenever AB and BA are defined. Let X be a $p \times p$ real positive definite matrix with distinct eigenvalues $\lambda_1 > \cdots > \lambda_p > 0$. Consider the following M-transform with the arbitrary parameter ρ.

Definition 5.6. Meijer's *G*-function of matrix argument in the real case. Let $f_1(X)$ be such that

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f_1(X)\mathrm{d}X = \phi(\rho) \tag{5.38}$$

where

$$\phi(\rho) = \frac{\left\{\prod_{j=1}^{m} \Gamma_p(b_j+\rho)\right\}\left\{\prod_{j=1}^{n} \Gamma_p\left(\frac{p+1}{2}-a_j-\rho\right)\right\}}{\left\{\prod_{j=m+1}^{s} \Gamma_p\left(\frac{p+1}{2}-b_j-\rho\right)\right\}\left\{\prod_{j=n+1}^{r} \Gamma_p(a_j+\rho)\right\}}. \tag{5.39}$$

Whenever the right side exists the class of functions defined by (5.38) and (5.39) will be called Meijer's G-function of matrix argument in the real case where $\Gamma_p(\cdot)$ is the real matrix-variate gamma function.

Note that when $p=1$, $f_1(X)$ reduces to Meijer's G-function in the real scalar variable case. One can extend the same idea and define a H-function of matrix argument as follows:

Definition 5.7. *H*-function of matrix argument in the real case. Let $f_2(X)$ be a symmetric function in the sense $f_2(AB) = f_2(BA)$ for all matrices A and B whenever AB and BA are defined. Let X be a $p\times p$ real symmetric positive definite matrix with distinct eigenvalues $\lambda_1 > \cdots > \lambda_p > 0$. Let ρ be an arbitrary parameter. Consider the following integral equation:

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f_2(X)\mathrm{d}X = \psi(\rho), \tag{5.40}$$

$$\psi(\rho) = \frac{\left\{\prod_{j=1}^{m} \Gamma_p(b_j+\beta_j\rho)\right\}\left\{\prod_{j=1}^{n} \Gamma_p\left(\frac{p+1}{2}-a_j-\alpha_j\rho\right)\right\}}{\left\{\prod_{j=m+1}^{s} \Gamma_p\left(\frac{p+1}{2}-b_j-\beta_j\rho\right)\right\}\left\{\prod_{j=n+1}^{r} \Gamma_p(a_j+\alpha_j\rho)\right\}}, \tag{5.41}$$

with $\alpha_j, j=1,\ldots,r$ and $\beta_j, j=1,\ldots,s$ real and positive. Whenever the right side in (5.41) exists the class of functions $f_2(X)$ determined by (5.40) and (5.41) will be called the H-function of matrix argument in the real case.

For $p=1$, (5.40) reduces to H-function in the real scalar case. For $\alpha_j=1, j=1,\ldots,r$ and $\beta_j=1, j=1,\ldots,s$ the class of functions $f_2(X)$ reduces to the class of functions $f_1(X)$ defined through (5.38) and (5.39) and the H-function reduces to a G-function.

5.7.1 Some Special Cases

When $m=1, n=0, r=0, s=1, b_1=0, \beta_1=1$, (5.40) reduces to the equation

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f_2(X)\mathrm{d}X = \Gamma_p(\rho). \tag{5.42}$$

One solution for (5.42) is obvious, namely,

$$f_2(X) = \mathrm{e}^{-\mathrm{tr}(X)}$$

because for $\Re(\rho) > \frac{p-1}{2}$

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X = \Gamma_p(\rho).$$

Hence we may define ${}_0F_0(\,;\,;-X)$ by the integral equation in (5.42). On the other hand if $m = 1, n = 0, r = 0, s = 1, b_1 = \alpha, \beta_1 = 1$, then (5.41) reduces to $\Gamma_p(\alpha + \rho)$. Then the equation

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f_2(X)\mathrm{d}X = \Gamma_p(\alpha+\rho), \text{ for } \Re(\alpha+\rho) > \frac{p-1}{2}$$

gives one solution as

$$f_2(X) = |X|^{\alpha}{}_0F_0(\,;\,;-X).$$

Let $m = 1, b_1 = 0, \beta_1 = 1, n = r, s$ is replaced by $s + 1$ then (5.40) becomes

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f_2(X)\mathrm{d}X = \Gamma_p(\rho)\frac{\left\{\prod_{j=1}^{r}\Gamma_p\left(\frac{p+1}{2} - a_j - \alpha_j\rho\right)\right\}}{\left\{\prod_{j=1}^{s}\Gamma_p\left(\frac{p+1}{2} - b_j - \beta_j\rho\right)\right\}}. \tag{5.43}$$

For $p = 1$, (5.43) corresponds to Wright's function and hence we will call the class of functions $f_2(X)$ determined by (5.43) as the Wright's function of matrix argument in the real case.

When $\alpha_j = 1, j = 1, \ldots, r$ and $\beta_j = 1, j = 1, \ldots, s$ then comparing (5.43) with (5.33) we have

$$f_2(X) = \frac{\left\{\prod_{j=1}^{r}\Gamma_p\left(\frac{p+1}{2} - a_j\right)\right\}}{\left\{\prod_{j=1}^{s}\Gamma_p\left(\frac{p+1}{2} - b_j\right)\right\}} \times {}_rF_s\left(\frac{p+1}{2} - a_1, \ldots, \frac{p+1}{2} - a_p; \frac{p+1}{2} - b_1, \ldots, \frac{p+1}{2} - b_s; -X\right) \tag{5.44}$$

or the hypergeometric function of matrix argument in the real case. When $r = 1$, $s = 1$ in (5.43) we may call the corresponding $f_2(X)$ as the generalized Mittag-Leffler function in the real matrix-variate case. Classes of other elementary functions can be defined by taking special cases in (5.39)–(5.44). The theory of H-functions of matrix argument can be extended to complex cases also, that is, when the matrices are hermitian positive definite. Some preliminaries in this direction may be seen from Mathai (1997).

Exercises

5.1. Let $x_1, \ldots, x_p$ be real scalar variables. Let $y_1 = x_1 + \cdots + x_p$, $y_2 = x_1x_2 + x_1x_3 + \cdots + x_{p-1}x_p$ (sum of products taken two at a time), $\cdots$, $y_p = x_1x_2\cdots x_p$. For $x_j > 0$, $j = 1, \ldots, p$ show that

$$\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_p = \left\{\prod_{i=1}^{p-1}\prod_{j=i+1}^{p}|x_i - x_j|\right\}\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p.$$

5.2. Consider the general polar coordinate transformation

$$\begin{aligned} x_1 &= r\sin\theta_1, \\ x_j &= r\cos\theta_1\cos\theta_2\cdots\cos\theta_{j-1}\sin\theta_j,\ j = 2, \ldots, p-1, \\ x_p &= r\cos\theta_1\cos\theta_2\cdots\cos\theta_{p-1}, \end{aligned}$$

for $r > 0, -\frac{\pi}{2} < \theta_j \le \frac{\pi}{2}, j = 1, \ldots, p-2, -\pi < \theta_{p-1} \le \pi$. Compute $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p$ in terms of $\mathrm{d}r \wedge \mathrm{d}\theta_1 \wedge \cdots \wedge \mathrm{d}\theta_{p-1}$.

5.3. For $X = X' > 0, Y = Y' > 0$ and $p \times p$ show that

$$\lim_{a\to\infty}|I + \frac{XY}{a}|^{-a} = \mathrm{e}^{-\mathrm{tr}(XY)} = \lim_{a\to\infty}|I - \frac{XY}{a}|^{a}.$$

5.4. Let X and A be $p \times p$ lower triangular matrices of distinct elements. Let $A = (a_{ij})$ be a constant matrix such that $a_{jj} > 0$, $j = 1, \ldots, p$. Then show that

$$Y = XA \Rightarrow \mathrm{d}Y = \left\{\prod_{j=1}^{p} a_{jj}^{p+1-j}\right\}\mathrm{d}X.$$

5.5. For the same X and A in Exercise 5.4 evaluate the Jacobians in the transformations (i) $Y = AX$, (ii) $Y = aX$ where a is a scalar quantity.

5.6. Redo Exercises 5.4 and 5.5 if the matrices X and A are upper triangular.

5.7. For real $X = X' > 0$ and $p \times p$ evaluate the following integrals:

$$\text{(i)}\ \ f_1 = \int_X \mathrm{d}X;\ \ \text{(ii)}\ \ f_2 = \int_X |X|\mathrm{d}X;\ \ \text{(iii)}\ \ f_3 = \int_X |I - X|\mathrm{d}X$$

$$\text{(iv)}\ \ f_4 = \int_X |X|^{\alpha}\mathrm{d}X;\ \ \text{(v)}\ \ \int_X |I - X|^{\alpha}\mathrm{d}X$$

and evaluate these explicitly for (vi) $p = 2$; (vii) $p = 3$.

5.8. By showing that both sides have the same M-transforms establish the following results for the class of functions defined through (5.44) where all are $p \times p$ real symmetric positive definite matrices.

$$\text{(i)} \quad {}_1F_1(a;c;-X) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)}|X|^{-(c-\frac{p+1}{2})} \times \int_{O<Y<X} \mathrm{e}^{-\mathrm{tr}(Y)}|Y|^{a-\frac{p+1}{2}}|X-Y|^{c-a-\frac{p+1}{2}}\mathrm{d}Y$$

$$\text{(ii)} \quad {}_2F_1(a,b;c;-X) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \int_{O<X<I} |Y|^{a-\frac{p+1}{2}}|I-Y|^{c-a-\frac{p+1}{2}} \times |I+YX|^{-b}\mathrm{d}Y$$

$$\text{(iii)} \quad {}_2F_1(a,b;c;-X) = |I-X|^{-b}{}_2F_1(c-a,b;c;-X(I-X)^{-1}).$$

5.9. It is seen that for real $X = X' > 0$ and $p \times p$

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X = \Gamma_p(\rho),$$

for arbitrary ρ such that $\Re(\rho) > \frac{p-1}{2}$. Suppose that

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f(X)\mathrm{d}X = \Gamma_p(\rho) \text{ for } \Re(\rho) > \frac{p-1}{2},$$

establish a set of sufficient conditions so that $f(X)$ is uniquely determined as $f(X) = \mathrm{e}^{-\mathrm{tr}(X)}$.

5.10. For real $X = X' > 0$ and $p \times p$ consider the equation

$$\int_{X>0} |X|^{\rho-\frac{p+1}{2}} f(X)\mathrm{d}X = \frac{\Gamma_p(\rho)\Gamma_p(\alpha-\rho)}{\Gamma_p(\alpha)},$$

for $\Re(\rho) > \frac{p-1}{2}, \Re(\alpha-\rho) > \frac{p-1}{2}$. One solution for $f(X)$ is seen to be

$$f(X) = |I+X|^{-\alpha}.$$

What are the sufficient conditions on f such that this is the only solution?

Chapter 6
Applications in Astrophysics Problems

6.1 Introduction

There are many areas in astrophysics where Meijer's G-function and H-function appear naturally. Some of these areas are analytic solar and stellar models, nuclear reaction rate theory and energy generation in stars, gravitational instability problems, nonextensive statistical mechanics, pathway analysis, input-output models and reaction-diffusion problems. Brief introductions to these areas will be given here so that the readers can develop the areas further and tackle more general and more complex situations.

6.2 Analytic Solar Model

The numerical approach to the study of solar structure is to go for the numerical solutions of the underlying system of differential equations. Even for a simple main sequence star in hydrostatic equilibrium at least four nonlinear differential equations are to be dealt with to obtain a good picture of the internal structure of the star. Our Sun is such a main-sequence star.

The simplest analytical procedure is to start with a simple mathematical model for the matter density distribution in the core of the Sun. Then, from there develop formulae for the mass, pressure, temperature, luminosity and other such critical parameters. Several such models were considered in a series of papers by Haubold and Mathai, some details may be seen from Mathai and Haubold (1988). A two-parameter model considered by them for the density $\rho(r)$, at an arbitrary distance of r from the center of the Sun is the following:

$$\rho(r) = \rho_c \left[1 - \left(\frac{r}{R_\odot}\right)^{\delta}\right]^{\gamma}, \delta > 0, \qquad (6.1)$$

γ is a positive integer, where ρ_c is the central density, $R_\odot$ is the radius of the Sun. Let $y = \frac{r}{R_\odot}$. Then for the solar core, that is, $0 \le y \le 0.3$, it is seen that the

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$\delta = 1.28$ and $\gamma = 10$ give a good fit to the observational data. Then the model for $u = \frac{\rho(r)}{\rho_c}$ is given by

$$u = (1 - y^{\delta})^{\gamma}, \text{ with } \delta = 1.28, \text{ and } \gamma = 10. \tag{6.2}$$

This is shown to give good estimates for the solar mass $M(r)$, pressure $P(r)$, temperature $T(r)$, and luminosity $\epsilon(r)$. From standard formula we have

$$\frac{\mathrm{d}}{\mathrm{d}r} M(r) = 4\pi r^2 \rho(r) \tag{6.3}$$

where $M(r)$ is the mass at the distance r from the center.

$$\begin{aligned} M(r) &= 4\pi \int_0^r t^2 \rho(t) \mathrm{d}t \\ &= 4\pi\rho_c \int_0^r t^2 \left[1 - \left(\frac{t}{R_\odot}\right)^{\delta}\right]^{\gamma} \mathrm{d}t \qquad (6.4) \\ &= \frac{4\pi\rho_c}{3} R_\odot^3 \left(\frac{r}{R_\odot}\right)^3 {}_2F_1\left[-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; \left(\frac{r}{R_\odot}\right)^{\delta}\right], \qquad (6.5) \end{aligned}$$

where ${}_2F_1$ is a Gauss' hypergeometric function, which is a special case of a H-function. For $r = R_\odot$ in (6.4) we have the total mass of the Sun, which works out to be the following:

$$\begin{aligned} M(R_\odot) &= \frac{4\pi\rho_c R_\odot^3}{3} {}_2F_1\left(-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; 1\right) \qquad (6.6) \\ &= \frac{4\pi\rho_c}{3} \frac{\gamma!}{\left(\frac{3}{\delta} + 1\right)\left(\frac{3}{\delta} + 2\right) \cdots \left(\frac{3}{\delta} + \gamma\right)}, \qquad (6.7) \end{aligned}$$

by using he expansion formula

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{6.8}$$

where $a > 0, c - a - b > 0$. Then

$$\frac{M(r)}{M(R_\odot)} = \frac{\left(\frac{3}{\delta} + 1\right) \cdots \left(\frac{3}{\delta} + \gamma\right)}{\gamma!} \left(\frac{r}{R_\odot}\right)^3 {}_2F_1\left(-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; \left(\frac{r}{R_\odot}\right)^{\delta}\right). \tag{6.9}$$

This is seen to be in good agreement with observational data for $\delta = 1.28$ and $\gamma = 10$. The internal pressure at arbitrary distance r from the center is available from standard formula

$$
\begin{aligned}
P(r) &= P_c - G\int_0^r \frac{M(t)\rho(t)}{t^2}\mathrm{d}t \\
&= P_c - \frac{4\pi G}{\delta^2}\rho_c^2 R_\odot^2 \sum_{m=0}^{\gamma} \frac{(-\gamma)_m \left(\frac{r}{R_\odot}\right)^{m\delta+2}}{\left(\frac{3}{\delta}+m\right)\left(\frac{2}{\delta}+m\right)} \\
&\quad \times {}_2F_1\left(-\gamma, \frac{2}{\delta}+m; \frac{2}{\delta}+m+1; \left(\frac{r}{R_\odot}\right)^{\delta}\right),
\end{aligned} \tag{6.10}
$$

where P_c is the pressure at the center and G is the gravitational constant. By using the fact that $P(R_\odot) = 0$ we can compute the pressure at the center P_c. Opening up the hypergeometric function we can write $P(r)$ in a closed form:

$$
\begin{aligned}
P(r) &= P_c - \frac{2}{3}\pi G\rho_c^2 r^2 \\
&\quad \times F_{1:2:0}^{1:3:1}\left[\begin{pmatrix} \left(\frac{r}{R_\odot}\right)^{\delta} \\ \left(\frac{r}{R_\odot}\right)^{\delta} \end{pmatrix} \middle| \begin{matrix} \frac{2}{3}; -\gamma, \frac{3}{\delta}, \frac{2}{\delta}; -\gamma \\ \frac{2}{\delta}+1; \frac{3}{\delta}+1, \frac{2}{\delta}+1; \end{matrix}\right],
\end{aligned} \tag{6.11}
$$

where $F_{1:2:0}^{1:3:1}(\cdot)$ is a Kampé de Fériet's function, see Srivastava and Karlsson (1985). The standard equation for temperature is the following:

$$
T(r) = \frac{\mu}{kN_A} P(r)\rho(r), \tag{6.12}
$$

where μ is the mean molecular weight, k is Boltzmann's constant and N_A is Avogadro's number. For the model in (6.1) it can be seen that

$$
T(r) = \frac{\mu}{kN_A} 4\pi G\rho_c R_\odot^2 \frac{g(r)}{[1-(\frac{r}{R_\odot})^{\delta}]^{\gamma}}, \tag{6.13}
$$

where

$$
\begin{aligned}
g(r) &= \frac{1}{\delta^2}\sum_{m=0}^{\gamma} \frac{(-\gamma)_m}{m!} \frac{1}{\left(\frac{3}{\delta}+m\right)\left(\frac{2}{\delta}+m\right)} \\
&\quad \times \left[\frac{\gamma!}{\left(\frac{2}{\delta}+m+1\right)\cdots\left(\frac{2}{\delta}+m+\gamma\right)}\right. \\
&\quad \left. - \left(\frac{r}{R_\odot}\right)^{m\delta+2} {}_2F_1\left(-\gamma, \frac{2}{\delta}+m; \frac{2}{\delta}+m+1; \left(\frac{r}{R_\odot}\right)^{\delta}\right)\right].
\end{aligned} \tag{6.14}
$$

From the computations in Haubold and Mathai (1994) it is seen that $M(r)$, $P(r)$, $T(r)$ and luminosity $L(r)$ are in good agreement with observational data for the model in (6.1) with $\delta = 1.28$ and $\gamma = 10$. Further details may be seen from Haubold and Mathai (1994).

Exercises 6.1

6.1.1. From the model in (6.1) derive expressions for solar mass $M(r)$, pressure $P(r)$, temperature $T(r)$ and luminosity $L(r)$ at an arbitrary distance r from the center.

6.1.2. Let $u = \frac{\rho(r)}{\rho_c}$ where $\rho(r)$ is the matter density at a distance r from the center of the Sun and ρ_c is the density at the center. Let $y = \frac{r}{R_\odot}$ for $0 \leq y \leq 0.3$, where $R_\odot$ is the solar radius. The following is the data from Sear (1964)

$$y : 0.0864, 0.1153, 0.1441, 0.1873, 0.2161, 0.2450, 0.2882$$
$$u : 0.6519, 0.5253, 0.3856, 0.2810, 0.1994, 0.1424, 0.0962$$

By using the method of least squares fit a polynomial of degree 3 to this data and show that the polynomial model is

$$y = 1 - 0.940y + 6.67y^2 - 2.73y^3.$$

Compute u by using this model and compare with Sear's data.

6.1.3. Consider the following three models for u of Exercise 6.1.2

$$u = 1 - 4y + 2y^2 + 2y^3 - y^4,$$
$$u = (1 - \sqrt{y})(1 - y^3)^{64},$$
$$u = (1 - y^{\frac{3}{2}})^{16}.$$

Compute u under these models and compare with Sear's data.

6.1.4. Consider the following four models for u in Exercise 6.1.2.

$$u = (1 - \sqrt{y})(1 - y^3)^{64}(1 - y),$$
$$u = (1 - y^{1.48})^{14},$$
$$u = (1 - y^{1.48})^{13},$$
$$u = (1 - y^{1.28})^{10}.$$

Compute u under these models and compare with Sear's data.

6.1.5. Show that the last model in Exercise 6.1.4 is the best among all the eight models in Exercises 6.1.2, 6.1.3 and 6.1.4.

6.3 Thermonuclear Reaction Rates

In nuclear reaction rate theory one comes across the following four reaction probability integrals in nonresonant reactions, reactions with high energy tail cut off, in screened case and in the depleted case:

$$I_1 = \int_0^\infty y^\nu \mathrm{e}^{-\left(y+zy^{-\frac{1}{2}}\right)} \mathrm{d}y \tag{6.15}$$

$$I_2 = \int_0^d y^\nu \mathrm{e}^{-\left(y+zy^{-\frac{1}{2}}\right)} \mathrm{d}y \tag{6.16}$$

$$I_3 = \int_0^\infty y^\nu \mathrm{e}^{-\left(y+\frac{z}{\sqrt{y+t}}\right)} \mathrm{d}y \tag{6.17}$$

$$I_4 = \int_0^\infty y^\nu \mathrm{e}^{-\left(y+by^\delta+zy^{-\frac{1}{2}}\right)} \mathrm{d}y. \tag{6.18}$$

These are the reaction rate probability integrals dealt with in Anderson et al. (1994). A more general case of I_1 is the following:

$$I_5 = \int_0^\infty y^\nu \mathrm{e}^{-(ay^\delta+by^{-\rho})} \mathrm{d}y \tag{6.19}$$

for $a > 0, b > 0, \delta > 0, \rho > 0$, where for $a = 1, b = z, \delta = 1, \rho = \frac{1}{2}$ we have the integral I_1. Observe that (6.19) is the limiting form of the versatile integral discussed in Chap. 4. Writing

$$f_1(x) = x^{\nu+1}\mathrm{e}^{-ax^\delta} \text{ and } f_2(x) = \mathrm{e}^{-x^\rho},$$

the integral in (6.19) can be written as

$$I_5 = \int_{v=0}^\infty \frac{1}{v} f_1(v) f_2\left(\frac{u}{v}\right) \mathrm{d}v, u = b^{\frac{1}{\rho}}. \tag{6.20}$$

Hence from Mellin convolution property, the Mellin transform of I_5 is the product of the Mellin transforms of $f_1(x)$ and $f_2(x)$ respectively. Denoting the Mellin transforms by $g_1(s)$ and $g_2(s)$, with s being the Mellin parameter, one has,

$$g_1(s) = \int_0^\infty x^{s-1}x^{\nu+1}\mathrm{e}^{-ax^\delta}\mathrm{d}x = \frac{1}{\delta}\frac{\Gamma\left(\frac{\nu+1+s}{\delta}\right)}{a^{\frac{\nu+1+s}{\delta}}}, \tag{6.21}$$

where $\Re(\nu + 1 + s) > 0$ and

$$g_2(s) = \int_0^\infty x^{s-1}\mathrm{e}^{-x^\rho}\mathrm{d}x = \frac{1}{\rho}\Gamma\left(\frac{s}{\rho}\right), \Re(s) > 0. \tag{6.22}$$

Then I_5 is available from the inverse Mellin transform of $g_1(s)g_2(s)$. That is,

$$I_5 = \frac{1}{2\pi i}\int_L \frac{1}{\delta\rho}\frac{\Gamma\left(\frac{\nu+1+s}{\delta}\right)}{a^{\frac{\nu+1+s}{\delta}}}\Gamma\left(\frac{s}{\rho}\right)u^{-s}\mathrm{d}s \tag{6.23}$$

$$= \frac{1}{\delta\rho a^{\frac{\nu+1}{\delta}}}\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{s}{\rho}\right)\Gamma\left(\frac{\nu+1}{\delta}+\frac{s}{\delta}\right)\left(a^{\frac{1}{\delta}}u\right)^{-s}\mathrm{d}s$$

$$= \frac{1}{\delta\rho a^{\frac{\nu+1}{\delta}}}H_{0,2}^{2,0}\left[a^{\frac{1}{\delta}}b^{\frac{1}{\rho}}\Big|_{(0,\frac{1}{\rho}),\left(\frac{\nu+1}{\delta},\frac{1}{\delta}\right)}\right]. \tag{6.24}$$

Thus the special cases of the integral in (6.20) are the special cases of the H-function in (6.24).

Some interesting special cases are the situations where (i): $\frac{1}{\delta}=m, \frac{1}{\rho}=n, m,n=1,2,\ldots$; (ii): $\frac{\rho}{\delta}=\lambda, \lambda=1,2,\ldots$; (iii) $\frac{\delta}{\rho}=\mu, \mu=1,2,\ldots$. In all these cases one can reduce the H-function in (6.24) to Meijer's G-function with the help of the multiplication formula for gamma functions, some details and computable representations are available from Mathai and Haubold (1988).

Exercises 6.2

6.2.1. Show that the reaction rate probability integral

$$\int_0^\infty x^{\nu-1}\mathrm{e}^{-ax-zx^{-\rho}}\mathrm{d}x = \frac{a^{-\nu}}{\rho}H_{0,2}^{2,0}\left[az^{\frac{1}{\rho}}\big|_{(0,\frac{1}{\rho}),(\nu,1)}\right],$$

for $a>0, z>0, \rho>0$.

6.2.2. For $\rho=\frac{1}{2}, a=1$ in Exercise 6.2.1 show that

$$\int_0^\infty x^{\nu-1}\mathrm{e}^{-x-zx^{-\frac{1}{2}}}\mathrm{d}x = \pi^{-\frac{1}{2}}G_{0,3}^{3,0}\left[\frac{z^2}{4}\big|_{0,\frac{1}{2},\nu}\right]$$

$$= \pi^{-\frac{1}{2}}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma\left(\frac{1}{2}+s\right)\Gamma(\nu+s)\left(\frac{z^2}{4}\right)^{-s}\mathrm{d}s.$$

6.2.3. Write down the conditions for the poles of the integrand in the Mellin–Barnes integral in Exercise 6.2.2 to be simple. Evaluate the Mellin–Barnes integral in Exercise 6.2.2 in the case of simple poles.

6.2.4. Write down the integral in Exercise 6.2.2 in series form when ν is an integer thereby the poles of the integrand can be up to order 2.

6.2.5. Write down the integral in Exercise 6.2.2 in series form when ν is a half-integer thereby the poles of the integrand can be up to order 2.

6.4 Gravitational Instability Problem

Gravitational condensation is believed to be the reason for the formation of the basic building blocks of the universe, that is, the stars and galaxies and systems of them at various scales. The universe is a multi-component medium. The influence of the components' relative motions upon the gravitational instability was investigated by many authors. Gravitational instability in a multi-component medium in an expanding universe under Newtonian approximation was studied by Mathai et al. (1988). Exact solutions of the differential equations connected with the gravitational instability problems in a two-component, and then in a multi-component medium, were considered by Mathai et al. (1988) by converting the basic equations to the equations satisfied by a Meijer's G-function. After a few substitutions, see Mathai et al. (1988), the basic equations for a two-component medium can be written as follows:

$$\Delta^2\delta_1 + (2\eta - 1)\Delta\delta_1 + k_1^2 t^{\alpha_1}\delta_1 = \frac{2}{3}(\Omega_1\delta_1 + \Omega_2\delta_2), \tag{6.25}$$

$$\Delta^2\delta_2 + (2\eta - 1)\Delta\delta_2 + k_2^2 t^{\alpha_2}\delta_2 = \frac{2}{3}(\Omega_1\delta_1 + \Omega_2\delta_2), \tag{6.26}$$

where Δ is the operator $\Delta = t\frac{\mathrm{d}}{\mathrm{d}t}$, $\Omega = \Omega_1 + \Omega_2 = 1, \Omega_i, i = 1, 2$ are constants, and other parameters have physical interpretations. Solving for δ_2 from (6.25) and then substituting for $\delta_2, \Delta\delta_2, \Delta^2\delta_2$ in (6.26) one has the following fourth degree equation:

$$\begin{aligned}
&\Delta^4\delta_1 + 2(2\eta - 1)\Delta^3\delta_1 + \left[k_1^2 t^{\alpha_1} + k_2^2 t^{\alpha_2} - \frac{2}{3} + (2\eta - 1)^2\right]\Delta^2\delta_1 \\
&\quad + \left[(2\eta - 1)k_1^2 t^{\alpha_1} + (2\eta - 1)k_2^2 t^{\alpha_2} + 2k_1^2\alpha_1 t^{\alpha_1} - (2\eta - 1)\frac{2}{3}\right]\Delta\delta_1 \\
&\quad + \left[k_1^2\alpha_1^2 t^{\alpha_1} + (2\eta - 1)k_1^2\alpha_1 t^{\alpha_1} - \frac{2}{3}\Omega_2 k_1^2 t^{\alpha_1}\right. \\
&\quad \left. - \frac{2}{3}\Omega_1 k_2^2 t^{\alpha_2} + k_1^2 k_2^2 t^{\alpha_1 + \alpha_2}\right]\delta_1 = 0.
\end{aligned} \tag{6.27}$$

Here (6.27) is the equation governing the growth and decay of gravitational condensation in the expanding two-fluid universe. An equation for δ_2, corresponding to (6.27) is available from symmetry. The following special cases of (6.27) have interesting solutions. We consider the following cases: (i) $k_1 = k_2 = 0$; (ii) $\alpha_1 = \alpha_2 = 0, k_1, k_2$ arbitrary; (iii) $\alpha_2 \neq 0, k_1 = 0$; (iv) $\alpha_1 \neq 0, k_2 = 0$; (v) $\alpha_1 = 0, \alpha_2 \neq 0$; (vi) $\alpha_1 \neq 0, \alpha_2 = 0$; (vii) $\alpha_2 = \alpha_1 = \alpha \neq 0, k_1 = k_2 = k \neq 0$.

In case (iii) by changing t to $x = \frac{k_2 t^{\alpha_2}}{\alpha_2^2}$ and $\tilde{\Delta} = x\frac{\mathrm{d}}{\mathrm{d}x}$, Eq. (6.27) reduces to

$$\begin{aligned}
&\left\{(\tilde{\Delta} - b_1)(\tilde{\Delta} - b_2)(\tilde{\Delta} - b_3)(\tilde{\Delta} - b_4)\right\}\delta_1 \\
&\quad + x\left\{(\tilde{\Delta} - a_1)(\tilde{\Delta} - a_2)\right\}\delta_1 = 0,
\end{aligned} \tag{6.28}$$

where

$$b_1 = 0, b_2 = -\frac{(2\eta - 1)}{\alpha_2},\ b_3 = \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_2} - \left[\frac{(\eta - \frac{1}{2})^2}{\alpha_2^2} + \frac{2}{3\alpha_2^2}\right]^{\frac{1}{2}}$$

$$b_4 = \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_2} + \left[\frac{(\eta - \frac{1}{2})^2}{\alpha_2^2} + \frac{2}{3\alpha_2^2}\right]^{\frac{1}{2}}, a_1 = \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_2} - \left[\frac{\left(\frac{1}{2} - \eta\right)^2}{\alpha_2^2} + \frac{1\Omega_1}{3\alpha_2^2}\right]^{\frac{1}{2}}$$

$$a_2 = \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_2} + \left[\frac{\left(\frac{1}{2} - \eta\right)^2}{\alpha_2^2} + \frac{2\Omega_1}{3\alpha_2^2}\right]^{\frac{1}{2}}.$$

Observe that (6.28) is a special case of the differential equation satisfied by a Meijer's G-function, see for example Mathai (1993c), so that the theory of G-function can be applied to (6.28). In all the particular cases it is seen that equation (6.27) reduces to the form

$$\{(\Delta - b_1)(\Delta - b_2)(\Delta - b_3)(\Delta - b_4)\}\delta_1 + x\{(\Delta - a_1)(\Delta - a_2)\}\delta_1 = 0 \quad (6.29)$$

where $a_1, a_2, b_1, \ldots, b_4$ and x change from case to case. Comparing (6.29) with a G-function differential equation for $G^{m,n}_{p,q}\left(x\Big|^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}\right)$ we have $q = 4, p = 2$, $(-1)^{p-m-n} = -1, a_1, a_2, b_1, \ldots, b_4$. From the standard solutions of the G-function equation, the solution near $x = 0$ is given by

$$\delta_1 = c_1 G_1 + c_2 G_2 + c_3 G_3 + c_4 G_4,$$

where c_1, c_2, c_3, c_4 are arbitrary constants and

$$G_j = \frac{\left[\prod_{k=1}^{2} \Gamma(-a_k + b_j)\right]}{\left[\prod_{k=1}^{4} \Gamma(1 - b_k + b_j)\right]} x^{b_j} \times {}_2F_3(-a_1 + b_j, -a_2 + b_j; 1 - b_1 + b_j, \ldots, *, \ldots, 1 - b_4 + b_j; -x), \quad (6.30)$$

where the $*$ indicates that parameter of the type $1 - b_j + b_j$ and the corresponding gamma are absent, and it is assumed that $b_i - b_j \neq 0, \pm 1, \pm 2, \ldots$ for all $i \neq j = 1, \ldots, 4$ and ${}_2F_3$ is a hypergeometric function.

Here in (6.29) the G-function parameters are $q = 4, p = 2$ or $q > p$. Hence the 4 fundamental solutions and the general solution for $x \to \infty$ are the following:

$$\delta_1 = c_1 G_1 + c_2 G_2 + c_3 G_3 + c_4 G_4, \quad (6.31)$$

where c_1, c_2, c_3, c_4 are arbitrary constants and

$$G_1 = G_{2,4}^{4,1}\left[x\Big|_{b_1,\ldots,b_4}^{1+a_1,1+a_2}\right],$$

$$G_2 = G_{2,4}^{4,1}\left[x\Big|_{b_1,\ldots,b_4}^{1+a_2,1+a_1}\right],$$

$$G_3 = G_{2,4}^{4,0}\left[x\mathrm{e}^{i\pi}\Big|_{b_1,\ldots,b_4}^{1+a_1,1+a_2}\right],$$

$$G_4 = G_{2,4}^{4,0}\left[x\mathrm{e}^{-i\pi}\Big|_{b_1,\ldots,b_4}^{1+a_1,1+a_2}\right], \quad i = \sqrt{-1}.$$

Computable series forms as well as explicit solutions for various cases of 3-component medium are available from Mathai et al. (1988).

Exercises 6.3

6.3.1. Derive Eq. (6.27) from Eqs. (6.25) and (6.26).

6.3.2. Derive an equation for δ_2 from Eq. (6.27).

6.3.3. Under the special case $k_1 = 0, k_2 = 0$ show that (6.27) reduces to the form $\{(\Delta - a_1)(\Delta - a_2)(\Delta - a_3)(\Delta_4 - a_4)\}\delta_1 = 0$ so that the general solution is $\delta_1 = c_1 + c_2 t^{a_2} + c_3 t^{a_3} + c_4 t^{a_4}$ where

$$a_1 = 0, a_2 = -(2\eta - 1), a_3 = \left(\frac{1}{2} - \eta\right) - \left[\left(\frac{1}{2} - \eta\right)^2 + \frac{2}{3}\right]^{\frac{1}{2}},$$

$$a_4 = \left(\frac{1}{2} - \eta\right) + \left[\left(\frac{1}{2} - \eta\right)^2 + \frac{2}{3}\right]^{\frac{1}{2}}.$$

6.3.4. Show that under case (iv): $\alpha_1 \neq 0, k_2 = 0$ Eq. (6.27) reduces to

$$\{(\tilde{\Delta} - b_1')(\tilde{\Delta} - b_2')(\tilde{\Delta} - b_3')(\tilde{\Delta} - b_4')\}\delta_1 + x\{(\tilde{\Delta} - a_1')(\tilde{\Delta} - a_2')\}\delta_1 = 0,$$

where $\tilde{\Delta} = x\frac{\mathrm{d}}{\mathrm{d}x}, \tilde{x} = \frac{k_1^2 t^{\alpha_1}}{\alpha_1^2}$. Show that

$$a_1' = \left[-1 + \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_1}\right] - \left[\frac{\left(\eta - \frac{1}{2}\right)^2}{\alpha_1^2} + \frac{2\Omega_2}{3\alpha_1^2}\right]^{\frac{1}{2}},$$

$$a_2' = \left[-1 + \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_1}\right] + \left[\frac{\left(\eta - \frac{1}{2}\right)}{\alpha_1^2} + \frac{2\Omega_2}{3\alpha_1^2}\right]^{\frac{1}{2}},$$

and $b_i' = b_i$ of case (iii).

6.3.5. Show that under Case (v): $\alpha_1 = 0, \alpha_2 \neq 0$ Eq. (6.27) reduces to the same form as in Exercise 6.3.4 with

$$a_1 = \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_2^2} - \left[\frac{\left(\frac{1}{2} - \eta\right)^2}{\alpha_2^2} + \frac{2\Omega_1}{3\alpha_2^2} - \frac{k_1^2}{\alpha_2^2}\right]^{\frac{1}{2}},$$

$$a_2 = \frac{\left(\frac{1}{2} - \eta\right)}{\alpha_2^2} + \left[\frac{\left(\frac{1}{2} - \eta\right)^2}{\alpha_2^2} + \frac{2\Omega_1}{3\alpha_2^2} - \frac{k_1^2}{\alpha_2^2}\right]^{\frac{1}{2}},$$

and the b_i's are the solutions of the equation

$$\alpha_2^4 b^4 + 2(2\eta - 1)\alpha_2^3 b^3 + \left[(2\eta - 1)^2 - \frac{2}{3} + k_1^2\right]\alpha_2^2 b^2$$
$$+ \left[-\frac{2}{3}(2\eta - 1) + (2\eta - 1)k_1^2\right]\alpha_2 b - \frac{2}{3}\Omega_2 k_1^2 = 0.$$

6.5 Generalized Entropies in Astrophysics Problems

Entropy is a measure of uncertainty in a probability scheme or in a probability density. If $P = (p_1, \ldots, p_k), p_i \geq 0, i = 1, \ldots, k, p_1 + \cdots + p_k = 1$ be the probabilities in a set $A = \{A_1, \ldots, A_k\}$ of mutually exclusive and totally exhaustive events then a measure of uncertainty in this scheme (A, P), proposed by Shannon in 1948, was

$$S = -G \sum_{i=1}^{k} p_i \ln p_i, \tag{6.32}$$

where G is a constant and ln is logarithm to the base e.

6.5.1 Generalizations of Shannon Entropy

Generalizations to Shannon's entropy were considered by many authors. A few of these are the following:

$$H-C = \frac{\sum_{i=1}^{k} p_i^{\alpha} - 1}{2^{1-\alpha}-1}, \quad \alpha \geq 0,\ \alpha \neq 1 \text{ (Havrda-Chárvat)}, \tag{6.33}$$

$$R = \frac{\ln\left(\sum_{i=1}^{k} p_i^{\alpha}\right)}{1-\alpha}, \quad \alpha \geq 0,\ \alpha \neq 1 \text{ (Rényi)}, \tag{6.34}$$

$$T = \frac{\sum_{i=1}^{k} p_i^{q} - 1}{1-q}, \quad q \geq 0,\ q \neq 1,\ \text{(Tsallis)}, \tag{6.35}$$

$$M = \frac{\sum_{i=1}^{k} p_i^{2-\alpha} - 1}{\alpha - 1}, \quad \alpha \leq 2,\ \alpha \neq 1 \text{ (Mathai)}. \tag{6.36}$$

All the α-generalized analogues, $H-C, R, T, M$ go to Shannon's entropy S when $\alpha \to 1$ and in this sense they are generalizations. Tsallis' entropy T is the basis for the current hot topic of nonextensive statistical mechanics and q-calculus. The corresponding measures in a probability density $f(x)$, $[f(x) \geq 0$ for all x, $\int_x f(x)\mathrm{d}x = 1]$ are the following:

$$S = -G\int_x f(x)\ln f(x)\mathrm{d}x, \tag{6.37}$$

$$H-C = \frac{\int_x [f(x)]^{\alpha}\mathrm{d}x - 1}{2^{1-\alpha}-1}, \quad \alpha \geq 0,\ \alpha \neq 1, \tag{6.38}$$

$$R = \frac{\ln \int_x [f(x)]^{\alpha}\mathrm{d}x}{1-\alpha}, \quad \alpha \geq 0,\ \alpha \neq 1, \tag{6.39}$$

$$T = \frac{\int_x [f(x)]^{q}\mathrm{d}x - 1}{1-q}, \quad q \geq 0,\ q \neq 1, \tag{6.40}$$

$$M = \frac{\int_x [f(x)]^{2-\alpha}\mathrm{d}x - 1}{\alpha - 1}, \quad 0 \leq \alpha \leq 2,\ \alpha \neq 1. \tag{6.41}$$

Tsallis' q-exponential function is derived from T of (6.40) by optimizing T subject to the conditions $\int_x f(x)\mathrm{d}x = 1$ and that the first moment is pre-assigned, that is, $\int_x xf(x)\mathrm{d}x =$ given. If the optimization of T is done in the escort density

$$g(x) = \frac{[f(x)]^q}{\int_x [f(x)]^q \mathrm{d}x}, \tag{6.42}$$

then one obtains Tsallis density or known as Tsallis' statistics

$$f_1(x) = c_1[1-(1-q)x]^{\frac{1}{1-q}}, \tag{6.43}$$

where c_1 is the normalizing constant such that $\int_x f(x)\mathrm{d}x = 1$. If Mathai's entropy (6.41) is optimized under the conditions of preassigning the δ-th moment and $(\gamma + \delta)$-th moment for some δ and γ, by using calculus of variation techniques, then one obtains a particular case of Mathai's pathway model in the scalar case

$$f_2(x) = c_2 x^{\gamma}[1 - a(1-\alpha)x^{\delta}]^{\frac{1}{1-\alpha}}, \delta > 0, a > 0 \tag{6.44}$$

where c_2 is the normalizing constant. Observe that c_2 will be different for the three cases $\alpha < 1, \alpha > 1, \alpha \to 1$. When $\alpha < 1$ then $f_2(x)$ for $1 - a(1-\alpha)x^{\delta} > 0$ remains in the generalized type-1 beta family of densities and when $\alpha > 1$, writing $1 - \alpha = -(\alpha - 1)$, $f_2(x)$ goes into the generalized type-2 beta family of densities. When $\alpha \to 1$, then $f_2(x)$ goes to $f_3(x)$ where

$$f_3(x) = c_3 x^{\gamma} \mathrm{e}^{-ax^{\delta}} \tag{6.45}$$

where c_3 is the normalizing constant. It may be mentioned here that M in (6.36) is also connected to the measure of directed divergence in discrete distributions. Observe that for $g_1(x) = f_1(x)/c_1$

$$\frac{\mathrm{d}}{\mathrm{d}x} g_1(x) = -[g_1(x)]^q \tag{6.46}$$

and hence $f_1(x)$, as a model, can describe situations of power function behavior, meaning that the rate of change of $g_1(x)$ is proportional to a power of $g_1(x)$.

When we study the properties (6.45), H-function comes in naturally as illustrated in Chap. 4, Sect. 4.3. These properties will not be repeated here. Thus, H-functions prop up when dealing with problems in nonextensive statistical mechanics, power laws, pathway analysis, generalized entropies and related areas.

Exercises 6.4

6.4.1. Consider the entropy measure in (6.41). By using calculus of variation techniques optimize M under the condition that the functional $f(x)$ is such that $f(x) \geq 0$ and $\int_x f(x)\mathrm{d}x = 1$ and show that the solution is a uniform density.

6.4.2. Optimize M in (6.41) for all densities $f(x)$ such that the first moment is a given or preassigned quantity. Show that the pathway model for $\gamma = 0$ and $\delta = 1$ is the resulting $f(x)$.

6.4.3. Redo Exercise 6.4.2 under the conditions $E(x^{\delta})$ and $E(x^{\delta+\gamma})$ are preassigned, where E denotes the expected value or δ-th moment and $(\delta+\gamma)$-th moments respectively. Show that the resulting density is the pathway model for the positive real scalar variable case.

6.4.4. Derive the density of $u = xy$ if x and y are independently distributed real scalar positive random variables where x is having the density in (6.44) with parameters as given there and y has the density in (6.45) with parameters $(\gamma_1, a_1, \delta_1)$.

6.4.5. Repeat Exercise 6.4.4 if x and y have the densities of the form in (6.44) with different parameters.

6.6 Input–Output Analysis

Input–output situations are many in nature. In a dam or storage capacity there is inflow and outflow and the difference or the residual part is the storage. In nuclear reactions, energy is produced and part of it is dissipated, destroyed or emitted out and the residual part is what is left out. In a human body a chemical called melatonin is produced every day. The production starts by evening, peaks by 1 am and the level of the chemical is back to normal by the morning. The body consumes or converts what is produced. There is a positive residual part during the night and the residual part is zero by the morning. In a growth–decay mechanism an item grows and part of it decays, and the residual part is the difference. In a stochastic process there is an input variable and after the process there is an output. In an industrial production process the total money value of raw materials plus operational cost is the input variable and the money value of the final product is the output variable.

A simple input–output model can be considered as a structure such as

$$u = x - y, \tag{6.47}$$

where x is the input variable and y is the output variable and u can be taken as the residual. Stochastic situations when x and y are independently distributed random variables, scalar variables or matrix variables, are considered by Mathai (1993c). Connections of a structure such as the one in (6.47) to distributions of bilinear forms and covariance structures are also established in Mathai (1993c). A model such as the one in (6.47) when both the input and output variables are gamma random variables can be used to model solar neutrino production or other such residual processes (Haubold and Mathai 1994).

In a reaction–diffusion process if $N(t)$ is the number density at time t and if the production rate is proportional to the original number, then

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \lambda N(t),\ \lambda > 0, \tag{6.48}$$

where λ is the rate of production. If the consumption or destruction rate is also proportional to the original number then

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = -\mu N(t),\ \mu > 0, \tag{6.49}$$

where μ is the destruction rate. Then the residual part is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = -cN(t),\ c = \mu - \lambda. \tag{6.50}$$

If c is free of t then the solution is the exponential model

$$N(t) = N_0\mathrm{e}^{-ct}, N_0 = N(t) \text{ at } t = t_0, \tag{6.51}$$

where t_0 is the starting time. Instead of the total derivative in (6.48)–(6.50) if we consider fractional derivative or fractional nature of reactions, that is, if we consider an equation of the form

$$N(t) - N_0 = -c^{\nu}{}_0D_t{}^{-\nu}N(t), \tag{6.52}$$

where ${}_0D_t{}^{-\nu}$ is the standard Riemann–Liouville fractional integral operator, then the solution for $N(t)$ is a Mittag-Leffler function

$$N(t) = N_0\sum_{k=0}^{\infty}\frac{(-1)^k(ct)^{\nu k}}{\Gamma(\nu k+1)} = N_0E_{\nu}(-(ct)^{\nu}) \tag{6.53}$$

where $E_{\nu}(\cdot)$ is the Mittag-Leffler function, which is a special case of a H-function. That is,

$$N(t) = N_0\frac{1}{2\pi i}\int_L\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\nu s)}[(ct)^{\nu}]^{-s}\mathrm{d}s = N_0H_{1,2}^{1,1}\left[(ct)^{\nu}\Big|_{(0,1),(0,\nu)}^{(0,1)}\right], \tag{6.54}$$

where L is a suitable contour. In such input–output models one can notice that under fractional rate of input or output can produce particular cases of H-functions as illustrated in (6.52)–(6.54). More of such situations will be examined in detail in the coming sections.

Exercises 6.5

6.5.1. Work out the density of $u = x - y$ if x and y are independently distributed with exponential densities with different parameters.

6.5.2. Repeat Exercise 6.5.1 if x has a gamma density and y has an exponential density.

6.5.3. Repeat Exercise 6.5.1 if both x and y have gamma densities with different parameters, for the cases (1): $x - y > 0$ and (2): general, where $x - y$ can be negative also.

6.5.4. Let x_j have the density

$$f_j(x_j) = c_j x_j^{\gamma_j - 1} \mathrm{e}^{-a_j x_j^{\delta_j}}, x_j > 0, a_j > 0, \delta_j > 0, j = 1, 2,$$

where $c_j, j = 1, 2$ are the normalizing constants. Let $u = \ln x_1 - \ln x_2$. When x_1 and x_2 are statistically independently distributed, evaluate the density of u by using Laplace transform of the density of u. Show that the density of u can be written as a H-function.

6.5.5. Work out the special cases in Exercise 6.5.4 when (1) x_j's are Weibull distributed with different parameters, (2) Weibull distributed with the same parameters, (3) gamma distributed (a) with different parameters, (b) with identical parameters, (4) exponentially distributed with (a) different parameters, (b) with identical parameters. Show that all the densities can be written as special cases of H-functions.

6.7 Application to Kinetic Equations

Fractional kinetic equations are studied to determine certain physical phenomena governing diffusion in porous media, reaction and relaxation processes in complex systems and anomalous diffusion, etc. In this connection, one can refer to the monographs by Hilfer (2000), Kilbas et al. (2006), Podlubny (1999), and the various works cited therein. Fractional kinetic equations are studied by Hille and Tamarkin (1930), Glöckle and Nonnenmacher (1991), Saichev and Zaslavsky (1997), Zaslavsky (1994) and Saxena et al. (2002, 2004, 2004b), among others, for their importance in the solution of certain applied problems. We now proceed to prove the following:

Theorem 6.1. *If $c > 0, \nu > 0$, then the solution of the integral equation*

$$N(t) - N_0 f(t) = -c^{\nu}{}_0D_t^{-\nu} N(t), \tag{6.55}$$

where $f(t)$ is any integrable function on the finite interval $[0.b]$, there holds the formula

$$N(t) = cN_0 \int_0^t H_{1,2}^{1,1}\left[c^{\nu}(t-\tau)^{\nu}\middle|_{(-\frac{1}{\nu},1),(0,\nu)}^{(-\frac{1}{\nu},1)}\right] f(\tau)\mathrm{d}\tau, \tag{6.56}$$

where $H_{1,2}^{1,1}(.)$ is the H-function defined by (1.2).

Proof 6.1. Applying the Laplace transform to (6.55) and using (3.65), it gives,

$$\tilde{N}(s) = L[N(t); s] = N_0 \frac{F(s)}{1 + (c/s)^{\nu}}. \tag{6.57}$$

Since (Mathai and Saxena 1978, p. 152)

$$\frac{s^{\nu}}{s^{\nu}+c^{\nu}} = H_{1,1}^{1,1}\left[(s/c)^{\nu}\left|\begin{matrix}(1,1)\\(1,1)\end{matrix}\right.\right], \tag{6.58}$$

then using (2.22), we obtain

$$L^{-1}\left[H_{1,1}^{1,1}\left[(s/c)^{\nu}\left|\begin{matrix}(1,1)\\(1,1)\end{matrix}\right.\right]\right] = t^{-1}H_{2,1}^{1,1}\left[(ct)^{-\nu}\left|\begin{matrix}(1,1),(0,\nu)\\(1,1)\end{matrix}\right.\right]. \tag{6.59}$$

If we use the property of the H-function (1.58), the above equation becomes

$$L^{-1}\left[H_{1,1}^{1,1}\left[(s/c)^{\nu}\left|\begin{matrix}(1,1)\\(1,1)\end{matrix}\right.\right]\right] = t^{-1}H_{1,2}^{1,1}\left[(ct)^{\nu}\left|\begin{matrix}(0,1)\\(0,1),(1,\nu)\end{matrix}\right.\right] \tag{6.60}$$

$$= cH_{1,2}^{1,1}\left[(ct)^{\nu}\left|\begin{matrix}(-1/\nu,1)\\(-1/\nu,1),(0,\nu)\end{matrix}\right.\right]. \tag{6.61}$$

□

The result (6.61) follows from (6.60), if we use the formula (1.60). Taking the inverse Laplace transform of (6.57) and applying the convolution theorem of the Laplace transform, we arrive at the desired result (6.56).

If we set $f(t) = t^{\mu-1}$, we obtain the result given by Saxena et al. (2002, p. 283, Eq. (15)). Theorem 6.1 was proved by Saxena et al. (2004).

Note 6.1. An alternative method for deriving the solution of fractional kinetic equations is recently given by Saxena and Kalla (2008).

6.8 Fickean Diffusion

We consider Fick's diffusion and establish the following:

Theorem 6.2. *The solution of the diffusion equation*

$$\frac{\partial}{\partial t}N(x,t) = C_1\frac{\partial^2}{\partial x^2}N(x,t), \tag{6.62}$$

with initial condition $N(x,t=0) = \delta(x)$, *where* $\delta(x)$ *is the Dirac delta function, is given by*

$$N(x,t) = \frac{1}{\sqrt{(4\pi C_1 t)}}\exp\left(-\frac{x^2}{4C_1 t}\right). \tag{6.63}$$

Proof 6.2. Applying Laplace transform to (6.62) with respect to the variable t and applying the given condition, it gives

$$s\tilde{N}(x,s) - \delta(x) = C_1 \frac{\partial^2}{\partial x^2}\tilde{N}(x,s). \tag{6.64}$$

Applying Fourier transform to the above equation with respect to x, we obtain

$$s\tilde{N}^*(k,s) - 1 = C_1(-k^2)\tilde{N}^*(k,s). \tag{6.65}$$

Solving for $\tilde{N}^*(k,s)$, it gives

$$N^{\sim *}(k,s) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{k^2 C_1}{s}\right)^r s^{-1}. \tag{6.66}$$

On inverting (6.66), the desired result (6.63) is obtained, where we have used the inverse Fourier transform formula

$$F^{-1}\left\{e^{-ak^2}; x\right\} = \frac{1}{\sqrt{4\pi a}} \exp\left(-\frac{x^2}{4a}\right). \tag{6.67}$$

□

Remark 6.1. Standard diffusion processes are described with the help of Fick's second law. The diffusion equation (6.62) can be derived by combining the continuity equation

$$\frac{\partial}{\partial t}N(x,t) = -S_x(x,t), \tag{6.68}$$

and the constitutive equation

$$S(x,t) = -C_1 N_x(x,t), \tag{6.69}$$

which is also called as Fick's first law. Here, $S(x,t)$ represents the flux, $N(x,t)$ the distribution function of the diffusing quantity, and C_1 a diffusion constant which is assumed to be a constant.

6.8.1 Application to Time-Fractional Diffusion

Theorem 6.3. *Consider the following time-fractional diffusion equation*

$$\frac{\partial^\alpha N(x,t)}{\partial t^\alpha} = D\frac{\partial^2 N(x,t)}{\partial x^2}, 0 < \alpha < 1, x \in R, R = (-\infty, \infty), \tag{6.70}$$

where D is the diffusion constant and $\in R\backslash\{0\}$

$$N(x, t = 0) = \delta(x), \lim_{x \to \pm\infty} N(x, t) = 0, \tag{6.71}$$

$\frac{\partial^\alpha}{\partial t^\alpha}$ *is the Caputo fractional derivative defined by (6.114) and* $\delta(x)$ *is the Dirac delta function. Then its fundamental solution is given by*

$$N(x, t) = \frac{1}{|x|} H_{1,1}^{1,0}\left[\frac{|x|^2}{Dt^\alpha}\bigg|_{(1,2)}^{(1,\alpha)}\right]. \tag{6.72}$$

Remark 6.2. It can be seen that Brownian motion takes place at $\alpha = 1$, which is irreversible. Wave propagation takes place at $\alpha = 2$ which is reversible.

Proof 6.3. In order to find a closed form representation of the solution of the equation (6.70) in terms of the H-function, we use the method of joint Laplace–Fourier transform, defined by

$$\tilde{N}^*(k, s) = \int_0^\infty \int_{-\infty}^\infty e^{-st+ikx} N(x, t) \mathrm{d}x \mathrm{d}t, \tag{6.73}$$

where, according to the convention followed, "$\sim$" will denote the Laplace transform and "$*$", the Fourier transform. Applying the Laplace transform with respect to time variable t, Fourier transform with respect to space variable x, using (3.75) and the given condition (6.71), we find that

$$s^\alpha \tilde{N}^*(k, s) - s^{\alpha-1} = -Dk^2 N^{\sim *}(k, s).$$

Solving for $N^{\sim *}(k, s)$, it gives

$$\tilde{N}^*(k, s) = \frac{s^{\alpha-1}}{s^\alpha + Dk^2}.$$

Inverting the Laplace transform, it yields

$$N^*(k, t) = L^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha + Dk^2}\right] = E_\alpha(-Dk^2 t^\alpha), \tag{6.74}$$

where $E_\alpha(.)$, is the Mittag-Leffler function defined by (1.44). □

In order to invert the Fourier transform, we will make use of the integral

$$\int_0^\infty \cos(kt) E_{\alpha,\beta}(-at^2) \mathrm{d}t = \frac{\pi}{k} H_{1,1}^{1,0}\left[\frac{k^2}{a}\bigg|_{(1,2)}^{(\beta,\alpha)}\right], \tag{6.75}$$

which follows from (2.51); where $\Re(\alpha) > 0, \Re(\beta) > 0, k > 0, a > 0$; and the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} f(k)\mathrm{d}k = \frac{1}{\pi}\int_{0}^{\infty} f(k)\cos(kx)\mathrm{d}k, \tag{6.76}$$

then it yields the required solution.

Note 6.2. When $\alpha = 1$, (6.72) reduces to (6.63) as

$$\begin{aligned} N(x,t) &= \frac{1}{|x|}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-2s)}{\Gamma(1-s)}\left(\frac{|x|^2}{Dt}\right)^s \mathrm{d}s \\ &= \frac{1}{|x|}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{1}{2}-s\right)\Gamma(1-s)2^{-2s}\pi^{-\frac{1}{2}}}{\Gamma(1-s)}\left(\frac{|x|^2}{Dt}\right)^s \mathrm{d}s \\ &= \frac{1}{(4\pi Dt)^{\frac{1}{2}}}\exp\left(-\frac{|x|^2}{4Dt}\right), \end{aligned} \tag{6.77}$$

which is a Gaussian density.

6.9 Application to Space-Fractional Diffusion

Notation 6.1. $\frac{\partial^\alpha}{\partial x^\alpha}N(x,t)$: Liouville fractional derivative of order α

Definition 6.1. The Liouville fractional derivative of order α is defined by

$$\frac{\partial^\alpha}{\partial x^\alpha}N(x,t) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{\partial}{\partial x}\right)^m \int_{-\infty}^{x} \frac{N(t,y)}{(x-y)^{\alpha-m+1}}\mathrm{d}y, x \in R, \alpha > 0, m = [\alpha]+1, \tag{6.78}$$

where $[\alpha]$ is the integral part of α.

Note 6.3. The operator defined by (6.78) is also denoted by

$$_{-\infty}D_x^\alpha N(x,t).$$

Its Fourier transform is given by

$$F\{_{-\infty}D_x^\alpha f(x,t)\} = (ik)^\alpha \Psi(k,t), \alpha > 0, \tag{6.79}$$

where $\Psi(k,t)$ is the Fourier transform of $f(x,t)$ with respect to the variable x of $f(x,t)$. Following the convention initiated by Compte (1996), we suppress the imaginary unit in Fourier space by adopting the slightly modified form of the above result in our investigations

$$F\{_{-\infty}D_x^\alpha f(x,t)\} = -|k|^\alpha \Psi(k,t), \alpha > 0 \tag{6.80}$$

instead of (6.79).

In this section, we will investigate the solution of the equation (6.81). The result is given in the form of the following:

Theorem 6.4. *Consider the following space-fractional diffusion equation*

$$\frac{\partial N(x,t)}{\partial t} = D\frac{\partial^{\alpha} N(x,t)}{\partial x^{\alpha}}, 0 < \alpha < 1, x \in R, \tag{6.81}$$

where D *is the diffusion constant and* $\in R\backslash\{0\}$, $\frac{\partial^{\alpha}}{\partial x^{\alpha}}N(x,t)$ *is the Liouville fractional derivative of order* α; $N(x,t=0)=\delta(x)$, *where* $\delta(x)$ *is the Dirac delta function and* $\lim_{x\to\pm\infty} N(x,t)=0$. *Then its fundamental solution is given by*

$$N(x,t) = \frac{1}{\alpha|x|}H_{2,2}^{1,1}\left[\frac{|x|}{(Dt)^{1/\alpha}}\middle|_{(1,1),(1,\frac{1}{2})}^{(1,1/\alpha),(1,\frac{1}{2})}\right]. \tag{6.82}$$

Proof 6.4. Applying the Laplace transform with respect to the time variable t, Fourier transform with respect to space variable x and using the given condition and the Eq. (6.80), it gives

$$s\tilde{N}^{*}(k,s) - 1 = -D|k|^{\alpha}\tilde{N}^{*}(k,s).$$

Solving for $\tilde{N}^{*}(k,s)$ and inverting the Laplace transform, it is seen that

$$\begin{aligned} N^{*}(k,t) &= L^{-1}\left[\sum_{r=0}^{\infty}(-1)^{r}s^{-r-1}(D|k|^{\alpha})^{r}\right] = \sum_{r=0}^{\infty}\frac{(-1)^{r}t^{r}(D|k|^{\alpha})^{r}}{\Gamma(r+1)} \\ &= \exp(-Dt|k|^{\alpha}) = H_{0,1}^{1,0}\left[Dt|k|^{\alpha}\middle|_{(0,1)}^{-}\right]. \end{aligned} \tag{6.83}$$

If we invert the Fourier transform with $\beta=\gamma=1, \theta=0$, the result (6.82) follows. □

6.10 Application to Fractional Diffusion Equation

In this section we present an alternative shorter method for deriving the solution of a diffusion equation discussed earlier by Kochubei (1990).

Theorem 6.5. *Consider the Cauchy problem*

$${}_{0}D_{t}^{\alpha}N(x,t) = -c^{\nu}\Delta N(x,t), 0 < \alpha < 1; x \in \Re^{n}; 0 < t \leq T, \tag{6.84}$$

with

$$N(x,t=0) = \delta(x), x \in \Re, \lim_{x\to\pm\infty} N(x,t) = 0. \tag{6.85}$$

${}_0D_t^{\alpha}$ *is the regularized* Caputo (1969) *partial fractional derivative with respect to* t*, defined by*

$${}_0D_t^{\alpha}N(x,t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t \frac{N(x,s)\mathrm{d}s}{(t-x)^{\alpha}} - \frac{N(x,0)}{t^{\alpha}}\right],$$

and Δ *is the Laplacian. The fundamental solution of the above Cauchy problem is given by*

$$N(x,t) = |x|^{-n}\pi^{-\frac{n}{2}}H_{1,2}^{2,0}\left[\frac{|x|^2 t^{-\alpha}}{4c^{\nu}}\middle|_{(\frac{n}{2},1),(1,1)}^{(1,\alpha)}\right], \tag{6.86}$$

where $H_{1,2}^{2,0}(.)$ *is the* H*-function (1.2).*

Proof 6.5. Applying the Laplace transform with respect to t, Fourier transform with respect to x to (6.84) and using the result (3.75), it gives

$$s^{\alpha}\tilde{N}^*(k,s) - s^{\alpha-1} = -c^{\nu}|k|^2\tilde{N}^*(k,s),$$

where the symbol "$\sim$" indicates the Laplace transform with respect to the time variable t and the symbol "$*$", the Fourier transform with respect to the space variable x. Solving for $\overset{\sim *}{N}(k,s)$, we have

$$\tilde{N}^*(k,s) = \frac{s^{\alpha-1}}{s^{\alpha}+c^{\nu}|k|^2}. \tag{6.87}$$

By virtue of the following Fourier transform formula

$$\left(F_x\left[|x|^{(2-n)/2}K_{(n-2)/2}(a|x|)\right]\right)(\tau) = \left(\frac{2\pi}{a}\right)^{n/2}\frac{a}{a^2+|\tau|^2},\ \tau\in\Re^n;\quad n\in\mathrm{N},\ a>0, \tag{6.88}$$

where the multidimensional Fourier transform with respect to $x\in\Re^n$ is defined by

$$(F_xN)(\tau,t) = \int_{\Re^n} N(x,t)\mathrm{e}^{ix\tau}\mathrm{d}x,\ \tau\in\Re^n,\ t>0, \tag{6.89}$$

and $K_{\nu}(.)$ is the modified Bessel function of the second kind, it yields

$$\tilde{N}(x,s) = c^{-\nu}s^{\alpha-1}(2\pi)^{-\frac{n}{2}}\left(\frac{|x|c^{\frac{\nu}{2}}}{s^{\frac{\alpha}{2}}}\right)^{1-\frac{n}{2}}K_{\frac{n-2}{2}}\left[\frac{|s^{\frac{\alpha}{2}}|x|}{c^{\frac{\nu}{2}}}\right]. \tag{6.90}$$

In order to invert the Laplace transform, we employ the following result given by the authors (Saxena et al. 2006)

$$L^{-1}\left\{s^{-\rho}K_{\nu}(zs^{\sigma});t\right\} = \frac{1}{2}t^{\rho-1}H_{1,2}^{2,0}\left[\frac{z^2t^{-2\sigma}}{4}\middle|_{(\frac{\nu}{2},1),(-\frac{\nu}{2},1)}^{(\rho,2\sigma)}\right], \tag{6.91}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind, $\Re(z^2) > 0$, $\Re(s) > 0$. Thus we obtain the solution in a closed form

$$N(x,t) = \frac{1}{2}(2\pi)^{-\frac{n}{2}} c^{-\frac{\nu}{2}-\frac{n\nu}{4}} |x|^{1-\frac{n}{2}} t^{-\frac{\alpha}{2}-\frac{\alpha n}{4}} H_{1,2}^{2,0}\left[\frac{t^{-\alpha}|x|^2}{4c^\nu}\middle|_{(\frac{n-2}{4},1),(\frac{2-n}{4},1)}^{(1-\frac{\alpha}{2}-\frac{\alpha n}{4},\alpha)}\right]. \tag{6.92}$$

By virtue of the H function identity (1.60), the power of the expression $\left[\frac{\{t^{-\nu}|x|^2\}}{4c^\nu}\right]$ can be absorbed inside the H-function and consequently we obtain

$$N(x,t) = |\pi^{\frac{1}{2}} x|^{-n} H_{1,2}^{2,0}\left[\frac{t^{-\alpha}|x|^2}{4c^\nu}\middle|_{(\frac{n}{2},1),(1,1)}^{(1,\alpha)}\right]. \tag{6.93}$$

□

Remark 6.3. If we employ the identity (1.58), the solution given by (6.93) can be expressed in the form

$$N(x,t) = \frac{1}{\alpha}|\pi^{\frac{1}{2}} x|^{-n} H_{1,2}^{2,0}\left[\frac{t^{-1}|x|^{2/\alpha}}{(4c^\nu)^{\frac{1}{\alpha}}}\middle|_{(\frac{n}{2},\frac{1}{\alpha}),(1,\frac{1}{\alpha})}^{(1,1)}\right], \tag{6.94}$$

where $\alpha > 0$.

Note 6.4. We note that the above form of the solution is due to Schneider and Wyss (1989). There is one importance of our result (6.91) that it includes the Lévy stable density in terms of the H-function as shown in (6.102). Similarly, using the identity (1.59), we arrive at

$$N(x,t) = \frac{1}{2}|\pi^{\frac{1}{2}} x|^{-n} H_{1,2}^{2,0}\left[\frac{t^{-\frac{\alpha}{2}}|x|}{2c^{\frac{\nu}{2}}}\middle|_{(\frac{n}{2},\frac{1}{2}),(1,\frac{1}{2})}^{(1,\frac{\alpha}{2})}\right], \tag{6.95}$$

where n is not an even integer. This form of the H-function is useful in determining its expansion in powers of x. Due to importance of the solution, we also discuss its series representation and behavior.

6.10.1 Series Representation of the Solution

Using the series expansion for the H-function given in the monograph (Mathai and Saxena, 1978), it follows that

$$H_{1,2}^{2,0}\left[x\middle|_{(\frac{n}{2},\frac{1}{\alpha}),(1,\frac{1}{\alpha})}^{(1,1)}\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma\left(\frac{n}{2}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{s}{\alpha}\right)}{\Gamma(1-s)} x^s \mathrm{d}s$$

$$= \alpha\left\{\sum_{\lambda=0}^{\infty}\frac{\Gamma\left(1-\frac{n}{2}-\lambda\right)(-1)^\lambda x^{\alpha\left(\frac{n}{2}+\lambda\right)}}{\Gamma\left(1-\frac{an}{2}-\alpha\lambda\right)(\lambda!)} + \sum_{\lambda=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}-1-\lambda\right)(-1)^\lambda x^{\alpha(1+\lambda)}}{\Gamma(1-\alpha-\alpha\lambda)(\lambda!)}\right\}, \tag{6.96}$$

where n is not an even integer. Thus for $n = 1$, we find that

$$N(x,t) = \frac{1}{2t^{\frac{\alpha}{2}}} \sum_{\lambda=0}^{\infty} (-1)^{\lambda} \frac{A^{\frac{\lambda}{2}}}{\Gamma(1-\alpha(\lambda+1)/2)(\lambda!)}, \tag{6.97}$$

where $A = \frac{x^2}{t^{\alpha}}$ and the duplication formula for the gamma function is used. For $n = 2$, H-function of (6.95) is singular and in this case, the result is explicitly given by Saichev (Barkai 2001) in the form

$$N(x,t) \sim \frac{1}{\pi\Gamma(1-\alpha)t^{\alpha}} \ln\left[\frac{t^{\alpha/2}}{x}\right]. \tag{6.98}$$

For $n = 3$, the series expansion is given by

$$N(x,t) = \frac{1}{4\pi t^{\frac{3\alpha}{2}} A^{\frac{1}{2}}} \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda} A^{\frac{\lambda}{2}}}{(\lambda)!\Gamma\left[1-\alpha\left(1+\frac{\lambda}{2}\right)\right]}. \tag{6.99}$$

From above it readily follows that for $n = 3$ and $\alpha \neq 1$,

$$N(x,t) \sim \frac{1}{x} \text{ as } x \to \infty. \tag{6.100}$$

It will not be out of place to mention that the one sided Lévy stable density $\varphi_{\rho}(t)$ can be obtained from Laplace inversion formula (6.91) by virtue of the identity

$$K_{\pm\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \mathrm{e}^{-x}, \tag{6.101}$$

and can be conveniently expressed in terms of the Laplace transform as

$$\int_0^{\infty} \mathrm{e}^{-ut}\varphi_{\rho}(t)\mathrm{d}t = \mathrm{e}^{-u^{\rho}}, \Re(u) > 0, \Re(\rho) > 0. \tag{6.102}$$

The result is,

$$\varphi_{\rho}(t) = \frac{1}{\rho} H_{1,1}^{1,0}\left[\frac{1}{t}\left|\begin{matrix}(1,1)\\ \left(\frac{1}{\rho},\frac{1}{\rho}\right)\end{matrix}\right.\right], \rho > 0. \tag{6.103}$$

Note 6.5. This result is obtained earlier by Schneider and Wyss (1989) by following a different procedure. Asymptotic behavior of $\varphi_{\alpha}(t)$ is also given by Schneider (1986).

6.11 Application to Generalized Reaction-Diffusion Model

6.11.1 Motivation

It is a known fact that reaction–diffusion models play a very important role in pattern formation in biology, chemistry and physics, see Wilhelmsson and Lazzaro (2001) and Frank (2005). These systems indicate that diffusion can produce the spontaneous formation of spatio-temporal patterns. For details, one can refer to the work of Nicolis and Prigogine (1977) and Haken (2004). A general model for reaction–diffusion systems is investigated by Henry and Wearne (2000, 2002) and Henry et al. (2005).

The simplest reaction–diffusion models are of the form

$$\frac{\partial N}{\partial t} = D\frac{\partial^2 N}{\partial x^2} + F(N), N = N(x,t), \tag{6.104}$$

where D is the diffusion constant and $F(N)$ is a nonlinear function representing reaction kinetics. It is interesting to observe that for $F(N) = \gamma N(1-N)$, (6.104) reduces to Fisher–Kolmogorov equation and if, however, we set $F(N) = \gamma N(1-N^2)$, it gives rise to the real Ginsburg–Landau equation. Del-Castillo-Negrete et al. (2002) studied the front propagation and segregation in a system of reaction–diffusion equations with cross-diffusion. Recently Del-Castillo-Negrete et al. (2003) discussed the dynamics in reaction–diffusion systems with non-Gaussian diffusion caused by asymmetric Lévy flights and solved the following model:

$$\frac{\partial N}{\partial t} = {}_{\eta}D_x^{\alpha} N + F(N), \quad N = N(x,t), \quad F(0) = 0. \tag{6.105}$$

Remark 6.4. It is interesting to observe that the Eq. (6.104) also represents the classical reproduction-dispersal equation for the growth and dispersal of biological species (Fisher 1937; Kolomogorov et al. 1937).

In this section, we present a solution of a more general model of fractional reaction–diffusion system (6.105) in which $\frac{\partial N}{\partial t}$ has been replaced by the Riemann–Liouville fractional derivative ${}_0D_t^{\beta}, \beta > 0$. The results derived are of general nature than those investigated earlier by many authors notably by Jespersen et al. (1999) for anomalous diffusion and by Del-Castillo-Negrete et al. (2003) for the reaction–diffusion systems with Lévy flights and fractional diffusion equation by Kilbas et al. (2004). The solution has been developed in terms of the H-function in a compact and elegant form with the help of Laplace and Fourier transforms and their inverses. Most of the results obtained are in a form suitable for numerical computation. The results reported in this section are in continuation of our earlier investigations, Haubold (1998), Haubold and Mathai (2000) and Saxena et al. (2002, 2004, 2004a,b, 2006, 2006a).

6.11.2 Mathematical Prerequisites

In order to present the results of this section, the definitions of the well-known Laplace and Fourier transforms of a function $N(x,t)$ and their inverses are described below:

Notation 6.2. $L\{N(x,s)\}$: Laplace transform of a function $N(x,t)$ with respect to t.

Notation 6.3. $F\{N(x,t)\}$: The Fourier transform of a function $N(x,t)$ with respect to x.

Definition 6.2. The Laplace transform of a function $N(x,t)$ with respect to t is defined by

$$\tilde{N}(x,s) = L\{N(x,t)\} = \int_0^{\infty} \mathrm{e}^{-st} N(x,t)\mathrm{d}t, \quad t > 0, \quad x \in R, \tag{6.106}$$

where $\Re(s) > 0$, and its inverse transform with respect to s is given by

$$L^{-1}\{\tilde{N}(x,s)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{e}^{st} \tilde{N}(x,s)\mathrm{d}s, \tag{6.107}$$

γ being a fixed real number.

Definition 6.3. The Fourier transform of a function $N(x,t)$ with respect to x is defined by

$$N^*(k,t) = F\{N(x,t)\} = \int_{-\infty}^{\infty} \mathrm{e}^{ikx} N(x,t)\mathrm{d}x \quad i = \sqrt{-1}. \tag{6.108}$$

The inverse Fourier transform with respect to k is given by the formula

$$N(x,t) = F^{-1}\{N^*(k,t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-ikx} N^*(k,t)\mathrm{d}k. \tag{6.109}$$

The space of functions for which the transforms defined by (6.106) and (6.108) exist is denoted by $LF = L(R_+) \times F(R)$.

Notation 6.4. ${}_0D_t^{-\nu} N(x,t)$: The Riemann–Liouville fractional integral of order ν.

Definition 6.4. The Riemann–Liouville fractional integral of order ν is defined by

$${}_0D_t^{-\nu} N(x,t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} N(x,u)\mathrm{d}u, \tag{6.110}$$

where $\Re(\nu) > 0$.

Notation 6.5. ${}_0D_t^{\alpha} N(x,t)$: The Riemann–Liouville fractional derivative of order $\alpha > 0$.

Definition 6.5. Following Samko et al. (1993, p. 37) we define the fractional derivative of order $\alpha > 0$ in the form

$$ {}_0D_t^{\alpha}N(x,t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{N(x,u)}{(t-u)^{\alpha-n+1}}\mathrm{d}u, \quad t>0, \quad n=[\alpha]+1, \quad (6.111) $$

where $[\alpha]$ means the integral part of the number α. From Erdélyi et al. (1954, Vol. II, p. 182) we have

$$ L\{{}_0D_t^{-\nu}N(x,t)\} = s^{-\nu}\tilde{N}(x,s), \quad (6.112) $$

where $\tilde{N}(x,s)$ is the Laplace transform with respect to t of $N(x,t)$, $\Re(s)>0$ and $\Re(\nu)>0$.

The Laplace transform of the fractional derivative, defined by (6.111) is given by Oldham and Spanier (1974, p. 134, Eq. (8.1.3)):

$$ L\{{}_0D_t^{\alpha}N(x,t)\} = s^{\alpha}\tilde{N}(x,s) - \sum_{r=1}^{n} s^{r-1}{}_0D_t^{\alpha-r}N(x,t)|_{t=0}, \quad n-1<\alpha\le n. \quad (6.113) $$

Notation 6.6. ${}^C{}_0D_t^{\alpha}f(x,t)$: Caputo fractional derivative of order $\alpha > 0$.

Definition 6.6. The following fractional derivative of order $\alpha > 0$ is introduced by Caputo (1969) in the form

$$ {}^C{}_0D_t^{\alpha}f(x,t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(x,\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau, \quad m-1<\alpha\le m. $$

The above formula is useful in deriving the solution of differential and integral equations of fractional order governing certain physical problems of reaction and diffusion. The Laplace transform of the Caputo derivative is given by

$$ L\{{}^C{}_0D_t^{\alpha}f(x,t)\} = s^{\alpha}\tilde{f}(x,s) - \sum_{r=0}^{n-1} s^{\alpha-r-1}f^{(r)}(x,0_+), n-1<\alpha\le n, \quad (6.114) $$

where $\alpha, s \in C, \Re(s)>0, \Re(\alpha)>0$.

Note 6.6. If there is no confusion, then this derivative ${}^C{}_0D_t^{\alpha}$ for simplicity will be denoted by ${}_0D_t^{\alpha}$.

Remark 6.5. Recently, Bagley (2007) has given the equivalence of Riemann–Liouville and Caputo fractional order derivatives in connection with modeling of linear viscoelastic materials.

6.11.3 Fractional Reaction–Diffusion Equation

In this section, we will investigate the solution of the generalized reaction–diffusion equation (6.115). The result is given in the form of the following result:

Theorem 6.6. *Consider the generalized fractional reaction–diffusion model*

$$ {}_0D_t^{\beta} N(x,t) = \eta_{-\infty} D_x^{\alpha} N(x,t) + \phi(x,t), \tag{6.115} $$

where $\eta > 0, t > 0, x \in R, 1 < \beta \leq 2,\ 0 \leq \alpha \leq 1$*, with the initial conditions*

$$ [{}_0D_r^{\beta-1} N(x,0)] = f(x),\ [{}_0D_t^{\beta-2} N(x,0)] = g(x), x \in R,\ \lim_{x\to\pm\infty} N(x,t) = 0, \tag{6.116} $$

where ${}_{-\infty}D_x^{\alpha} N(x,t)$ *is defined in (6.78);* $[{}_0D_t^{\beta-1} N(x,0)]$ *means the Riemann-Liouville fractional derivative of order* $\beta-1$ *with respect to* t *evaluated at* $t=0$*. Similarly* $[{}_0D_t^{\beta-2} N(x,0)]$ *means the Riemann–Liouville fractional derivative of order* $\beta-2$ *with respect to* t *evaluated at* $t=0$*.* η *is a diffusion constant and* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction kinetics. Then for the solution of (6.115), subject to the initial conditions (6.116), there holds the formula*

$$ \begin{aligned} N(x,t) &= \frac{t^{\beta-1}}{2\pi} \int_{-\infty}^{\infty} f^*(k) E_{\beta,\beta}(-\eta)|k|^{\alpha} t^{\beta} \exp(-ikx)\mathrm{d}k \\ &+ \frac{t^{\beta-2}}{2\pi} \int_{-\infty}^{\infty} g^*(k) E_{\beta,\beta-1}(-\eta|k|^{\alpha} t^{\beta}) \exp(-ikx)\mathrm{d}x \\ &+ \frac{1}{2\pi} \int_0^t \zeta^{\beta-1} \int_{-\infty}^{\infty} \tilde{\phi}(k,t-\zeta) E_{\beta,\beta}(-\eta|k|^{\alpha}\zeta^{\beta}) \exp(-ikx)\mathrm{d}k\mathrm{d}\zeta, \end{aligned} \tag{6.117} $$

where $*$ *indicates the Fourier transform with respect to space variable* x*.*

Proof 6.6. If we apply the Laplace transform with respect to the time variable t and use the formula (6.113), the given equation (6.115) becomes

$$ s^{\beta}\tilde{N}(x,s) - f(x) - sg(x) = \eta_{-\infty} D_x^{\alpha} \tilde{N}(x,s) + \tilde{\phi}(x,s). \tag{6.118} $$

□

As is customary, it is convenient to employ the symbol $\tilde{N}(x,s)$ to indicate the Laplace transform of $N(x,t)$ with respect to the variable t.

Now we apply the Fourier transform with respect to space variable x to the above equation, use the initial conditions and the result (6.80), then the above equation transforms into the form

$$ \tilde{N}^*(k,s) = \frac{f^*(k)}{s^{\beta}+\eta|k|^{\alpha}} + \frac{sg^*(k)}{s^{\beta}+\eta|k|^{\alpha}} + \frac{\tilde{\phi}^*(k)}{s^{\beta}+\eta|k|^{\alpha}}. \tag{6.119} $$

On taking the inverse Laplace transform of (6.119) and using the result

$$L^{-1}\left\{\frac{s^{\beta-1}}{a+s^{\alpha}};t\right\} = t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-at^{\alpha}), \tag{6.120}$$

where $\Re(s) > 0, \Re(\alpha-\beta) > -1$, it is seen that

$$\begin{aligned} N^*(k,t) &= f^*(k)t^{\beta-1}E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta}) + g^*(k)t^{\beta-2}E_{\beta,\beta-1}(-\eta|k|^{\alpha}t^{\beta}) \\ &\quad + \int_0^t \tilde{\phi}(k,t-\zeta)\zeta^{\beta-1}E_{\beta,\beta}(-\eta|k|^{\alpha}\zeta^{\beta})\mathrm{d}\zeta. \end{aligned} \tag{6.121}$$

The required solution (6.121) now readily follows by taking the inverse Fourier transform of (6.117). Thus, we have

$$\begin{aligned} N(x,t) &= \frac{t^{\beta-1}}{2\pi}\int_{-\infty}^{\infty} f^*(k)E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta})\exp(-ikx)\mathrm{d}k \\ &\quad + \frac{t^{\beta-2}}{2\pi}\int_{-\infty}^{\infty} g^*(k)E_{\beta,\beta-1}(-\eta|k|^{\alpha}t^{\beta})\exp(-ikx)\mathrm{d}k \\ &\quad + \frac{1}{2\pi}\int_0^t \zeta^{\beta-1}\int_{-\infty}^{\infty}\tilde{\phi}(k,t-\zeta)E_{\beta,\beta}(-\eta|k|^{\alpha}\zeta^{\beta})\exp(-ikx)\mathrm{d}k\,\mathrm{d}\zeta. \end{aligned} \tag{6.122}$$

This completes the proof of the Theorem 6.6.

Note 6.7. It may be noted here that by virtue of the identity (1.136), the solution (6.117) can be expressed in terms of the H-function as can be seen from the solutions given in the special cases of the theorem in the next section. Further, we observe that (6.117) is not an explicit solution, special cases are interesting, general solution is not.

6.11.4 Some Special Cases

When $g(x) = 0$, then applying the convolution theorem of the Fourier transform to the solution (6.117), the theorem yields the following result:

Corollary 6.1. *The solution of fractional reaction-diffusion equation*

$${}_0D_t^{\beta}N(x,t) = \eta\,{}_{-\infty}D_x^{\alpha}N(x,t) + \phi(x,t), \quad t > 0, \eta > 0, \tag{6.123}$$

subject to the conditions

$$[{}_0D_t^{\beta-1}N(x,t)]_{t=0} = f(x), \quad [{}_0D_t^{\beta-2}N(x,t)]_{t=0} = 0, \tag{6.124}$$

for $x \in R, \lim_{x\to\pm\infty} N(x,t) = 0,\ 1 < \beta \leq 2,\ 0 \leq \alpha \leq 1$, *when* η *is a diffusion constant and* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction kinetics is given by*

$$N(x,t) = \int_{-\infty}^{\infty} G_1(x-\tau,t) f(\tau)\mathrm{d}\tau + \int_0^t (t-\zeta)^{\beta-1} \int_0^x G_2(x-\tau,t-\zeta)\phi(\tau,\zeta)\mathrm{d}\tau\mathrm{d}\zeta, \tag{6.125}$$

where,

$$\begin{aligned} G_1(x,t) &= \frac{t^{\beta-1}}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx) E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta})\mathrm{d}k \\ &= \frac{t^{\beta-1}}{\pi\alpha}\int_0^{\infty} \cos(kx) H_{1,2}^{1,1}\left[k\eta^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}\middle|\begin{matrix}(0,\frac{1}{\alpha})\\(0,\frac{1}{\alpha}),(1-\beta,\frac{\beta}{\alpha})\end{matrix}\right]\mathrm{d}k \\ &= \frac{t^{\beta-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}}\middle|\begin{matrix}(1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})\end{matrix}\right], \Re(\alpha) > 0, \end{aligned} \tag{6.126}$$

$$\begin{aligned} G_2(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx) E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta})\mathrm{d}k \\ &= \frac{1}{\pi\alpha}\int_0^{\infty} \cos(kx) H_{1,2}^{1,1}\left[k\eta^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}\middle|\begin{matrix}(0,\frac{1}{\alpha})\\(0,\frac{1}{\alpha}),(1-\beta,\frac{\beta}{\alpha})\end{matrix}\right]\mathrm{d}k \\ &= \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}}\middle|\begin{matrix}(1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})\end{matrix}\right],\ \Re(\alpha) > 0. \end{aligned} \tag{6.127}$$

If we set $f(x) = \delta(x),\ \phi = 0$, where $\delta(x)$ is the Dirac-delta function, then we arrive at the following result:

Corollary 6.2. *Consider the following reaction–diffusion model*

$$\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}} N(x,t) = \eta_{-\infty} D_x^{\alpha} N(x,t),\ \ \eta > 0,\ \ x \in R, \tag{6.128}$$

with the initial condition

$$[{}_0D_t^{\beta-1} N(x,t)]_{t=0} = \delta(x),\ \lim_{x\to\pm\infty} N(x,t) = 0,\ 0 < \beta \leq 1,$$

where η *is a diffusion constant and* $\delta(x)$ *is the Dirac-delta function. Then the fundamental solution of (6.128) under the given initial conditions is given by*

$$N(x,t) = \frac{t^{\beta-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{(\eta t^{\beta})^{1/\alpha}}\middle|\begin{matrix}(1,1/\alpha),(\beta,\beta/\alpha),(1,1/2)\\(1,1),(1,1/\alpha),(1,1/2)\end{matrix}\right], \tag{6.129}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$.

When $\beta = \frac{1}{2}$ the above corollary reduces to the following interesting result: Consider the following reaction–diffusion model

$$\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} N(x,t) = \eta_{-\infty} D_x^{\alpha} N(x,t), \; \eta > 0, x \in R, \tag{6.130}$$

with the initial condition

$$[{}_0D_t^{-\frac{1}{2}} N(x,t)]_{t=0} = \delta(x), \quad \lim_{x \to \pm\infty} N(x,t) = 0,$$

where η is a diffusion constant and $\delta(x)$ is the Dirac-delta function. Then the fundamental solution of (6.130) under the given initial conditions is given by

$$N(x,t) = \frac{1}{\alpha |x| t^{1/2}} H_{3,3}^{2,1} \left[\frac{|x|}{(\eta t^{1/2})^{1/\alpha}} \middle|_{(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})}^{(1,\frac{1}{\alpha}),(\frac{1}{2},\frac{1}{2\alpha}),(1,\frac{1}{2})} \right], \tag{6.131}$$

where $\Re(\alpha) > 0$.

Remark 6.6. The solution of the Eq. (6.128), as given by Kilbas et al. (2004) is in terms of the inverse Laplace and inverse Fourier transforms of certain functions whereas the solution of the same equation is obtained here in an explicit closed form in terms of the H-function.

An interesting case occurs when $\beta \to 1$. Then in view of the cancelation law for the H-function (1.57), the equation (6.128) provides the following result given by Jespersen et al. (1999) and recently by Del-Castillo-Negrete et al. (2003) in an entirely different form.

For the solution of fractional reaction–diffusion equation

$$\frac{d}{dt} N(x,t) = \eta_{-\infty} D_x^{\alpha} N(x,t), \tag{6.132}$$

with initial condition

$$N(x, t = 0) = \delta(x), \quad \lim_{x \to \pm\infty} N(x,t) = 0, \tag{6.133}$$

there holds the relation

$$N(x,t) = \frac{1}{\alpha |x|} H_{2,2}^{1,1} \left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{1}{\alpha}}} \middle|_{(1,1),(1,\frac{1}{2})}^{(1,\frac{1}{\alpha}),(1,\frac{1}{2})} \right], \tag{6.134}$$

where $\Re(\alpha) > 0$. In passing, it may be noted that the equation (6.134) is a closed form representation of a Lévy stable law, see Metzler and Klafter (2000, 2004). It is interesting to note that as $\alpha \to 2$, the classical Gaussian solution is recovered as

$$
\begin{aligned}
N(x,t) &= \frac{1}{2|x|} H_{2,2}^{1,1}\left[\frac{|x|}{(\eta t)^{\frac{1}{2}}}\middle|_{(1,1),(1,\frac{1}{2})}^{(1,\frac{1}{2})}\right] \\
&= \frac{1}{2|x|} H_{1,1}^{1,0}\left[\frac{|x|}{(\eta t)^{\frac{1}{2}}}\middle|_{(1,1)}^{(1,\frac{1}{2})}\right] \\
&= (4\pi\eta t)^{-\frac{1}{2}} \exp\left(-\frac{|x|^2}{4\eta t}\right). \qquad (6.135)
\end{aligned}
$$

It is useful to study the solution (6.131) due to its occurrence in certain fractional diffusion models. Now we will find the fractional order moments of (6.131) in the next section.

Remark 6.7. Applying Fourier transform with respect to x in (6.128), it is found that

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\Psi(k,t) = -\eta|k|^\alpha \Psi(k,t),\ 0 < \beta \leq 1, \qquad (6.136)$$

which is the generalized Fourier transformed diffusion equation, since for $\alpha = 2$ and for $\beta \to 1$, it reduces to Fourier transformed diffusion equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(k,t) = -\eta|k|^2 \Psi(k,t), \qquad (6.137)$$

being a diffusion equation, for a fixed wave number k (Metzler and Klafter 2000, 2004). Here $\Psi(k,t)$ is the Fourier transform of $N(x,t)$ with respect to x.

Remark 6.8. It is interesting to observe that the method employed for deriving the solution of the Eqs. (6.115) and (6.116) in the space $LF = L(R_+) \times F(R)$ can also be applied in the space $LF' = L'(R_+) \times F'$, where $F' = F'(R)$ is the space of Fourier transforms of generalized functions. As an illustration, we can choose $F' = S'$ or $F' = D'$. The Fourier transforms in S' and D' are introduced by Gelfand and Shilov (1964). S' is the dual of the space S, which is the space of all infinitely differentiable functions which together with their derivatives approach zero more rapidly than any power of $1/|x|$ as $|x| \to \infty$. D' is the dual of the space D, which consists of all smooth functions with compact supports. In this connection, see the monographs by Gelfand and Shilov (1964) and Brychkov and Prudnikov (1989).

6.11.5 Fractional Order Moments

In this section, we will calculate the fractional order moments, defined by

$$< |x|^\delta >= \int_{-\infty}^{\infty} |x|^\delta N(x,t)\mathrm{d}x. \qquad (6.138)$$

Using the definition of the Mellin transform

$$M\{f(t);s)\} = \int_0^\infty t^{s-1} f(t)\mathrm{d}t, \tag{6.139}$$

we find from (6.138) that

$$< |x(t)|^\delta > = \int_{-\infty}^{\infty} |x|^\delta N(x,t)\mathrm{d}x. \tag{6.140}$$

$$< |x|^\delta(t) > = \frac{2t^{\beta-1}}{\alpha} \int_0^\infty x^{\delta-1} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}}\left|\begin{matrix}(1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\frac{1}{2})\\(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})\end{matrix}\right.\right]\mathrm{d}x. \tag{6.141}$$

Applying the Mellin transform formula for the H-function (2.8) we see that

$$< |x|^\delta(t) > = \frac{2}{\alpha}\eta^{\frac{\delta}{\alpha}} t^{\beta\left(\frac{\delta}{\alpha}+1-\frac{1}{\beta}\right)} \frac{\Gamma\left(-\frac{\delta}{\alpha}\right)\Gamma(1+\delta)\Gamma\left(1+\frac{\delta}{\alpha}\right)}{\Gamma\left(-\frac{\delta}{2}\right)\Gamma\left(\beta+\frac{\beta\delta}{\alpha}\right)\Gamma\left(1+\frac{\delta}{2}\right)}, \tag{6.142}$$

whenever the gammas exist, $\Re(\delta) > -1$ and $\Re(\delta+\alpha) > 0$.

Two interesting special cases of (6.142) are worth mentioning.

(i) As $\delta \to 0$, then by using the result $\frac{1}{\Gamma(z)} \sim z$ for $z << 1$, we find that

$$\lim_{\delta\to 0} < |x|^\delta(t)| > = \beta t^{\beta-1}. \tag{6.143}$$

(ii) When $\alpha = 2, \delta = 2$, the linear time dependence

$$\lim_{\delta\to 2,\alpha\to 2} < |x(t)|^\delta > = \frac{2\eta t^{2\beta-1}}{\Gamma(2\beta)}, \tag{6.144}$$

of the mean squared displacement is recovered.

6.11.6 Some Further Applications

This section deals with the investigation of the solution of an unified fractional reaction–diffusion equation associated with the Caputo derivative as the time-derivative and Riesz–Feller fractional derivative as the space-derivative. The solution is derived by the application of the Laplace and Fourier transforms in a compact and closed form in terms of the H-function.

6.11.7 Background

The theory and applications of reaction–diffusion systems are contained in many books and articles. In recent works (Saxena et al. 2006a–c), the authors have demonstrated the depth of mathematics and related physical issues of reaction–diffusion equations such as nonlinear phenomena, stationary and spatio-temporal dissipative pattern formation, oscillation, waves, etc. (Frank 2005; Grafiychuk et al. 2006, 2007). In recent time, interest in fractional reaction–diffusion equations has increased because the equation exhibits self-organization phenomena and introduces a new parameter, the fractional index, into the equation. Additionally, the analysis of fractional reaction–diffusion equations is of great interest from the analytic and numerical point of view.

The object of this section is to derive the solution of an unified model of reaction–diffusion system, associated with the Caputo derivative and the Riesz–Feller derivative. This new model provides the extension of the models discussed earlier by Mainardi et al. (2001), Mainardi et al. (2005), and Saxena et al. (2006). The advantage of using Riesz–Feller derivative lies in the fact that the solution of the fractional reaction–diffusion equation containing this derivative includes the fundamental solution for space-time fractional diffusion, which itself is a generalization of neutral fractional diffusion, space-fractional diffusion, and time-fractional diffusion. These specialized type of diffusions can be interpreted as spatial probability density functions evolving in time and are expressible in terms of the H-functions in compact form.

Notation 6.7. ${}_xD_0^{\alpha}$: Riesz–Feller space-fractional derivative of order α.

Definition 6.7. Following Feller (1952, 1966) it is conventional to define the Riesz–Feller space-fractional derivative of order α and skewness θ in terms of its Fourier transform as

$$F\{{}_xD_\theta^{\alpha};k\} = -\psi_\alpha^\theta(k) f^*(k), \tag{6.145}$$

where,

$$\psi_\alpha^\theta(k) = |k|^\alpha \exp[i(sign\ k)\frac{\theta\pi}{2}],\ \ 0<\alpha\leq 2,\ \ |\theta|\leq \min\{\alpha, 2-\alpha\}. \tag{6.146}$$

When $\theta = 0$, then (6.145) reduces to

$$F\{{}_xD_0^{\alpha} f(x);k\} = -|k|^\alpha f^*(k), \tag{6.147}$$

which is the Fourier transform of the Liouville fractional derivative, defined by

$${}_{-\infty}D_x^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-\infty}^{t} \frac{f(u)}{(t-u)^{\alpha-n+1}} \mathrm{d}u. \tag{6.148}$$

This shows that Riesz–Feller space-fractional derivative may be regarded as a generalization of Liouville fractional derivative.

Note 6.8. Further, when $\theta = 0$, we have a symmetric operator with respect to x which can be interpreted as

$$ {}_xD_0^{\alpha} = -\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right]^{\frac{\alpha}{2}}. \tag{6.149}$$

This can be formally deduced by writing $-(k)^{\alpha} = -(k^2)^{\frac{\alpha}{2}}$. For $0 < \alpha < 2$ and $|\theta| \leq \min\{\alpha, 2-\alpha\}$, the Riesz–Feller derivative can be shown to possess the following integral representation in x domain:

$$ {}_xD_{\theta}^{\alpha} f(x) = \frac{\Gamma(1+\alpha)}{\pi}\Bigg\{ \sin[(\alpha+\theta)\frac{\pi}{2}] \int_0^{\infty} \frac{f(x+\zeta)-f(x)}{\zeta^{1+\alpha}}\mathrm{d}\zeta + \sin\left[(\alpha-\theta)\frac{\pi}{2}\right] \int_0^{\infty} \frac{f(x-\zeta)-f(x)}{\zeta^{1+\alpha}}\mathrm{d}\zeta\Bigg\}. \tag{6.150}$$

6.11.8 Unified Fractional Reaction–Diffusion Equation

In this section, we will investigate the solution of the reaction–diffusion equation (6.151) under the initial conditions (6.153). The result is given in the form of the following result:

Theorem 6.7. *Consider the following unified fractional reaction–diffusion model*

$$ {}_0D_t^{\beta} N(x,t) = \eta\, {}_xD_{\theta}^{\alpha} N(x,t) + \phi(x,t), \tag{6.151}$$

where $\eta, t > 0, x \in R; \alpha, \theta, \beta$ *are real parameters with the constraints*

$$ 0 < \alpha \leq 2, |\theta| \leq \min(\alpha, 2-\alpha),\ 0 < \beta \leq 2, \tag{6.152}$$

and the initial conditions

$$ N(x,0) = f(x), N_i(x,0) = g(x)\ for\ x \in R,\ \lim_{x\to\pm\infty} N(x,t) = 0, t > 0. \tag{6.153}$$

Here $N_i(x,0)$ *means the first partial derivative of* $N(x,t)$ *with respect to* t *evaluated at* $t = 0$, η *is a diffusion constant and* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction–diffusion. Further,* ${}_xD_{\theta}^{\alpha}$ *is Riesz–Feller space-fractional derivative of order* α *and asymmetry* θ. ${}_0D_t^{\beta}$ *is the Caputo time-fractional derivative of order* β. *Then for the solution of (6.151), subject to the above constraints, there holds the formula*

$$N(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k)E_{\beta,1}(-\eta t^{\beta}\Psi_{\alpha}^{\theta}(k))\exp(-ikx)\mathrm{d}k$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} tg^*(k)E_{\beta,2}(-\eta k^{\alpha}t^{\beta}\Psi_{\alpha}^{\theta}(k))\exp(-ikx)\mathrm{d}k$$
$$+\frac{1}{2\pi}\int_0^t \zeta^{\beta-1}\int_{-\infty}^{\infty}\phi^*(k,t-\zeta)E_{\beta,\beta}(-\eta k^{\alpha}t^{\beta}\Psi_{\alpha}^{\theta}(k))\exp(-ikx)\mathrm{d}k\ \mathrm{d}\zeta. \tag{6.154}$$

Proof 6.7. If we apply the Laplace transform with respect to the time variable t, Fourier transform with respect to the space variable x, and use the initial conditions (6.153) and the formulae (6.114) and (6.147), then the given equation transforms into the form

$$s^{\beta}\tilde{N}^*(k,s)-s^{\beta-1}f^*(k)-s^{\beta-2}g^*(k) = -\eta\Psi_{\alpha}^{\theta}(k)\tilde{N}^*(k,s)+\tilde{\phi}^*(k,s), \tag{6.155}$$

where according to the conventions followed, the symbol $\tilde{N}(x,s)$ will stand for the Laplace transform with respect to time variable t and $*$ represents the Fourier transform with respect to space variable x. Solving for $\tilde{N}^*(k,s)$, it yields

$$\tilde{N}^*(k,s) = \frac{f^*(k)s^{\beta-1}}{s^{\beta}+\eta\Psi_{\alpha}^{\theta}(k)}+\frac{g^*(k)s^{\beta-2}}{s^{\beta}+\eta\Psi_{\alpha}^{\theta}(k)}+\frac{\tilde{\phi}^*(k)}{s^{\beta}+\eta\Psi_{\alpha}^{\theta}(k)}. \tag{6.156}$$

On taking the inverse Laplace transform of (6.156) and applying the formula (6.120), it is seen that

$$N^*(k,t) = f^*(k)E_{\beta,1}(-\eta t^{\beta}\Psi_{\alpha}^{\theta}(k))+g^*(k)tE_{\beta,2}(-\eta t^{\beta}\Psi_{\alpha}^{\theta}(k))$$
$$+\int_0^t \phi^*(k,t-\zeta)\zeta^{\beta-1}E_{\beta,\beta}(-\eta\Psi_{\alpha}^{\theta}(k)\zeta^{\beta})\mathrm{d}\zeta. \tag{6.157}$$

□

The required solution (6.154) is now obtained by taking the inverse Fourier transform of (6.157). This completes the proof of the Theorem 6.7.

6.11.9 Some Special Cases

When $g(x) = 0$ then by the application of the convolution theorem of the Fourier transform to the solution (6.154) of the Theorem 6.7, it readily yields the following result:

Corollary 6.3. *The solution of fractional reaction–diffusion equation*

$$\frac{\partial^{\beta}}{\partial t^{\beta}}N(x,t)-\eta\frac{\partial^{\alpha}}{\partial x^{\alpha}}N(x,t) = \phi(x,t), x\in R, t>0, \eta>0, \tag{6.158}$$

with initial conditions

$$N(x,0) = f(x), N(x,0) = 0 \textit{ for } x \in R, 1 < \beta \leq 2, \lim_{x\to\pm\infty} N(x,t) = 0, t > 0, \tag{6.159}$$

where η is a diffusion constant and $\phi(x,t)$ is a nonlinear function belonging to the area of reaction–diffusion, is given by

$$N(x,t) = \int_{-\infty}^{\infty} G_1(x-\tau,t) f(\tau)\mathrm{d}\tau + \int_0^t (t-\zeta)^{\beta-1} \int_0^x G_2(x-\tau,t-\zeta)\phi(\tau,\zeta)\mathrm{d}\tau\,\mathrm{d}\zeta, \tag{6.160}$$

where,

$$\rho = \frac{\alpha-\theta}{2\alpha}$$
$$G_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx) E_{\beta,1}\left(-\eta t^{\beta}\Psi_{\alpha}^{\theta}(k)\right)\mathrm{d}k = \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}}\middle|\begin{matrix}(1,\frac{1}{\alpha}),\left(1,\frac{\beta}{\alpha}\right),(1,\rho)\\(1,\frac{1}{\alpha}),(1,1),(1,\rho)\end{matrix}\right], \alpha > 0, \tag{6.161}$$

and

$$G_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx) E_{\beta,\beta}(-\eta t^{\beta}\Psi_{\alpha}^{\theta}(k))\mathrm{d}k = \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}}\middle|\begin{matrix}(1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\rho)\\(1,\frac{1}{\alpha}),(1,1),(1,\rho)\end{matrix}\right], \alpha > 0. \tag{6.162}$$

In deriving the above results, we have used the inverse Fourier transform formula

$$F^{-1}[E_{\beta,\gamma}(-\eta t^{\beta}\Psi_{\theta}^{\alpha}(k)); x] = \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}}\middle|\begin{matrix}(1,\frac{1}{\alpha}),(\gamma,\frac{\beta}{\alpha}),(1,\rho)\\(1,\frac{1}{\alpha}),(1,1),(1,\rho)\end{matrix}\right] \tag{6.163}$$

where $\rho = \frac{\alpha-\theta}{2\alpha}, \Re(\beta) > 0, \Re(\gamma) > 0$, which can be established by following a procedure similar to that employed by Mainardi et al. (2001).

Next, if we set $f(x) = \delta(x), \phi = 0, g(x) = 0$, where $\delta(x)$ is the Dirac delta function, then we arrive at the following interesting result given by Mainardi et al. (2005).

Corollary 6.4. *Consider the following space-time fractional diffusion model*

$$\frac{\partial^{\beta}}{\partial t^{\beta}} N(x,t) = \eta\ {}_xD_{\theta}^{\alpha} N(x,t),\ \eta > 0, x \in R, 0 < \beta \leq 2, \tag{6.164}$$

with the initial conditions $N(x, t = 0) = \delta(x), N_t(x, 0) = 0, \lim_{x\to\pm\infty} N(x, t) = 0$ *where* η *is a diffusion constant and* $\delta(x)$ *is the Dirac delta function. Then for the fundamental solution of (6.164) with initial conditions, there holds the formula*

$$N(x,t) = \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{(\eta t^{\beta})^{\frac{1}{\alpha}}}\middle|_{(1,\frac{1}{\alpha}),(1,1),(1,\rho)}^{(1,\frac{1}{\alpha}),(1,\frac{\beta}{\alpha}),(1,\rho)}\right], \rho = \frac{\alpha-\theta}{2\alpha}. \tag{6.165}$$

Some interesting special cases of (6.164) are enumerated below.

(i) We note that for $\alpha = \beta$, Mainardi et al. (2005) have shown that the corresponding solution of (6.165), denoted by N_{α}^{θ}, which we call as the neutral fractional diffusion, can be expressed in terms of elementary function and can be defined for $x > 0$ as **Neutral fractional diffusion**: $0 < \alpha = \beta < 2; \theta \leq \min\{\alpha, 2-\alpha\}$,

$$N_{\alpha}^{\theta}(x) = \frac{1}{\pi}\frac{x^{\alpha-1}\sin[\left(\frac{\pi}{2}\right)(\alpha-\theta)]}{1+2x^{\alpha}\cos[\left(\frac{\pi}{2}\right)(\alpha-\theta)]+x^{2\alpha}}. \tag{6.166}$$

The neutral fractional diffusion is not studied at length in the literature.
Next we derive some stable densities in terms of the H-functions as special cases of the solution of the equation (6.164).

(ii) If we set $\beta = 1,\ 0 < \alpha < 2; \theta \leq \min\{\alpha, 2-\alpha\}$, then (6.164) reduces to space-fractional diffusion equation, which we denote by $L_{\alpha}^{\theta}(x)$, and we obtain the fundamental solution of the following **space-time fractional diffusion model**:

$$\frac{\partial}{\partial t}N(x,t) = \eta\, {}_xD_{\theta}^{\alpha}N(x,t),\ \eta > 0, x \in R, \tag{6.167}$$

with the initial conditions $N(x, t = 0) = \delta(x), \lim_{x\to\pm\infty} N(x, t) = 0$, where η is a diffusion constant and $\delta(x)$ is the Dirac delta function. Hence for the fundamental solution of (6.167) there holds the formula

$$L_{\alpha}^{\theta}(x) = \frac{1}{\alpha(\eta t)^{\frac{1}{\alpha}}} H_{2,2}^{1,1}\left[\frac{(\eta t)^{\frac{1}{\alpha}}}{|x|}\middle|_{(\frac{1}{\alpha},\frac{1}{\alpha}),(\rho,\rho)}^{(1,1),(\rho,\rho)}\right], 0 < \alpha < 1, |\theta| \leq \alpha, \tag{6.168}$$

where $\rho = \frac{\alpha-\theta}{2\alpha}$. The density represented by the above expression is known as α-stable Lévy density. Another form of this density is given by

$$L_{\alpha}^{\theta}(x) = \frac{1}{\alpha(\eta t)^{\frac{1}{\alpha}}} H_{2,2}^{1,1}\left[\frac{|x|}{(\eta t)^{\frac{1}{\alpha}}}\middle|_{(0,1),(1-\rho,\rho)}^{(1-\frac{1}{\alpha},\frac{1}{\alpha}),(1-\rho,\rho)}\right], \tag{6.169}$$

where $1 < \alpha < 2, |\theta| \leq 2-\alpha$.

(iii) Next, if we take $\alpha = 2,\ 0 < \beta < 2, \theta = 0$ then we obtain the time-fractional diffusion, which is governed by the following time fractional diffusion model:

$$\frac{\partial^\beta}{\partial t^\beta} N(x,t) = \eta \frac{\partial^2}{\partial x^2} N(x,t),\ \eta > 0, x \in R, 0 < \beta \le 2, \tag{6.170}$$

with the initial conditions $N(x, t = 0) = \delta(x), N_t(x, 0) = 0, \lim_{x\to\pm\infty} N(x, t) = 0$ where η is a diffusion constant and $\delta(x)$ is the Dirac delta function, whose fundamental solution is given by the equation

$$N(x,t) = \frac{1}{2|x|} H_{1,1}^{1,0}\left[\frac{|x|}{(\eta t^\beta)^{\frac{1}{2}}}\middle|_{(1,1)}^{(1,\frac{\beta}{2})}\right] \tag{6.171}$$

which is same as (6.72).

(iv) Further, if we set $\alpha = 2, \beta = 1$, and $\theta \to 0$ then for the fundamental solution of the standard diffusion equation

$$\frac{\partial}{\partial t} N(x,t) = \eta \frac{\partial^2}{\partial x^2} N(x,t), \tag{6.172}$$

with initial condition

$$N(x, t = 0) = \delta(x), \lim_{x\to\pm\infty} N(x,t) = 0, \tag{6.173}$$

there holds the formula

$$N(x,t) = \frac{1}{2|x|} H_{1,1}^{1,0}\left[\frac{|x|}{\eta^{\frac{1}{2}} t^{\frac{1}{2}}}\middle|_{(1,1)}^{(1,\frac{1}{2})}\right] = (4\pi\eta t)^{-\frac{1}{2}} \exp\left[-\frac{|x|^2}{4\eta t}\right], \tag{6.174}$$

which is the classical Gaussian density. For further details and importance of these special cases based on the Green function, one can refer to the papers by Mainardi et al. (2001, 2005).

Remark 6.9. Fractional order moments and the asymptotic expansion of the solution (6.165) are discussed by Mainardi et al. (2001).

Finally, for $\beta = \frac{1}{2}$ and $g(x) = 0$ in (6.151) we arrive at the following result:

Corollary 6.5. *Consider the following fractional reaction–diffusion model*

$$D^{\frac{1}{2}} N(x,t) = \eta_x D_\theta^\alpha N(x,t) + \phi(x,t), \tag{6.175}$$

where $\eta, t > 0, x \in R; \alpha, \theta$ *are real parameters with the constraints* $0 < \alpha \le 2$, $|\theta| \le \min(\alpha, 2-\alpha)$, *and the initial conditions*

$$N(x,0) = f(x), N_t(x,0) = 0 \textit{ for } x \in R, \lim_{x\to\pm\infty} N(x,t) = 0. \tag{6.176}$$

Here η *is a diffusion constant and* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction–diffusion. Further,* ${}_xD_\theta^\alpha$ *is the Riesz–Feller space fractional* *derivative of order* α *and asymmetry* θ *and* $D_t^{\frac{1}{2}}$ *is the Caputo time-fractional derivative of order* $\frac{1}{2}$*. Then for the solution of (6.175), subject to the above constraints, there holds the formula*

$$N(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k)E_{\frac{1}{2}}(-\eta t^{\frac{1}{2}}\Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}x + \frac{1}{2\pi}\int_0^t \zeta^{-\frac{1}{2}}\int_{-\infty}^{\infty}\phi^*(k,t-\zeta)E_{\frac{1}{2},\frac{1}{2}}(-\eta k^\alpha t^{\frac{1}{2}}\Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k\,\mathrm{d}\zeta. \tag{6.177}$$

If we set $\theta = 0$ in Theorem 6.7, then it reduces to the result recently obtained by Saxena et al. (2006) for the fractional reaction–diffusion equation.

Following a similar procedure, we can derive the solution of the fractional reaction–diffusion system (6.178) given below under the given initial conditions (6.179) associated with Riemann–Liouville fractional derivative and the Riesz–Feller fractional derivative. The result is given in the form of the following result:

Theorem 6.8. *Consider the unified fractional reaction–diffusion model associated with Riemann–Liouville fractional derivative* ${}_0D_t^\alpha$ *defined by (6.111) and the Riesz–Feller space fractional derivative* ${}_xD_\theta^\alpha$ *of order* α *and asymmetry* θ *defined by (6.145) in the form*

$${}_0D_t^\beta N(x,t) = \eta\,{}_xD_\theta^\alpha N(x,t) + \phi(x,t), \tag{6.178}$$

where $\eta, t > 0, x \in R, \alpha, \theta, \beta$ *are real parameters with the constraints* $0 < \alpha \le 2$, $|\theta| \le \min(\alpha, 2-\alpha),\ 1 < \beta \le 2$*, and the initial conditions*

$$[{}_0D_t^{\beta-1}N(x,0)] = f(x),\ \ [{}_0D_t^{\beta-2}N(x,0)] = g(x) \text{ for } x \in R, \quad \lim_{|x|\to\infty} N(x,t) = 0, t > 0. \tag{6.179}$$

Here $[{}_0D_t^{\beta-1}N(x,0)]$ *means the Riemann–Liouville fractional partial derivative of* $N(x,t)$ *with respect to* t *of order* $\beta-1$ *evaluated at* $t=0$*. Similarly,* $[{}_0D_t^{\beta-2}N(x,0)]$ *is the Riemann–Liouville fractional partial derivative of* $N(x,t)$ *with respect to* t *of order* $\beta - 2$ *evaluated at* $t = 0, \eta$ *is a diffusion constant and* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction–diffusion. Then for the solution of (6.178), subject to the above constraints, there holds the formula*

$$N(x,t) = \frac{t^{\beta-1}}{2\pi}\int_{-\infty}^{\infty} f^*(k)E_{\beta,\beta}(-\eta t^\beta\Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k + \frac{t^{\beta-2}}{2\pi}\int_{-\infty}^{\infty} tg^*(k)E_{\beta,\beta-1}(-\eta t^\beta\Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k + \frac{1}{2\pi}\int_0^t \zeta^{\beta-1}\int_{-\infty}^{\infty}\phi^*(k,t-\zeta)E_{\beta,\beta}(-\eta\zeta^\beta\Psi_\alpha^\theta(k))\exp(-ikx)\mathrm{d}k\,\mathrm{d}\zeta. \tag{6.180}$$

6.11.10 More Special Cases

When $g(x) = 0$ then by the application of the convolution theorem of the Fourier transform to the solution (6.180) of the theorem, it readily yields the following result:

Corollary 6.6. *The solution of fractional reaction–diffusion equation*

$$ {}_0D_t^{\beta} N(x,t) - \eta_x D_\theta^{\alpha} N(x,t) = \phi(x,t),\ x \in R, t > 0, \eta > 0, \tag{6.181}$$

with initial conditions

$$\begin{aligned} [{}_0D_t^{\beta-1} N(x,t)] = f(x),\ [{}_0D_t^{\beta-2} N(x,0)] = 0 \textit{ for } x \in R, \\ 0 \le \alpha \le 1, 1 < \beta \le 2,\ \lim_{x\to\pm\infty} N(x,t) = 0, \end{aligned} \tag{6.182}$$

where η is a diffusion constant and $\phi(x,t)$ is a nonlinear function belonging to the area of reaction–diffusion; $\eta, t > 0, x \in R; \alpha, \theta, \beta$ are real parameters with the constraints $0 < \alpha \le 2$, $|\theta| \le \min(\alpha, 2-\alpha)$, $1 < \beta \le 2$, is given by

$$\begin{aligned} N(x,t) = & \int_{-\infty}^{\infty} G_1(x-\tau,t) f(\tau) \mathrm{d}\tau \\ & + \int_0^t (t-\zeta)^{\beta-1} \int_0^x G_2(x-\tau, t-\zeta) \phi(\tau,\zeta) \mathrm{d}\tau\ \mathrm{d}\zeta, \end{aligned} \tag{6.183}$$

where $\rho = \frac{\alpha-\theta}{2\alpha}$;

$$\begin{aligned} G_1(x,t) &= \frac{t^{\beta-1}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) E_{\beta,\beta}(-\eta t^{\beta} \Psi_\alpha^\theta(k)) \mathrm{d}k \\ &= \frac{t^{\beta-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}} \middle| \begin{matrix} (1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\rho) \\ (1,\frac{1}{\alpha}),(1,1),(1,\rho) \end{matrix} \right], \alpha > 0, \end{aligned} \tag{6.184}$$

and

$$\begin{aligned} G_2(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) E_{\beta,\beta}(-\eta t^{\beta} \Psi_\alpha^\theta(k)) \mathrm{d}k \\ &= \frac{1}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}} \middle| \begin{matrix} (1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\rho) \\ (1,\frac{1}{\alpha}),(1,1),(1,\rho) \end{matrix} \right], \alpha > 0. \end{aligned} \tag{6.185}$$

In deriving the above results, we have used the inverse Fourier transform formula (6.163) given by Haubold et al. (2007).

Remark 6.10. It is interesting to observe that for $\theta = 0$, Theorem 6.8 reduces to (6.117) given by the authors Saxena et al. (2006b). On the other hand, if we set $f(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta function, it yields the following result:

Corollary 6.7. *Consider the following reaction–diffusion model*

$$ {}_0D_t^{\beta} N(x,t) = \eta_x D_\theta^\alpha N(x,t), \tag{6.186}$$

with the initial conditions

$$ [{}_0D_t^{\beta-1} N(x,0) = \delta(x), 0 \le \beta \le 1, \quad \lim_{x\to\pm\infty} N(x,t) = 0, \tag{6.187}$$

where η is a diffusion constant; $\eta, t > 0, x \in R; \alpha, \theta, \beta$ are real parameters with the constraints $0 < \alpha \le 2$, $|\theta| \le \min(\alpha, 2-\alpha)$, and $\delta(x)$ is the Dirac delta function. Then for the fundamental solution of (6.186) with initial conditions in (6.187), there holds the formula

$$ N(x,t) = \frac{t^{\beta-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{(\eta t^\beta)^{\frac{1}{\alpha}}} \middle| \begin{matrix} (1,\frac{1}{\alpha}),(\beta,\frac{\beta}{\alpha}),(1,\rho) \\ (1,\frac{1}{\alpha}),(1,1),(1,\rho) \end{matrix} \right], \alpha > 0, \tag{6.188}$$

where $\rho = \frac{\alpha-\theta}{2\alpha}$.

Exercises 6.10

6.10.1. Consider the fractional reaction–diffusion equation connected with nonlinear waves

$$ \begin{aligned} &{}_0D_t^{\alpha} N(x,t) + \alpha_0 D_t^{\beta} N(x,t) \\ &\quad = \nu^2 {}_{-\infty}D_x^{\gamma} N(x,t) + \zeta^2 N(x,t) + \phi(x,t), \end{aligned} $$

for $x \in R, t > 0, 0 \le \alpha \le 1, 0 \le \beta \le 1$ with initial conditions

$$ N(x,0) = f(x), \quad \lim_{x\to\pm\infty} N(x,t) = 0, \quad x \in R $$

where the operator ${}_{-\infty}D_x^{\gamma}$ is defined in (6.78); ${}_0D_t^{\alpha}$ and ${}_0D_t^{\beta}$ are the Caputo fractional order derivatives, ν^2 is a diffusion constant, ζ is a constant which describes the nonlinearity in the system, and $\phi(x,t)$ is nonlinear function which belongs to the area of reaction–diffusion, then show that there holds the following formula for the solution of the above mentioned reaction–diffusion model.

$$
\begin{aligned}
N(x,t) &= \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_{-\infty}^{\infty} t^{(\alpha-\beta)r} f^*(k) \exp(-ikx) \\
&\times \left[E_{\alpha,(\alpha-\beta)r+1}\left(-bt^{\alpha}\right) + t^{\alpha-\beta} E_{\alpha,(\alpha-\beta)(r+1)+1}\left(-bt^{\alpha}\right) \right] \mathrm{d}k \\
&+ \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_0^t \zeta^{\alpha+(\alpha-\beta)r-1} \\
&\times \int_{-\infty}^{\infty} \phi(k, t-\zeta) \exp(-ikx) E_{\alpha,\alpha+(\alpha-\beta)r}(-b\zeta^{\alpha}) \mathrm{d}k \mathrm{d}\zeta,
\end{aligned}
$$

where $\alpha > \beta$ and $E^{\delta}_{\beta,\gamma}(\cdot)$ is the generalized Mittag-Leffler function, defined by (1.39) and $b = \nu^2|k|^{\gamma} - \zeta^2$.

6.10.2. Consider the following fractional reaction–diffusion model

$$
\frac{\partial^{\beta}}{\partial t^{\beta}} N(x,t) = \eta\ {}_{-\infty}D^{\alpha}_{x} N(x,t) + \phi(x,t); \eta, t > 0, x \in R, 0 < \beta \leq 2,
$$

with the initial conditions

$$
N(x,0) = f(x), \quad N_t(x,0) = g(x), \quad x \in R, \quad \lim_{x \to \pm\infty} N(x,t) = 0,
$$

where the operator ${}_{-\infty}D^{\alpha}_{x}$ is defined in (6.78); $N_t(x,0)$ means the first derivative of $N(x,t)$ with respect to t evaluated at $t = 0$, η is a diffusion constant, $\phi(x,t)$ is a nonlinear function belonging to the area of reaction diffusion and $\frac{\partial^{\beta}}{\partial t^{\beta}}$ is the Caputo fractional derivative. Then show that for the solution of reaction–diffusion model, subject to the initial conditions, there holds the formula

$$
\begin{aligned}
N(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(k) E_{\beta,1}(-\eta|k|^{\alpha} t^{\beta}) \exp(-ikx) \mathrm{d}k \\
&+ \frac{1}{2\pi} \int_{-\infty}^{\infty} t g^*(k) E_{\beta,2}(-\eta|k|^{\alpha} t^{\beta}) \exp(-ikx) \mathrm{d}k \\
&+ \frac{1}{2\pi} \int_0^t \zeta^{\beta-1} \int_{-\infty}^{\infty} \tilde{\phi}(k, t-\zeta) E_{\beta,\beta}(-\eta|k|^{\alpha} \zeta^{\beta}) \exp(-ikx) \mathrm{d}k \mathrm{d}\zeta.
\end{aligned}
$$

Hence or otherwise derive the solution of the next exercise.

6.10.3. Consider the following reaction–diffusion model

$$
\frac{\partial^{\beta}}{\partial t^{\beta}} N(x,t) = \eta\ {}_{-\infty}D^{\alpha}_{x} N(x,t), \alpha > 0, -\infty < x < \infty, 0 < \beta \leq 1,
$$

with the initial condition $N(x,t=0)=\delta(x)$, $\lim_{x\to\pm\infty}N(x,t)=0$, $\frac{\partial^\beta}{\partial t^\beta}$ is the Caputo fractional derivative, the operator ${}_{-\infty}D_x^\gamma$ is defined in (6.78), η is a diffusion constant and $\delta(x)$ is the Dirac delta function. Then show that for the solution of the above equation there holds the formula

$$N(x,t)=\frac{1}{\alpha|x|}H_{3,3}^{2,1}\left[\frac{|x|}{(\eta t^\beta)^{\frac{1}{\alpha}}}\left|\begin{matrix}(1,\frac{1}{\alpha}),(1,\frac{\beta}{\alpha}),(1\frac{1}{2})\\(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})\end{matrix}\right.\right].$$

6.10.4. Show that the solution of the following boundary value problem for the one-dimensional fractional diffusion equation associated with the Riemann–Liouville fractional derivative ${}_0D_t^\alpha$

$${}_0D_t^\alpha N(x,t)=\lambda^2\frac{\partial^2}{\partial x^2}N(x,t),t>0,-\infty<x<\infty,$$

with the initial conditions

$$\lim_{x\to\pm\infty}N(x,t)=0,\ \ [{}_0D_t^{\alpha-1}N(x,t)]_{t=0}=\phi(x),0<\alpha<1,$$

is given by

$$N(x,t)=\int_{-\infty}^{\infty}G(x-\zeta,t)\phi(\zeta)\mathrm{d}\zeta,$$

where

$$G(x,t)=\frac{t^{\alpha-1}}{|x|}H_{1,1}^{1,0}\left[\frac{|x|^2}{\lambda^2t^\alpha}\left|\begin{matrix}(\alpha,\alpha)\\(1,2)\end{matrix}\right.\right].$$

(Nigmatullin 1986)

Remark 6.11. Nigmatullin (1986) derived the solution of the above fractional diffusion equation in terms of the following integral:

$$G(x,t)=\frac{1}{\pi}\int_0^\infty t^{\alpha-1}E_{\alpha,\alpha}(-\lambda^2k^2t^\alpha)\cos(kx)\mathrm{d}k,$$

whereas, the solution of this problem given here is in terms of the H-function in an explicit form.

6.10.5. Consider the fractional diffusion equation

$${}_0D_t^\nu N(x,t)-\frac{t^{-\nu}}{\Gamma(1-\nu)}\delta(x)=c^\nu\frac{\partial^2}{\partial x^2}N(x,t),$$

with the initial condition

$$D_t^{\nu-k}N(x,t)|_{t=0}=0,k=1,\dots,n,$$

where $n = [\Re(\nu)] + 1$, c^{ν} is a diffusion constant and $\delta(x)$ is a Dirac delta function. Then show that for the solution of the diffusion equation, there exists the formula

$$N(x,t) = \frac{1}{(4\pi c^{\nu} t^{\nu})^{\frac{1}{2}}} H_{1,2}^{2,0}\left[\frac{|x|^2}{4c^{\nu}t^{\nu}}\left|{}^{(1-\frac{\nu}{2},\nu)}_{(0,1),(\frac{1}{2},1)}\right.\right].$$

(Metzler and Klafter 2000; Jorgenson and Lang 2001).

6.10.6. Consider the generalized free electron laser equation

$$ {}_0D_{\tau}^{\alpha} f(\tau) = \lambda \int_0^{\tau} t^{\sigma} f(\tau - t)\phi(b, \sigma+1; i\nu t)\mathrm{d}t + k\tau^{\gamma}\phi(\beta, \gamma+1; i\nu t),\ 0 \le \tau \le 1 \tag{6.189}$$

with $\lambda, k \in C; \nu, b, \beta \in R, \alpha > 0, \gamma > -1, \sigma > -1$ with initial condition

$$ {}_0D_{\tau}^{\alpha-\tau} f(\tau)|_{\tau=0} = b_r, r = 1, \ldots, N, \tag{6.190}$$

where $N = [\alpha] + 1$ is a positive integer, $N - 1 \le \alpha < N$ and b_r's are real numbers. Then show that there exists an unique solution of the Cauchy-type problem (6.189)–(6.190), given by

$$f(\tau) = f_0(\tau) + \int_0^{\tau} f(\zeta)\left[\sum_{m=1}^{\infty} P_1(m,\tau,\zeta)\right]\mathrm{d}\zeta + k\Gamma(\gamma+1)\sum_{m=0}^{\infty} P_2(m,\tau), \tag{6.191}$$

where,

$$f_0(\tau) = \sum_{j=1}^{N} \frac{b_j}{\Gamma(\alpha - j + 1)}\tau^{N-j}, \tag{6.192}$$

$$P_1(m,\tau,\zeta) = [\lambda\Gamma(\sigma+1)]^m(\tau-\zeta)^{m(\alpha+\sigma+1)-1}\phi^*[bm, m(\alpha+\sigma+1); i\nu(\tau-\zeta)], \tag{6.193}$$

$$\begin{aligned} P_2(m,\tau) = {} & [\lambda\Gamma(\sigma+1)]^m \tau^{\alpha(m+1)+m(\sigma+1)+\gamma}\phi^*(bm+\beta, \alpha(m+1) \\ & + m(\sigma+1)+\gamma+1; i\nu\tau), \end{aligned} \tag{6.194}$$

and

$$\phi^*(a,c;z) = \frac{1}{\Gamma(c)}\phi(a,c;z). \tag{6.195}$$

(Saxena and Kalla 2003).

6.10.7. Let $\alpha, \rho, \sigma, \gamma, \omega, \lambda \in C, \min\{\Re(\alpha), \Re(\rho), \Re(\sigma)\} > 0$. If $f(x) \in L(a,b)$, then show that the Cauchy-type problem

$$(D_{a+}^{\alpha} f)(x) = \lambda \int_a^x (x-t)^{\sigma-1} E_{\rho,\sigma}^{\gamma}[\omega(x-t)^{\rho}] f(t)\mathrm{d}t + h(x), a \le x \le b, \tag{6.196}$$

and

$$\lim_{x \to +a} (D_{a+}^{\alpha-r} f)(x) = b_r, r = 1, \ldots, n = -[-\Re(\alpha)], \tag{6.197}$$

is solvable in the space $L(a, b)$ and its unique solution is given by

$$f(x) = \sum_{r=1}^{n} b_r f_r(x) + \int_a^x \Omega(x-t) h(t)\mathrm{d}t, \tag{6.198}$$

where

$$\begin{aligned} f_r(x) = (x-a)^{\alpha-r} & \sum_{j=0}^{\infty} \lambda^j (x-a)^{\sigma+\alpha} \\ & \times E_{\rho+(\sigma+\alpha)j+\alpha-r+1}^{\gamma j}[\omega(x-a)^{\rho}], r = 1, \ldots, n, \end{aligned} \tag{6.199}$$

and

$$\Omega(u) = \sum_{j=0}^{\infty} \lambda^j u^{(\sigma+\alpha)j+\alpha-1} E_{\rho,(\sigma+\alpha)j+\alpha}^{\gamma j}[\omega u^{\rho}], \tag{6.200}$$

where $E_{\rho,\sigma}^{\gamma}(z)$ is the generalized Mittag-Leffler function defined in (1.46) (Kilbas et al. 2002).

Appendix

A.1 H-Function of Several Complex Variables

Notation A.1. $H(z_1, \ldots, z_n)$: Multivariable H-function or H-function of several complex variables.

Definition A.1. The multivariable H-function is defined in terms of multiple Mellin–Barnes type contour integral as

$$\begin{aligned} H[z_1, \ldots, z_r] &= H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} (a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p}:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ \\ (b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q}:(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix} \right. \right] \\ &= \frac{1}{(2\pi w)^r} \int_{L_1} \cdots \int_{L_r} \Psi(\zeta_1, \ldots, \zeta_r) \left\{ \prod_{i=1}^{r} \phi_i(\zeta_i) z_i^{\zeta_i} \right\} \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_r, \end{aligned} \quad \text{(A.1)}$$

where

$$\Psi(\zeta_1, \ldots, \zeta_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \zeta_i)}{\left[\prod_{j=n+1}^{p} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} \zeta_i)\right] \left[\prod_{j=1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} \zeta_i)\right]}, \quad \text{(A.2)}$$

$$\phi_i(\zeta_i) = \frac{\left[\prod_{\lambda=1}^{m_i} \Gamma(d_\lambda^{(i)} - \delta_\lambda^{(i)} \zeta_i)\right] \left[\prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \zeta_i)\right]}{\left[\prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \zeta_i)\right] \left[\prod_{\lambda=m_i+1}^{q_i} \Gamma(1 - d_\lambda^{(i)} + \delta_\lambda^{(i)} \zeta_i)\right]}, \quad \text{(A.3)}$$

for $i = 1, \ldots, r$, and $L_i = L_{w\tau_i\infty}$, $w = (-1)^{\frac{1}{2}}$ represents the contours which start at the point $\tau_i - w\infty$ and goes to the point $\tau_i + w\infty$ with $\tau_i \in R = (-\infty, \infty), i = 1, \ldots, r$ such that all the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)} \zeta_i), j = 1, \ldots, m_i; i = 1, \ldots, r$ are separated from those of $\Gamma(1 - c_j^{(i)} - \gamma_j^{(i)} \zeta_i), j = 1, \ldots, n_i; i = 1, \ldots, r$ and $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \zeta_i), j = 1, \ldots, n$. Here, the integers n, p, q, m_i, n_i, p_i and

q_i, satisfy the inequalities $0 \le n \le p; q \ge 0, 1 \le m_i \le q_i$ and $1 \le n_i \le p_i, i = 1, \ldots, r$. Further, we suppose that the parameters

$$\begin{aligned} &a_j, j = 1, \ldots, p; c_j^{(i)}, j = 1, \ldots, p_i; i = 1, \ldots, r, \\ &b_j, j = 1, \ldots, q; d_j^{(i)}, j = 1, \ldots, q_i; i = 1, \ldots, r, \end{aligned} \tag{A.4}$$

are complex numbers and the associated coefficients

$$\begin{aligned} &\alpha_j^{(i)}, j = 1, \ldots, p; i = 1, \ldots, r; \gamma_j^{(i)}, j = 1, \ldots, p_i, i = 1, \ldots, r, \\ &\beta_j^{(i)}, j = 1, \ldots, q; i = 1, \ldots, r; \delta_j^{(i)}, j = 1, \ldots, q_i; i = 1, \ldots, r, \end{aligned} \tag{A.5}$$

are positive real numbers, such that

$$\begin{aligned} \Lambda_i &= \sum_{j=1}^{p} \alpha_j^{(i)} + \sum_{j=1}^{p_i} \gamma_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} - \sum_{j=1}^{q_i} \delta_j^{(i)} \le 0, \\ i &= 1, \ldots, r \end{aligned} \tag{A.6}$$

$$\begin{aligned} \Omega_i = &\sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} \\ &+ \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0, i = 1, \ldots, r. \end{aligned} \tag{A.7}$$

It is assumed that the poles of the integrand of (A.1) are simple. We know that the integral in (A.1) converges absolutely, under the conditions (A.7), (Srivastava et al. (1982), p. 251) with

$$|\arg(z_i)| < \frac{\pi}{2}\Omega_i, i = 1, \ldots, r, \tag{A.8}$$

and the points $z_i = 0, i = 1, \ldots, r$ and various exceptional paramater values being tacitly excluded. From Srivastava and Panda (1976b, p. 131) we have

$$H(z_1, \ldots, z_r) = O(|z_1|^{e_1}, \ldots, |z_r|^{e_r}), \max_{1 \le j \le r} [|z_j|] \to 0, \tag{A.9}$$

where

$$e_i = \min_{1 \le j \le m_i} \left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}} \right], i = 1, \ldots, r. \tag{A.10}$$

For $n = 0$, there holds the following asymptotic expansion (Srivastava and Panda, 1976b, p. 131):

$$H[z_1, \ldots, z_r] = O(|z_1|^{g_1}, \ldots, |z_r|^{g_r}), \min_{1 \le j \le r} [|z_j|] \to \infty, \tag{A.11}$$

where

$$g_i = \max_{1 \le j \le n_i} \left[\frac{\Re(c_j^{(i)}) - 1}{\gamma_j^{(i)}} \right], i = 1, \ldots, r, \tag{A.12}$$

provided that each of the inequalities in (A.6), (A.7), and (A.8) hold true.

Remark A.1. When $n = 2$ the multivariable H-function defined by (A.1) reduces to the H-function of two variables studied by Mittal and Gupta (1972). H-function of two variables are also defined and studied by Munot and Kalla (1971) and Verma (1971), and others. A comprehensive and detailed account of the H-function of two variables is available from the monograph by Hai and Yakubovich (1992).

It is interesting to observe that for $n = p = q = 0$, the multivariable H-function breaks up into product of r H-functions and consequently there holds the following result (Saxena 1977):

$$H^{0,0:m_1,n_1;\cdots;m_r,n_r}_{0,0:p_1,q_1;\cdots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} -:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ -:(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix}\right] = \prod_{i=1}^{r} H^{m_i,n_i}_{p_i,q_i}\left[z_i \middle| \begin{matrix} (c_j^{(i)},\gamma_j^{(i)})_{1,p_i} \\ (d_j^{(i)},\delta_j^{(i)})_{1,q_i} \end{matrix} \right]. \tag{A.13}$$

Remark A.2. The function defined by (A.1) was introduced and studied by Srivastava and Panda (1976a, p. 271).

When $\alpha_j^{(1)} = \cdots = \alpha_j^{(r)}, j = 1, \ldots, p; \beta_j^{(1)} = \cdots = \beta_j^{(r)}, j = 1, \ldots, q$ in (A.1) the multivariable H-function defined and studied by Saxena (1974, 1977) is obtained. In case all the Greek letters are assumed to be unity, the H-function of several complex variables (A.1) reduces to the G-function of several complex variables studied by Khadia and Goyal (1970, 1975).

Remark A.3. Fractional integrals involving multivariable H-functions are given in a series of papers by Saigo and Saxena (1999, 1999a, 2001). Srivastava and Hussain (1995) Saigo et al. (2005), and others.

A.2 Kampé de Fériet Function and Lauricella Functions

A.2.1 Kampé de Fériet Series in the Generalized Form

Definition A.2. Kampé de Fériet series in the generalized form is defined by

$$F^{p;q;r}_{k;m;n}\left[\begin{matrix}(a_p): & (b_q), & (c_r): \\ (d_k) & : (e_m), & (g_n); \end{matrix} x, y\right] = \sum_{\tau,\nu=0}^{\infty} \frac{\left[\prod_{j=1}^{p}(a_j)_{\tau+\nu}\right]\left[\prod_{j=1}^{q}(b_j)_{\tau}\right]\left[\prod_{j=1}^{r}(c_j)_{\nu}\right]}{\left[\prod_{j=1}^{k}(d_j)_{\tau+\nu}\right]\left[\prod_{j=1}^{m}(e_j)_{\tau}\right]\left[\prod_{j=1}^{n}(g_j)_{\nu}\right]} \frac{x^\tau}{\tau!}\frac{y^\nu}{\nu!}, \tag{A.14}$$

where, for convergence

$$(i)\;\; p+q<k+m+1;\, p+r<k+n+1, |x|<\infty, |y|<\infty, \tag{A.15}$$

or

$$(ii)\;\; p+q=k+m+1;\, p+r=k+n+1,$$

and

$$\begin{cases} |x|^{1/(p-k)}+|y|^{1/(p-k)}<1, \text{ if } p>k, \\ \max\{|x|,|y|\}<1, \text{ if } p\le k. \end{cases} \tag{A.16}$$

The above series reduces to the original Kampé de Fériet series (Kampé de Fériet 1921), when $q=r$ and $m=n$, and is also called Kampé de Fériet series.

Remark A.4. A generalization of the series (A.14) is given by Srivastava and Daoust (1969), which is indeed the extension of Wright's generalized hypergeometric series ${}_p\Psi_q(z)$. This generalization is further extended by Srivastava and Daoust (1969a), which is described in the next subsection.

Three interesting special cases of the reducibility of (A.14) to generalized hypergeometric series ${}_pF_q(z)$, are given below. For further cases of reducibility of the series defined by (A.14) in terms of the generalized hypergeometric series, see the monograph by Srivastava and Karlsson (1985, pp. 28–32) and references of special cases given therein.

$$F^{p:0,0}_{q:0,0}\begin{bmatrix} a_1,\cdots,a_p; & ; & ; \\ b_1,\cdots,b_q; & ; & ; \end{bmatrix} x,y = {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x+y), \tag{A.17}$$

$$F^{0:p,m}_{0:q,n}\begin{bmatrix} ;a_1,\cdots,a_p; & c_1,\cdots,c_m \\ ;b_1,\cdots,b_q; & d_1,\cdots,d_n; \end{bmatrix} x,y = {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) \times {}_mF_n(c_1,\ldots,c_m;d_1,\ldots,d_n;y), \tag{A.18}$$

$$F^{p:1,1}_{q:0,0}\begin{bmatrix} a_1,\cdots,a_p; & c; & d; \\ b_1,\cdots,b_q; & ; & ; \end{bmatrix} x,x = {}_{p+1}F_q(a_1,\ldots,a_p;c+d;b_1,\ldots,b_q;x). \tag{A.19}$$

A.2.2 Generalized Lauricella Function

Notation A.2. $F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$: Generalized Lauricella function of n complex variables.

Definition A.3. The generalized Lauricella series (Srivastava and Daoust 1969a) is defined in the following manner:

$$F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$$

$$= F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\left[\begin{matrix}[(a):\theta^{(1)},\ldots,\theta^{(n)}]:[(b^{(1)}):\phi^{(1)})];\cdots;[(b^{(n)});\phi^{(n)})\\ [(c):\psi^{(1)},\ldots,\psi^{(n)}]:[(d^{(1)}:(\delta^{(1)})];\cdots;[(d^{(n)});\delta^{(n)})\end{matrix} x_1,\ldots,x_n\right]$$

$$= \sum_{m_1=0,\ldots,m_n=0}^{\infty}\chi(m_1,\ldots,m_n)\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \tag{A.20}$$

where, for convenience,

$$\chi(m_1,\ldots,m_n) = \frac{[\prod_{j=1}^{A}(a_j)_{m_1\theta_1^{(1)}+\cdots+m_n\theta_j^{(n)}}][\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_1^{(1)}}]\cdots[\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}]}{[\prod_{j=1}^{C}(c_j)_{m_1\psi_1^{(1)}+\cdots+m_n\psi_j^{(n)}}][\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}]\cdots[\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}]}, \tag{A.21}$$

the coefficients

$$\begin{cases}\theta_j^{(k)}, j=1,\ldots,A; \phi_j^{(k)}, j=1,\ldots,B^{(k)}; \Psi_k^{(j)}, j=1,\ldots,C,\\ \delta_j^{(k)}, j=1,\ldots,D^{(k)}, k=1,\ldots,n,\end{cases} \tag{A.22}$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\ldots,a_A$; $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}, j=1,\ldots,B^{(k)}, k=1,\ldots,n. \tag{A.23}$$

Similar interpretations hold for the remaining parameters. For precise conditions under which this multiple series (A.20) converges, see Srivastava and Daoust (1972, pp. 153–157), also see Exton (1976, Sect. 3.7) and Exton (1978, Sect. 1.4).

When each of the positive numbers given in (A.22) takes the value unity, the generalized Lauricella series (A.20) gives rise to a direct multivariable extension of Kampé de Fériet series (A.14). Thus the multivariable generalization of the Kampé de Fériet series defined by (A.14) is given by (see, Srivastava and Panda, 1975, p. 1127; Srivastava and Karlsson 1985, p. 38):

$$F^{p:p_1,\ldots,p_n}_{k:q_1,\ldots,q_n}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = F^{p:p_1,\ldots,p_n}_{k:q_1,\ldots,q_n}\left[\begin{matrix}(a_p):(b^{(1)}_{p_1});\cdots;(b^{(n)}_{p_n}):\\(\alpha_k):(\beta^{(1)}_{q_1});\cdots;(\beta^{(n)}_{q_n});\end{matrix} x_1,\ldots,x_n\right], \tag{A.24}$$

$$= \sum_{m_1=0,\ldots,m_n=0}^{\infty} \theta(m_1,\ldots,m_n)\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \tag{A.25}$$

where

$$\theta(m_1,\ldots,m_n) = \frac{[\prod_{j=1}^{p}(a_j)_{m_1+\cdots+m_n}]\left[\prod_{j=1}^{p_1}(b_j^{(1)})_{m_1}\right]\cdots\left[\prod_{j=1}^{p_n}(b_j^{(n)})_{m_n}\right]}{\left[\prod_{j=1}^{k}(\alpha_j)_{m_1+\cdots+m_n}\right]\left[\prod_{j=1}^{q_1}(\beta_j^{(1)})_{m_1}\right]\cdots\left[\prod_{j=1}^{q_n}(\beta_j^{(n)})_{m_n}\right]}, \tag{A.26}$$

and, for convergence of the series (A.25),

$$1 + k + q_r - p - p_r \geq 0, r = 1,\ldots,n. \tag{A.27}$$

The equality holds when, in addition, either

$$p > k \text{ and } |x_1|^{1/(p-k)} + \cdots + |x_n|^{1/(p-k)} < 1, \tag{A.28}$$

or

$$p \leq k \text{ and } \max\{|x_1|,\ldots,|x_n|\} < 1. \tag{A.29}$$

Remark A.5. Karlsson (1973) has considered a special case of (A.24) when

$$p_r = q, q_r = m_r, r = 1,\ldots,n. \tag{A.30}$$

A relation connecting generalized Lauricella function and the multivariable H-function is given by Srivastava and Panda (1976a, p. 272)

$$H^{0,p:1,p_1;\cdots;1,p_r}_{p,q:p_1,q_1+1;\cdots;p_r,q_r+1}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p}:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q}:(0,1),(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\cdots;(0,1),(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right.\right]$$

$$= \frac{[\prod_{j=1}^{p}\Gamma(1-a_j)][\prod_{j=1}^{p_1}\Gamma(1-c_j^{(1)})]\cdots[\prod_{j=1}^{p_r}\Gamma(1-c_j^{(r)})]}{[\prod_{j=1}^{q}\Gamma(1-b_j)][\prod_{j=1}^{q_1}\Gamma(1-d_j^{(1)})]\cdots[\prod_{j=1}^{q_r}\Gamma(1-d_j^{(r)})]}$$

$$\times F^{p:p_1;\cdots;p_r}_{q:q_1;\cdots;q_r}\left[\begin{matrix}[(1-a_j:\alpha_j^{(1)},\ldots,\alpha_j^{(r)})]_{1,p}:[(1-c_j^{(1)},\gamma_j^{(1)})]_{1,p_1};\cdots\\ [(1-b_j:\beta_j^{(1)},\ldots,\beta_j^{(r)})]_{1,q}:[(1-d_j^{(1)},\delta_j^{(1)})]_{1,q_1};\cdots\end{matrix}\right.$$

$$\left.\begin{matrix};[(1-c_j^{(r)};\gamma_j^{(r)})]_{1,p_r}\\ ;[(1-d_j^{(r)};\delta_j^{(r)})]_{1,q_r}\end{matrix} -z_1,\ldots,-z_n\right]. \tag{A.31}$$

A.3 Appell Series

Notation A.3. $F_1(a, b, b'; c; x, y)$: Appell function of the first kind.

Notation A.4. $F_2(a, b, b'; c, c'; x, y)$: Appell function of the second kind.

Notation A.5. $F_3(a, b, b'; c; x, y)$: Appell function of the third kind.

Notation A.6. $F_4(a, b'; c, c'; x, y)$: Appell function of the fourth kind.

Following Appell (1880) we define the four Appell series as follows:

Definition A.4.

$$F_1(a, b, b'; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m!n!} = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1(a+m, b'; c+m; y)\frac{x^m}{m!}, \tag{A.32}$$

where $\max\{|x|, |y|\} < 1$.

Definition A.5.

$$F_2(a, b, b'; c, c'; x, y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m y^n}{m!n!} = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1(a+m, b'; c'; y)\frac{x^m}{m!}, \tag{A.33}$$

where $|x| + |y| < 1$.

Definition A.6.

$$F_3(a, b, b'; c; x, y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m!n!} = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1(a', b'; c+m; y)\frac{x^m}{m!}, \tag{A.34}$$

where $\max\{|x|, |y|\} < 1$.

Definition A.7.

$$F_4(a, b'; c, c'; x, y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m y^n}{m!n!} = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1(a+m, b+m; c'; y)\frac{x^m}{m!}, \tag{A.35}$$

where $\sqrt{|x|} + \sqrt{|y|} < 1$. Here the denominator parameters c and c' are neither zero nor a negative integer.

The above defined functions are discovered while considering the product of two Gauss series. In this analysis, we also come across the following interesting result:

$$ {}_2F_1(a, b; c; x + y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_{m+n}} \frac{x^m y^n}{m!n!}. \tag{A.36} $$

A multiple integral representation for the generalized hypergeometric series is given by (Saigo and Saxena 1999)

$$ \begin{aligned} &\frac{\prod_{j=1}^{P} \Gamma(A_j)}{\prod_{j=1}^{Q} \Gamma(B_j)} {}_P F_Q[(A_P); (B_Q); -(x_1 + \cdots + x_n)] \\ &= \left(\frac{1}{2\pi i}\right)^n \int_{L_1} \cdots \int_{L_n} \frac{[\prod_{j=1}^{P} \Gamma(A_j + s_1 + \cdots + s_n)]}{[\prod_{j=1}^{Q} \Gamma(B_j + s_1 + \cdots + s_n)} \\ &\quad \times \Gamma(-s_1) \cdots \Gamma(-s_r) x_1^{s_1} \cdots s_n^{s_n} \mathrm{d}s_1 \cdots \mathrm{d}s_n, \end{aligned} \tag{A.37} $$

where the contours are of Barnes type with indentations, if necessary, such that the poles of $\Gamma(A_j + s_1 + \cdots + s_n)$, $j = 1, \ldots, p$ are separated from those of $\Gamma(-s_j)$, $j = 1, \ldots, n$.

A.3.1 Confluent Hypergeometric Function of Two Variables

Definition A.8.

$$ \phi_1(a, b; c; x, y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c)_{m+n}} \frac{x^m y^n}{m!n!}, |x| < 1, |y| < \infty. \tag{A.38} $$

Definition A.9.

$$ \phi_2(b, b'; c; x, y) = \sum_{m=0,n=0}^{\infty} \frac{(b)_m(b')_m}{(c)_{m+n}} \frac{x^m y^n}{m!n!}, |x| < \infty, |y| < \infty. \tag{A.39} $$

Definition A.10.

$$ \phi_3(b; c; x, y) = \sum_{m=0,n=0}^{\infty} \frac{(b)_m}{(c)_{m+n}} x^m y^n, |x| < \infty, |y| < \infty. \tag{A.40} $$

Definition A.11.

$$\psi_1(a,b;c,c';x,y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(c)_m(c')_n}\frac{x^m y^n}{m!n!}, |x| < 1, |y| < \infty. \quad \text{(A.41)}$$

Definition A.12.

$$\psi_2(a;c,c';x,y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_{m+n}}{(c)_m(c')_n}\frac{x^m y^n}{m!n!}, |x| < \infty, |y| < \infty. \quad \text{(A.42)}$$

Definition A.13.

$$\Xi_1(a,a',b;c;x,y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m}{(c)_{m+n}}\frac{x^m y^n}{m!n!}, |x| < 1, |y| < \infty. \quad \text{(A.43)}$$

Definition A.14.

$$\Xi_2(a,b;c;x,y) = \sum_{m=0,n=0}^{\infty} \frac{(a)_m(b)_m}{(c)_{m+n}}\frac{x^m y^n}{m!n!}, |x| < 1, |y| < \infty. \quad \text{(A.44)}$$

A.4 Lauricella Functions of Several Variables

The four Appell series F_1, F_2, F_3, F_4 are generalized by Lauricella (1893) in terms of multiple hypergeometric series as given below.

Definition A.15.

$$F_A^{(n)}[a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n] = \sum_{m_1=0,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \quad \text{(A.45)}$$

where $|x_1| + |x_2| + \cdots + |x_n| < 1$.

Definition A.16.

$$F_B^{(n)}[a_1,\ldots,a_n;b_1,\ldots,b_n;c;x_1,\ldots,x_n] = \sum_{m_1=0,\ldots,m_n=0}^{\infty} \frac{(a_1)_{m_1}\cdots(a_n)_{m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \quad \text{(A.46)}$$

where $\max\{|x_1|,\ldots,|x_n|\} < 1$.

Definition A.17.

$$F_C^{(n)}[a, b; c_1, \ldots, c_n; x_1, \ldots, x_n] = \sum_{m_1=0,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \tag{A.47}$$

where $\sqrt{|x_1|} + \cdots + \sqrt{|x_n|} < 1$.

Definition A.18.

$$F_D^{(n)}[a, b_1, \ldots, b_n; c; x_1, \ldots, x_n] = \sum_{m_1=0,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \tag{A.48}$$

where $\max\{|x_1|, \ldots, |x_n|\} < 1$.

For $n = 2$ we have the following relations:

$$F_A^{(2)} = F_2, F_B^{(2)} = F_3, F_C^{(2)} = F_4, F_D^{(2)} = F_1, \tag{A.49}$$

where the Appell series are defined in the previous section. An interesting result is the following reduction formula (Lauricella, 1893)

$$F_D^{(n)}[a, b_1, \ldots, b_n; c; x, \ldots, x] = {}_2F_1(a, b_1 + \cdots + b_n; c; x). \tag{A.50}$$

We also have (Lauricella 1893)

$$F_D^{(n)}[a, b_1, \ldots, b_n; c; 1, \ldots, 1] = \frac{\Gamma(c)\Gamma(c - a - b_1 - \cdots - b_n)}{\Gamma(c - a)\Gamma(c - b_1 - \cdots - b_n)}, \tag{A.51}$$

where $c \neq 0, -1, -2, \ldots; \Re(c - a - b_1 - \cdots - b_n) > 0$.

Single integral representations for the function $F_D^{(n)}$ is given by

$$F_D^{(n)}[a, b_1, \ldots, b_n; c; x_1, \ldots, x_n] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 u^{a-1}(1 - u)^{c-a-1}(1 - ux_1)^{-b_1}\cdots(1 - ux_n)^{-b_n}\,\mathrm{d}u, \tag{A.52}$$

where $\Re(a) > 0, \Re(c - a) > 0$.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(f_1 t+g_1)^{\sigma_1}\cdots(f_k t+g_k)^{\sigma_k}\,\mathrm{d}t$$
$$= (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(af_1+g_1)^{\sigma_1}\cdots(af_k+g_k)^{\sigma_k}$$
$$\times F_D^{(n)}\left[\alpha,-\sigma_1,\ldots,-\sigma_k;\alpha+\beta;-\frac{(b-a)f_1}{af_1+g_1},\ldots,-\frac{(b-a)f_k}{af_k+g_k}\right], \quad \text{(A.53)}$$

where $a,b\in\Re(a<b)$, $f_i,g_i,\sigma_i\in C, i=1,\ldots,k$, $\min\{\Re(\alpha),\Re(\beta)\}>0$ and

$$\max\left[\left|\frac{(b-a)f_1}{af_1+g_1}\right|,\ldots,\left|\frac{(b-a)f_k}{af_k+g_k}\right|\right]<1.$$

A.4.1 Confluent form of Lauricella Series

Definition A.19.

$$\Phi_2^{(n)}[b_1,\ldots,b_n;c;x_1,\ldots,x_n]=\sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}. \quad \text{(A.54)}$$

Definition A.20.

$$\Psi_2^{(n)}[a;c_1,\ldots,c_n;x_1,\ldots,x_n]=\sum_{m_1=0,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}. \quad \text{(A.55)}$$

Definition A.21.

$$M_{k,\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n)=x_1^{\mu_1+\frac{1}{2}}\cdots x_n^{\mu_n+\frac{1}{2}}\exp\left[-\frac{x_1+\cdots+x_n}{2}\right]$$
$$\times\Psi_2^{(n)}[\mu_1+\cdots+\mu_n-k+\frac{n}{2};2\mu_1+1,\ldots,2\mu_n+1;x_1,\ldots,x_n]. \quad \text{(A.56)}$$

For a detailed definition and other properties of these functions, see the original paper by Pierre Humbert [La fonction $W_{k,\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n)$. Comptes Rendus.t. CLXXI, 1920, p. 328] and Appell and Kampé de Fériet (1926).

A.5 The Generalized H-Function (The $\bar{H}$-Function)

Notation A.7. $\bar{H}(z)$, $\bar{H}_{p,q}^{m,n}[x]$; H bar function

Definition A.22. In an attempt to derive certain Feynman integrals in two different ways which arise in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions, Inayat-Hussain (1987b) investigated a generalization of the H-function as

$$\bar{H}(z) = \bar{H}^{m,n}_{p,q}(z) = \bar{H}^{m,n}_{p,q}\left[x\left|\begin{matrix}(\alpha_j,A_j,a_j)_{1,n},(\alpha_j,A_j)_{n+1,p}\\(\beta_j,B_j)_{1,m},(\beta_j,B_j,b_j)_{m+1,q}\end{matrix}\right.\right] \tag{A.57}$$

$$= \frac{1}{2\pi i}\int_L \chi(s)z^s \mathrm{d}s, \tag{A.58}$$

where

$$\chi(s) = \frac{\left[\prod_{j=1}^{m}\Gamma(\beta_j - B_j s)\right]\left[\prod_{j=1}^{n}\{\Gamma(1-\alpha_j + A_j)\}^{a_j}\right]}{\left[\prod_{j=M+1}^{q}\{\Gamma(1-\beta_j + B_j s)\}^{b_j}\right]\left[\prod_{j=n+1}^{p}\Gamma(\alpha_j - A_j)\right]}, \tag{A.59}$$

which contains fractional powers of some of the gamma functions. $L = L_{i\tau\infty}$ is a contour starting at the point $\tau - i\infty$, and going to the point $\tau + i\infty$ with $\gamma \in R = (-\infty, \infty)$. For a detailed definition, convergence and existence conditions, and for the computable representation of the $\bar{H}$-function, the reader is referred to the original papers of Buschman and Srivastava (1990) and Saxena (1998). It is interesting to note that for $a_j = b_j = 1$ for all j, the $\bar{H}$-function reduces to the familiar H-function defined by Fox (1961), see also Mathai and Saxena (1978) and Kilbas and Saigo (2004).

A.5.1 Special Cases of $\bar{H}$-Function

A few interesting special cases of the $\bar{H}$-function, which cannot be obtained from the H-function are given below.

$$g_1 = (-1)^p g(\gamma,\eta,\zeta,p;z) = \frac{K_{d-1}\Gamma(1+p)\Gamma(1+\frac{\zeta}{2})B\left(\frac{1}{2},\frac{1}{2}+\frac{\zeta}{2}\right)}{2^{2+p}\pi\Gamma(\gamma)\Gamma(\gamma-\frac{\zeta}{2})}$$

$$\times\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\mathrm{d}s(-z)^s\Gamma(-s)\Gamma(\gamma+s)\Gamma(\gamma-\frac{\zeta}{2}+s)}{(\eta+s)^{1+p}\Gamma(1+\frac{\zeta}{2}+s)} \tag{A.60}$$

$$= \frac{K_{d-1}\Gamma(1+p)\Gamma(\frac{1}{2}+\frac{\zeta}{2})}{2^{2+p}\pi\Gamma(\gamma)\Gamma(\gamma-\frac{\zeta}{2})}$$

$$\times \bar{H}^{1,3}_{3,3}\left[-z\left|\begin{matrix}(1-\gamma,1;1),(1-\gamma+\frac{\zeta}{2},1;1),(1-\eta,1;1+p)\\(0,1),\left(-\frac{\zeta}{2},1;1\right),(-\eta,1;1+p)\end{matrix}\right.\right], \tag{A.61}$$

where $K_d = [2^{1-d}\pi^{-\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right)]$ (Inayat-Hussain 1987a, Eq. (5)). The above integral is connected with certain class of Feynman integrals.

$$\beta F(d,\epsilon) = -\frac{1}{4\pi^{\frac{d}{2}}(1+\epsilon)^2} \times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}s[-(1+\epsilon)^{-2}]^s\Gamma(-s)[\Gamma(1+s)]^2[\Gamma\left(\frac{3}{2}+s\right)]^d}{[\Gamma(2+s)]^{1+d}} \tag{A.62}$$

$$= -\frac{1}{4\pi^{\frac{d}{2}}(1+\epsilon)^2}\bar{H}_{2,2}^{1,2}\left[-(1+\epsilon)^{-2}\middle|_{(0,1),(-1,1;1+d)}^{(0,1;2),\left(-\frac{1}{2},1;d\right)}\right] \tag{A.63}$$

$$= -\frac{1}{4\pi^{\frac{d}{2}}(1+\epsilon)^2}\bar{H}_{3,2}^{1,3}\left[-(1+\epsilon)^{-2}\middle|_{(0,1),(-1,1;1+d)}^{(0,1;1),(0,1;1),\left(-\frac{1}{2},1;d\right)}\right]. \tag{A.64}$$

The above function is the exact partition function of the Gaussian model in statistical mechanics.

For further example of a function, which is not a special case of the H-function is the poly-logarithm of complex order ν, denoted by $L^{\nu}(z)$. Its relation with $\bar{H}$-function is given by Saxena (1998, eq. (1.12)) as

$$L^{\nu}(z) = \bar{H}_{2,2}^{1,2}\left[-z\middle|_{(0,1),(0,1;\nu-1)}^{(0,1,1),(1,1;\nu)}\right]. \tag{A.65}$$

An account of $L^{\nu}(z)$ is available from the book by Marichev (1983).

The function due to Nagarsenker and Pillai (1973, 1974) also furnishes an example of a function, which is not a special case of Fox's H-function. Yet another function, which is not a special case of the H-function is the generalized Riemann-zeta function defined by

$$\phi(z,q,\eta) = \sum_{k=0}^{\infty}\frac{z^k}{(\eta+k)^q} = \bar{H}_{2,2}^{1,2}\left[-z\middle|_{(0,1),(-\eta,1,q)}^{(0,1,1),(1-\eta,1,q)}\right]. \tag{A.66}$$

The above function is a generalization of the well-known generalized (Hurwitz's) zeta function $\zeta(q,\eta), q \neq 0,-1,-2,\ldots$ and the Riemann zeta function $\zeta(q)$, $\Re(q) > 1$. It has been shown by Buschman and Srivastava (1990, p. 4708) that the sufficient condition for absolute convergence of the contour integral (A.58) is given by

$$A = \sum_{j=1}^{m}|B_j| + \sum_{j=1}^{n}|a_jA_j| - \sum_{j=m+1}^{q}|b_jB_j| - \sum_{j=n+1}^{p}|A_j| > 0. \tag{A.67}$$

This condition provides exponential decay of the integrand in (A.58), and region of absolute convergence of the contour integral (A.58) is given by

$$|\arg z| < \frac{\pi A}{2}. \tag{A.68}$$

Remark A.6. In a series of papers, abelian theorems, complex inversion formulas and characterizations for the distributional $\bar{H}$-function transformation are established by Saxena and Gupta (1994, 1995, 1997). Functional relations for the $\bar{H}$-function are given by Saxena (1998). Unified fractional integration operators associated with the $\bar{H}$-function are defined and studied by Saxena and Soni (1997). Fractional integral formulas for this function are investigated by Gupta and Soni (2001). Fractional integral formulas associated with Saigo–Maeda operators of fractional integration are given by Saxena et al. (2002). Application of this function in bivariate probability distributions is demonstrated by Saxena et al. (2002).

A.6 Representation of an H-Function in Computable Form

Case I: When the poles of $\prod_{j=1}^{m}\Gamma(b_j - sB_j)$ are simple, that is, where $B_h(b_j + \lambda) \neq B_j(b_h + \nu)$ for $j \neq h, j, h = 1, \dots, m; \lambda, \nu = 0, 1, 2, \dots$; then we obtain the following expansion for the H-function.

$$\begin{aligned} H_{p,q}^{m,n}(z) &= \sum_{h=1}^{m}\sum_{\nu=0}^{\infty} \frac{[\prod_{j=1,j\neq h}^{m}\Gamma(b_j - B_j(b_h+\nu)/B_h)][\prod_{j=1}^{n}\Gamma(1-a_j - A_j(b_h+\nu)/B_h)]}{[\prod_{j=m+1}^{q}\Gamma(b_j - B_j(b_h+\nu)/B_h)][\prod_{j=n+1}^{p}\Gamma(a_j - A_j(b_h+\nu)/B_h)]} \\ &\quad \times \frac{(-1)^{\nu} z^{(b_h+\nu)/B_h}}{\nu! B_h}, \end{aligned} \tag{A.69}$$

which exists for all $z \neq 0$ if $\mu > 0$ and for $0 < |z| < \frac{1}{\beta}$ if $\mu = 0$, where β and μ are defined in (1.8) and (1.9) respectively.

Case II. When the poles of $\prod_{j=1}^{n}\Gamma(1-a_j + sA_j)$ are simple, that is, where $A_h(1 - a_j + \nu) \neq A_j(1 - a_h + \lambda)$ for $j \neq h, j, h = 1, \dots, n; \lambda, \nu = 0, 1, 2, \dots$ then we obtain the following expansion for the H-function.

$$\begin{aligned} H_{p,q}^{m,n}(z) &= \sum_{h=1}^{n}\sum_{\nu=0}^{\infty} \frac{[\prod_{j=1,j\neq h}^{n}\Gamma(1-a_j - A_j(1-a_h+\nu)/A_h)]}{[\prod_{j=m+1}^{q}\Gamma(1-b_j - B_j(1-a_h+\nu)/A_h)]} \\ &\quad \times \frac{[\prod_{j=1}^{m}\Gamma(b_j + B_j(1-a_h+\nu)/A_h)]}{[\prod_{j=n+1}^{p}\Gamma(a_j + A_j(1-a_h+\nu)/A_h)]} \frac{(-1)^{\nu}\left(\frac{1}{z}\right)^{\frac{1-a_h+\nu}{A_h}}}{\nu! A_h}, \end{aligned} \tag{A.70}$$

which exists for all $z \neq 0$ if $\mu < 0$ and for $|z| > \frac{1}{\beta}$ if $\mu = 0$, β and μ are defined in (1.8) and (1.9) respectively.

A.7 Further Generalizations of the H-Function

Notation A.8. **I-function**:

$$I^{m,n}_{p_i,q_i}[z], \quad I^{m,n}_{p_i,q_i}\left[z\middle|\begin{matrix}(a_j,A_j)_{1,n},\dots,(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m},\dots,(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right].$$

Definition A.23. The I-function is defined, like the H-function in terms of a Mellin–Barnes type integral in the following form (Saxena 1982):

$$I^{m,n}_{p_i,q_i}\left[z\middle|\begin{matrix}(a_j,A_j)_{1,n},\dots,(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m},\dots,(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right] = \frac{1}{2\pi i w}\int_L \chi(s) z^{-s}\mathrm{d}s, \tag{A.71}$$

where

$$\chi(s) = \frac{\left[\prod_{j=1}^m \Gamma(b_j + B_j s)\right]\left[\prod_{j=1}^n \Gamma(1-a_j - A_j s)\right]}{\left[\sum_{i=1}^r \left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji} - B_{ji}s)\right]\left[\prod_{j=n+1}^{p_i}\Gamma(a_{ji} + A_{ji}s)\right]\right]}, \tag{A.72}$$

where m, n, p_i, q_i are nonnegative integers satisfying $0 \leq n \leq p_i, 1 \leq m \leq q_i$, $i = 1, \dots, r$ with r being finite and $w = (-1)^{\frac{1}{2}}$. The existing conditions for the defining integral (A.71) are given below:

$$(i) \qquad \alpha_i > 0, |\arg z| < \frac{1}{2}\alpha_i \pi, \tag{A.73}$$

$$(ii) \qquad \alpha_i \geq 0, |\arg z| \leq \frac{1}{2}\alpha_i \pi \text{ and } \Re(\beta + 1) < 0, \tag{A.74}$$

where

$$\alpha_i = \sum_{j=1}^n A_j - \sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=1}^m B_j - \sum_{j=m+1}^{q_i} B_{ji}, \ \ i = 1, \dots, r, \tag{A.75}$$

and

$$\beta = \sum_{j=1}^m b_j + \sum_{j=m+1}^{q_i} - \sum_{j=1}^n a_j - \sum_{j=n+1}^{p_i} a_{ji} + \frac{1}{2}(p_i - q_i), i = 1, \dots, r. \tag{A.76}$$

Note A.1. For $r = 1$ in (A.72), the definition of the H-function (1.2) is recovered.

Note A.2. We note that integral operators involving I-function are defined and studied by Saxena et al. (1993). A basic analogue of the I-function is given by Saxena et al. (1995). Saigo–Maeda operators of the product of I-function and a general class of polynomials are discussed by Saxena et al. (2002).

Remark A.7. I-function is further generalized by Südland et al. (1998) in a different notation with a modified definition of slightly general nature and call it Aleph functions. Aleph functions occur naturally in certain problems of fractional driftless Fokker–Planck equations. For further details in this regard, one can refer to the original paper Südland et al. (2001).

Bibliography

Abiodun RFA, Sharma BL (1971) (Also see Sharma BL) Summation of series involving generalized hypergeometric functions of two variables. Glasnik Mat Ser III 6(26):253–264

Abiodun RFA, Sharma BL (1973) Fourier series for generalized function of two variables. Univ Nac Tucumán Rev Ser A 23:25–33

Agal SN, Koul CI (1983) Weyl fractional calculus and Laplace transform. Proc Indian Acad Sci (Math Sci)92:167–170

Agarwal RP (1965) An extension of Meijer's G-function. Proc Nat Inst Sci India Part A 31: 536–546

Agarwal BM (1968a) Application of Δ and E operators to evaluate certain integrals. Proc Cambridge Philos Soc 64:99–104

Agarwal BM (1968b) On generalized Meijer H-functions satisfying the Truesdell F-equations. Proc Nat Acad Sci India Sect A 38:259–264

Agarwal RP (1969) Certain q-integrals and q-derivatives. Proc Cambridge Philos Soc 66:365–370

Agarwal RP (1970) On certain transformation formulae and Meijer's G-function of two variables Indian J Pure Appl Math 1(4):537–551

Agarwal RP (1973) Contributions to the theory of generalized hypergeometric series. J Math Phys Sci Madras 7(Jubilaums-Sonderheft):S93–S100

Agarwal I, Saxena RK (1969) Integrals involving Bessel functions. Univ Nac Tucumán Rev Ser A 19:245–254

Agarwal I, Saxena RK (1972) An infinite integral involving Meijer's G-function. Riv Mat Univ Parma 1(3):15–21

Agarwal BM, Singhal BM (1974) A transformation from G-functions to H-functions. Vijnana Parishad Anusandhan Patrika 17:137–142

Aggarwala I, Goyal AN (1973) On some integrals involving generalized Lommel, Maitland and A*-functions. Indian J Pure Appl Math 4:798–805

Al-Musallam F, Tuan YK (2001) H-function with complex parameters; evaluation. Internat J Math Math Sci 25:727–743

Al-Musallam F, Tuan YK (2001a) H-function with complex parameters: existence. Internat J Math Math Sci 25:571–586

Al-Salam WA (Al-Salam Waleed A; Also see Carlitz L) (1966–67) Some fractional q -integrals and q-derivatives. Proc Edinburgh Math Soc 15:135–140

Al-Saqabi BN (1995) Solution of a class of differintegral equations by means of Riemann-Liouville operator. J Fract Calc 8:95–102

Al-Saqabi BN, Tuan YK (1996) Solution of a fractional differintegral equation. Integral Transform Spec Funct 4:321–326

Al-Saqabi BN, Kalla SL, Srivastava HM (1990) A certain family of infinite series associated with digamma functions. J Math Anal Appl 159:361–372

Al-Shammery AH, Kalla SL, Khajah HG (2000) On a generalized fractional integro-differential equation of Volterra-type. Integral Transform Spec Funct 9(2):81–90

Anandani P (1967) Some expansion formulae for H-function II. Ganita 18:89–101

Anandani P (1968) Some integrals involving products of Meijer's G-function and H-function. Proc Indian Acad Sci Sect A 67:312–321
Anandani P (1968a) Summation of some series of products of H-functions. Proc Nat Inst Sci India Part A 34:216–223
Anandani P (1968b) Fourier series for H-functions. Proc Indian Acad Sci Sect A 68:291–295
Anandani P (1969) On some integrals involving generalized associated Legendre's functions and H-functions. Proc Natl Acad Sci India Sect A 39:341–348
Anandani P (1969a) On some recurrence formulae for the H-function. Ann Polon Math 21: 113–117
Anandani P (1969b) On finite summation, recurrence relations and identities of H-functions. Ann Polon Math 21:125–137
Anandani P (1969c) Some infinite series of H-function-I. Math Student 37:117–123
Anandani P (1969d) Some integrals involving products of generalized Legendre's associated functions and the H-function. J Sci Eng Res 13:274–279
Anandani P (1969e) Some integrals involving generalized associated Legendre's functions and the H-function. Proc Natl Acad Sci India Sect A 39:127–136
Anandani P (1969f) Some expansion formulae for the H-function-III. Proc Natl Acad Sci India Sect A 39:23–34
Anandani P (1969g) Some expansion formulae for H-function-IV: Rend. Cir Mat Palermo 18(2):197–214
Anandani P (1969h) On some identities of H-function. Proc Indian Acad Sci Sect A 70:89–91
Anandani P (1969i) Some integrals involving H-functions. Lebdev J Sci Tech Part A 7:62–66
Anandani P (1970) Some integrals involving associated Legendre functions of the first kind and the H-function. J Natur Sci Math 10:97–104
Anandani P (1970a) Some infinite series of H-functions-II. Vijnana Parishad Anusandhan Patrika 13:57–66
Anandani P (1970b) Integration of products of generalized Legendre function with respect to parameters. Lebdev J Sci Tech Part A 9:13–19
Anandani P (1970c) On the derivative of H-function. Rev Roum Math Pures et Appl 15:189–191
Anandani P (1970d) Use of generalized Legendre associated function and the H-function in heat production in a cylinder. Kyungpook Math J 10:107–113
Anandani P (1970e) Some integrals involving Jacobi polynomials and H-function. Lebdev J Sci Tech Part A 8:145–149
Anandani P (1970f) An expansion formula for the H-function involving associated Legendre function J Natur Sci Math 10(1):49–51
Anandani P (1970g) An expansion for the H-function involving generalized Legendre's associated functions. Glasnik Mat Ser III 25(5):55–58
Anandani P (1970h) Expansion of the H-function involving generalized Legendre's associated function and H-function. Kyungpook Math J 10:53–57
Anandani P (1970i) Some expansion formulae for the H-function. Lebdev J Sci Tech India Part A 8:80–87
Anandani P (1970j) Some integrals involving H-functions of generalized arguments. Math Education 4:32–38
Anandani P (1970k) Some infinite series of H-function-II. Vijnana Parishad Anusandhan Patrika 13:57–66
Anandani P (1971) Some integrals involving associated Legendre functions and the H-function. Univ Nac Tucumán Rev Ser A 21:33–41
Anandani P (1971a) An expansion formula for the H-function involving products of associated Legendre functions and H-functions. Univ Nac Tucumán Rev Ser A 21:95–99
Anandani P (1971b) Some integrals involving H-function. Rend Circ Mat Palermo 20(2):70–82
Anandani P (1971c) An expansion formula for the H-function involving generalized Legendre associated functions. Portugal Math 30:173–180
Anandani P (1971d) Integration of products of generalized Legendre functions and the H-function with respect to parameters. Lebdev J Sci Tech Part A 9:13–19

Anandani P (1972) On some generating functions for the H-functions. Lebdev J Sci Tech Part A 10:5–8

Anandani P (1972a) Some infinite series of H-functions. Ganita 23(2):11–17

Anandani P (1973) Integrals involving products of generalized Legendre functions and the H-function. Kyungpook Math J 13:21–25

Anandani P (1973a) Some integrals involving the H-function and generalized Legendre functions. Bull Soc Math Phys Macedoine 24:33–38

Anandani P (1973b) Expansion theorems for the H-function involving associated Legendre functions. Bull Soc Math Phys Macedoine 24:39–43

Anandani P (1973c) On some results involving generalized Legendre's associated Legendre functions. Ganita 24(1):41–48

Anandani P, Srivastava HSP (1972/73) On Mellin transform of product involving Fox's H-function and a generalized function of two variables. Comment Math Univ St Paul 21(2):35–42

Anderson WJ Haubold HJ, Mathai AM (1994) Astrophysical thermonuclear functions. Astrophysics & Space Science 214:49–70

Andrews GE (1974) Applications of basic hypergeometric functions. Siam Review 16:441–484

Andrews GE, Askey R, Roy R (1999) Special functions encyclpedia. In: Rota GC (eds) Mathematics and its applications, vol 71. Cambridge University Press, Cambridge

Anh VV, Leonenko NN (2001) Spectral analysis of fractional kinetic equations with random data. J Statist Phys 104 nO 516:1349–1387

Anh VV, Heyde CC, Leonenko NN (2002) Dynamic models driven by Levy noise. Journal of Applied Probability 39:730–747

Appell P (1880) Sur les series hypergeometriques de deux variables, et sur des equations differentielles lineaires aux derives partielles. CR Acad Sci Paris 90:296–298

Appell P, Kampé de Fériet J (1926) Fonctions Hypergéometriques et Hyperphériques: Polynomes d'Hermites. Gauthier-Villars, Paris

Askey R (1965) Orthogonal expansion with positive coefficients. Proc Amer Math Soc 16: 1191–1194

Atanackovic TN, Stankovic B (2007) On a differential equation with left and right fractional derivatives. Fract Calc Appl Anal 10(2):139–150

Baeumer B, Kurita S, Meerschaert MM (2005) Inhomogeneous fractional diffusion equations. Frac Calc Appl Anal 8(4):371–386

Baeumer B, Kovács M, Meerschaert MM (2007) Fractional reproduction-dispersal equations and heavy tail dispersal kernels. Bulletin of Mathematical Biology 69(7):2281–2297

Bagley RL (1990) On the fractional order initial value problem and its engineering applications. In: Nishimoto K (ed) Fractional Calculus and Its Applications. Proceedings of the International Conference held at the Nihon University Centre at Tokyo May 29–June 1, 1989 Nihon University, Koriyama, pp 11–20

Bagley RL, Torvik HJ (1986) On the fractional calculus model of viscoelastic behavior. J Rheol 30:133–135

Bailey WN (1933) A reducible case of the fourth type of Appell's hypergeometric function of two variables. Quart J Math Oxford Ser 4(2):305–308

Bailey WN (1934) On the reducibility of Appell's function F4. Quart J Math Oxford Ser 5(2): 291–292

Bailey WN (1935) Generalized Hypergeometric Series, Cambridge Tracts in Mathematics and Mathematical Physics No 32. Cambridge University, Cambridge & New York

Bailey WN (1936a) Some integrals involving Bessel functions. Proc London Math Soc 40:37–48

Bailey WN (1936b) Some infinite integrals involving Bessel functions-II. J London Math Soc 11:16–20

Bajpai SD (1969a) An integral involving Fox's H-function and Whittaker functions. Proc Cambridge Philos Soc 65:709–712

Bajpai SD (1969b) On some results involving Fox's H-function and Jacobi polynomials. Proc Cambridge Philos Soc 65:697–701

Bajpai SD (1969c) Fourier series of generalized hypergeometric functions. Proc Cambridge Philos Soc 65:703–707

Bajpai SD (1969d) An expansion formula for Fox's H-function. Proc Cambridge Philos Soc 65:683–685

Bajpai SD (1969e) An integral involving Fox's H-function and heat Conduction. Math Education 3:1–4

Bajpai SD (1969f) An expansion formula for H-function involving Bessel functions. Lebdev J Sci Tech Part A 7:18–20

Bajpai SD (1969/70) An integral involving Fox's H-function and its application. Univ Lisboa Revista Fac Ci II, Ser A 13:109–114

Bajpai SD (1970a) Some expansion formulae for Fox's H-function involving exponential functions. Proc Cambridge Philos Soc 67:87–92

Bajpai SD (1970b) On some results involving Fox's H-function and Bessel function. Proc Indian Acad Sci Sect A 72:42–46

Bajpai SD (1970c) Transformation of an infinite series of Fox's H-function. Portugal Math 29: 141–144

Bajpai SD (1971) Some results involving Fox's H-function. Portugal Math 30:45–52

Bajpai SD (1972) Some results involving G-function of two variables. Gaz Mat Lisboa 33:13–24

Bajpai SD (1974) Expansion formulae for the products of Meijer's G-function and Bessel functions. Portugal Math 33:35–41

Banerji PK, Saxena RK (1971) Integrals involving Fox's H-function. Bull Math Soc Sci Math RS Roumanie 63(15):263–269

Banerji PK, Saxena RK (1973a) On some results involving products of H-functions. An Sti Univ Al I Cuza Iasi Sect La Mat (NS) 19:175–178

Banerji PK, Saxena RK (1973b) Contour integral involving Legendre polynomial and Fox's H-function. Univ Nac Tucumán Rev Ser A 23:193–198

Banerji PK, Saxena RK (1976) Expansions of generalized H-functions. Indian J Pure Appl Math 7(3):337–341

Barkai E (2001) Fractional Fokker-Planck equation, solution and application. Phys Rev E 63:046118–17

Barnes EW (1908) A new development of the theory of the hypergeometric functions. Proc London Math Soc 6(2):141–177

Barrios JA, Betancor JJ (1991) The Krätzel integral transformation of distributions. Math Nachr 154:11–26

Beck C (2006) Stretched exponentials from superstatistics. Physica A 365:96–101

Beck C , Cohen EGD (2003) Superstatistics. Physica A 322:267–275

Berbaren-Santos MN (2003) Properties of the Mittag-Leffler relaxation function. Journal of Mathematical Chemistry 38:629–635

Berbaren-Santos MN (2005) Properties of the Mittag-Leffler relaxation function. Journal of Mathematical Chemistry 38:629–635

Betancor JJ, Jerez DC (1994) Boundedness and range of H-transformation of certain weighted L_p-spaces. Serdica 20:269–297

Betancor JJ, Jerez DC (1997) Weighted norm inequalities for the H-transformation. Internat J Math Math Sci 20:647–656

Bhagchandani LK, Mehra KN (1970) Some results involving generalized Meijer function and Jacobi polynomials. Univ Nac Tucumán Rev Ser A 20:167–174

Bhatnagar PL (1973) Numerical integration of Lommel type of integrals involving products of three Bessel functions. Indian J Math 15:77–97

Bhatt RC (1966) Certain integrals involving the products of hypergeometric functions. Mathematische (Catania) 21:6–10

Bhise VM (1964) Some finite and infinite series of Meijer-Laplace transform. Math Ann 154: 267–272

Bhise VM (1967) Certain properties of Meijer-Laplace transform. Comp Math 18:1–6

Bhonsle BR (1962) Some series and recurrence relations for MacRobert's E-function. Proc Glasgow Math Assoc 5:116-117

Bhonsle BR (1966) Jacobi polynomials and heat production in a cylinder. Math Japon 11(1):83–90

Bhonsle BR (1967) Steady state heat flow in a shell enclosed between two prolate spheroids Math Japon 12(1):83–90

Bochner S (1952) Bessel functions and modular relations of higher type and hyperbolic differential equations. Comm Sem Math De l'Univ de Lund Tome Supplementaire dedié á Marcel Riez 12–20

Bochner S (1958) On Riemann's functional equation with multiple gamma factors. Ann Math 67(2):29–41

Boersma J (1962) On a function which is a special case of Meijer's G-function. Comp Math 15: 34–63

Bonilla B, Kilbas AA, Rivero M, Rodriguez L, Trujillo JJ (1998) Modified Bessel-type transform in $L_{\nu,r}$-space. Rev Acad Canaria Cienc 10(1):45–58

Bonilla B, Kilbas AA, Rivero M, Rodriguez L, Trujillo JJ (2000) Modified Bessel-type function and solution of differential and integral equations. Indian J Pure Appl Math 31(1):93–109

Bora SL (1970) An infinite integral involving generalized function of two variables. Vijnana Parishad Anusandhan Patrika 13:95–100

Bora SL, Kalla SL (1970) Some results involving generalized function of two variables. Kyungpook Math J 10:133–140

Bora SL, Kalla SL (1971a) Some recurrence relations for the H-function. Vijnana Parishad Anusandhan Patrika 14:9–12

Bora SL, Kalla SL (1971b) An expansion formula for the generalized function of two variables. Univ Nac Tucumán Rev Ser A 21:53–58

Bora SL, Saxena RK (1971) Integrals involving product of Bessel functions and generalized hypergeometric functions. Publ Inst Math (Beograd) 25(11):23–28

Bora SL, Kalla SL, Saxena RK (1970) On integral transforms. Univ Nac Tucumán Rev Ser A 20:181–188

Bora SL Saxena RK, Kalla SL (1972) An expansion formula for Fox's H-function of two variables. Univ Nac Tucumán Rev Ser A 22:43–48

Bouzeffour F (2007) Inversion formulas for q-Riemann-Liouville and q-Weyl transforms. J Math Anal Appl 336:833–848

Boyadjiev J, Kalla SL (2001) Series representations of analytic functions and applications. Frac Calc Appl Anal 3:379–408

Braaksma BLJ (1964) Asymptotic expansions and analytic continuations for a class of Barnes integrals. Comp Math 15:239–341

Bromwich TJ (1909) An asymptotic formula for the generalized hypergeometric series. Proc London Math Soc 7(2):101–106

Brychkov YA, Prudnikov AP (1989) Integral Transforms of Generalized Functions translated and revised from the second Russian edition. Gordon and Breach Science, New York

Brychkov YA, Glaeske H-J, Prudnikov AP, Tuan YK (1992) Multidimensional Integral Transformations. Gordon and Breach Science, Philadelphia

Buckwar E, Luchko Y (1998) Invariance of partial differential equation of fractional order under the Lie group and scaling transformations. J Math Anal Appl 237(2):81–97

Burchnall JL (1939) The differential equations of Appell's function F_4. Quart J Math Oxford Ser 10:145–150

Burchnall JL (1942) Differential equations associated with hypergeometric functions. Quart J Math Oxford Ser 13:90–106

Burchnall JL, Chaundy TW (1940) Expansions of Appell's double hypergeometric functions. Quart J Math Oxford Ser 11:249–270

Burchnall JL, Chaundy TW (1941) Expansions of Appell's double hypergeometric functions-II. Quart J Math Oxford Ser 12:112–128

Buschman RG (1972) Contiguous relations and related formulas for the H-function of Fox. Jnanabha Sect A 2:39–47

Buschman RG (1974a) The asymptotic expansion of an integral. Rend del Cir Mat7(3) Ser 6: 481–486

Buschman RG (1974b) Partial derivatives of the H-function with respect to parameters expressed as finite sums and as integrals. Univ Nac Tucumán Rev Ser A 24:149–155

Buschman RG (1974c) Finite sum representations for partial derivatives of special functions with respect to parameters. Math Comp 28(127):817–824

Buschman RG (1978) H-function of two variables-I. Indian J Math 20:139–153

Buschman RG (1982) Analytic domains for multivariable H-functions. Pure Appl Math Soc 18: 23–27

Buschman RG, Gupta KC (1975) Contiguous relations for the H-functions of two variables. Indian J Pure Appl Math 6(12):1416–1421

Buschman RG, Srivastiva HM (1975) Inversion formulas for the integral transformation with the H-function as kernel. Indian J Pure Appl Math 6(6):583–590

Buschman RG, Srivastava HM (1986) Convergence regions for some multiple Mellin-Barnes contour integrals representing generalized hypergeometric functions. Int J Math Educ Sci Technol 17:605–609

Buschman RG, Srivastava HM (1990) The $\bar{H}$-function associated with certain class of Feynman integrals. J Phys A Math Gen 23:4707–4710

Butzer PL, Westphal U (2000) An introduction to fractional calculus. In: Hilfer R (ed) (2000) Applications of Fractional Calculus in Physics. World Scientific, Singapore, pp 1–85

Caputo M (1967) Linear model of dissipation whose Q is almost frequency independent II. Geophys J R Astro Soc 13:529–539

Caputo M (1969) Elasticitá e Dissipazione. Zanichelli, Bologna

Carlitz L (1962) Summation of some series of Bessel functions. Neder Akad Wetensch Proc Ser A 65 Indag Math 24:47–54

Carlitz L, Al Salam WA (1963) Some functions associated with Bessel functions. J Math Mech 12:911–933

Carlson BC (1963) Lauricella's hypergeometric function F_D. J Math Anal Appl 7:452–470

Carmichael RD, Pathak RS (1987) Asymptotic behaviour of the H-transform in the complex domain. Math Proc Cambridge Philos Soc 102:533–552

Carmichael RD, Pathak RS (1990) Asymptotic analysis of the H-function transform. Glas Mat Ser III 45(25):103–127

Chak AM (1970) Some generalization of Laguerre polynomials, I, II. Math Vesnik 7(22):7–13, 14–18

Chamati H, Tonchev NS (2006) Generalized Mittag-Leffler functions in the theory of finite-size scaling for systems with strong anisotropy and/or long-range interaction. J Phys A Math Gen 39:469–478

Chandel RCS (1969) Generalized Laguerre polynomials and the polynomials related to them. Indian J Math 11:57–66

Chandel RCS (1971) A short note on generalized Laguerre polynomials and the polynomials related to them. Indian J Math 13:25–27

Chandel RCS (1972) Generalized Laguerre polynomials and the polynomials related to them II. Indian J Math 14:149–155

Chandel RCS (1973) On some multiple hypergeometric functions related to Lauricella functions. Jnanabha A 3:119–136

Chandel RCS, Agarwal RD (1971) On the G-functions of two variables. Jnanabha Sect A 1(1): 83–91

Chandrasekharan K, Narasimhan R (1962) Functional equations with multiple gamma factors and the average order of arithmetical functions. Ann Math 76:93–136

Chatterjea SK (1964) On a generalization of Laguerre polynomials. Rend Sem Math Univ Padova 34:180–190

Chaturvedi KK, Goyal AN (1972) A*-function-I. Indian J Pure Appl Math 3:357–360

Chaturvedi KK, Goyal AN (1973) Integrals involving A*-function. Ganita 26:1–18

Chaudhry KL (1975) Fourier series of Fox's H-function. Math Education Sect A 9:53–56

Chaudhry MA (1999) Transformation of the extended gamma function $\Gamma_{0,2}^{2,0}[(B,x)]$ with applications to astrophysical thermonuclear functions. Astrophysics & Space Science 262:263–270
Chaudhry, M Aslam (2000) Analytical study of thermonuclear reaction probability integrals. Astrophysics & Space Science 273:43–52
Chaurasia VBL (1976a) On some integrals involving Kampé de Fériet function and the H-function (Hindi). Vijnana Parishad Anusandhan Patrika 19:163–167
Chaurasia VBL (1976b) On the H-function. Jnanabha 6:9–14
Chaurasia VBL (2004) Equation of the internal blood pressure and the H-function. Acta Ciencia Indica 30M(4):719–720
Chaurasia VBL, Gupta N (1999) General fractional integral operators, general class of polynomials and Fox's H-function. Soochow J Math 25:333–339
Chaurasia VBL, Patni R (1999) Simultaneous operational calculus involving a product of two general class of polynomials, Fox's H-function and the H-function of several complex variables. Kyungpook Math J 39:47–55
Chhabra SP, Singh F (1969) An integral involving product of a G-function and a generalized hypergeometric function. Proc Cambridge Philos Soc 65:479–482
Churchill RV (1941) Fourier Series and Boundary Value Problems. McGraw-Hill, New York
Compte A (1998) Stochastic foundation of fractional dynamics. Physical Review E 53:4191–4193
Condes S Kalla SL, Saxena RK (1981) A note on a short table of the generalized hypergeometric distribution. Metrika 28:197–201
Constantine AG (1963) Some non-central distribution problems in multivariate analysis. Ann Math Statist 34:1270–1285
Constantine AG, Muirhead RJ (1972) Partial differential equations of hypergeometric functions of two arguments matrices. J Mult Anal 2:332–338
Cross MC, Hohenberg PC (1990) Pattern formation outside of equilibrium. Rev Modern Phys 65:851–912
Dahiya RS (1971a) Multiple integrals and the transformations involving H-functions and Tchebichef polynomials. Acta Mexicana Ci Tech 5:192–197
Dahiya RS (1971b) On integral representation of Fox's H-function for evaluating double integrals. An Fac Ci Univ Porto 54:363–367
Dahiya RS (1971/72) On an integral relation involving Fox's H-function. Univ Lisboa Revista Fac Ci A 14(2):105–111
Dahiya RS, Singh B (1971) Fourier series of Meijer's G-function of higher order. An Sti Univ Al I Cuza N Ser Sect 1 17:111–116
Dahiya RS, Singh B (1972) On Fox's H-function. Indian J Pure Appl Math 3(3):493–495
D'Angelo IG, Kalla SL (1973) Algunos resultados que involucran la function H de Fox. Univ Nac Tucumán Rev Ser A 33:83–87
Dattoli G, Gianessi L Mezi L, Tocci D, Coloi R (1991) FEL time-evolution operator Instru Methods A 304:541–544
Dattolic G, Lorezutta S, Maino G, Torre A (1996) Analytical treatment of the high gain free electron laser equation. Radiat Phys Chem 48:29–40
Davis HT (1927) The application of fractional operators to functional equations. Amer J Math 49:123–142
Davis HT (1936) The Theory of Linear Operators. Principia, Bloomington, Indianna
De Amin LH, Kalla SL (1973) Integrales que involucran productos de funciones hipergeometricas generalizadas y la funcion H de dos variables. Univ Nac Tucumán Rev Ser A 23:131–141
De Anguio MEF, Kalla SL (1972) The Laplace transform of the product of two Fox's H-functions. Univ Nac Tucumán Rev Ser A 22:171–175
De Anguio MEF, Kalla SL (1973) Sobre integracion con respecto a parametros. Univ Nac Tucumán Rev Ser A 23:103–110
De Batting NEF, Kalla SL (1971a) Some results involving generalized hypergeometric function of two variables. Rev Ci Mat Univ Laurenco Marques Ser A 2:47–53
De Batting NEF, Kalla SL (1971b) On certain finite integrals involving the hypergeometric, H-function of two variables. Acta Mexicana Ci Tecn 5:142–148

De Anguio MEF De Gomez LAMM, Kalla SL (1972a) Integrals that involve the H-function of two variables. Acta Mexicana Ci Tech 6:30–41

De Anguio MEF De Gomez LAMM, Kalla SL (1972b) Integrales que involucrana la function H de dos variables. Acta Mexicana Cien Ten 6(2):30–41

Debnath L (2000) Integral transforms and their applications. CRC, Boca Raton, FL

Debnath L (2003) Fractional integral and fractional differential equations in fluid mechanics. Frac Calc Appl Anal 6:119–155

De Galindo SM, Kalla SL (1975) Sobre una extension de la function generalizada de dos variables. Univ Nac Tucumán Rev SerA 25:221–229

De Gomez Lopez AMM, Kalla SL (1972) Integrals that involve Fox's H-function. Univ Nac Tucumán Rev Ser A 22:165–170

De Gomez Lopez AMM, Kalla SL (1973) On a generalized integral transform. Kyungpook Math J 13(2):275–280

Del-Castillo-Negrete D, Carreras BA, Lynch VE (2003) Front dynamics in reaction-diffusion systems with Lévy flights: A fractional diffusion approach. Physical Review Letters 91:018302

Del-Castillo-Negrete D, Carreras BA, Lynch VE (2002) Front propagation and segregation in a reaction-diffusion model with cross-diffusion. Physica D 168:45–60

Denis RY (1968) Certain transformations of bilateral cognate trigonometrical series of hypergeometric type. Proc Cambridge Philos Soc 64:421–424

Denis RY (1969) A general expansion theorem for products of generalized hypergeometric series. Proc Nat Inst Sci India Part A 35:70–76

Denis RY (1970) Certain integrals involving G-function of two variables. Ganita 21(2):1–10

Denis RY (1972) Certain expansions of generalized hypergeometric series. Math Student 40A: 82–86

Denis RY (1973) On certain double series involving generalized hypergeometric series. Bull Soc Math Phys Macédoine, 23:33–35

Deora Y, Banerji PK (1994) An application of fractional calculus to the solution of Euler-Darboux equation in terms of the Dirichlet averages. J Frac Calc 5:91–94

Deora Y, Banerji PK, Saigo M (1994) Fractional integral and Dirichlet averages. J Frac Calc 6: 55–59

Deshpande VL (1971) On the derivatives of G-function of two variables. Proc Nat Acad Sci India Sect A 41:60–68

Deshpande VL (1991) Expansion theorems for the Kampé de Fériet function. Neder Akad Wetensch Proc 74 = Indag Math 33:39–45

Dhawan GK (1969) Series and expansion formulae for G-function of two variables. J MACT 2: 88–94

Dixon AL, Ferrar WL (1936) A class of discontinuous integrals. Quart J Math Oxford Ser 7:81–96

Doetsch G (1943) Theorie und Answendung der Laplace-transformation. Dover, New York

Doetsch G (1956) Anleitung zum Praktischen Gebrauch der Laplace transformaton. Oldenbourg, Munich

Doetsch G (1958) Einfuhrung in Theorie und Anwendung der Laplace-Transformation. Birkhäuser-Verlag, Basel

Dotsenko MR (1991) On some applications of Wright's hypergeometric function. CR Acad Bulgare Sci 44:13–16

Dotsenko MR (1993) On an integral transform with Wright's hypergeometric function Mat Fiz Nelilein Mekh 18(52):17–52

Dubey GK, Sharma CK (1972) On Fourier series for generalized Fox H-functions. Math Student 40A:147–156

Dzherbashyan MM (1960) On the integral transformations generated by the generalized Mittag-Leffler function (in Russian). Izv AN Arm SSR 13(3):21-63

Dzherbashyan MM (1966) Integral transforms and representation of functions in complex domain (in Russian). Nauka, Moscow

Dzherbashyan MM (1993) Harmonic analysis and boundary value problems in the complex domain. Operator Theory Adv Appl, Birkhaüser-Verlag, Basel

Dzrbasjan VA (1964) On a theorem of Whipple. Z Wycysl Mat i Mat Fiz 4:348–351
Edelstien LA (1964) On the one-centre expansion of Meijer's G-function. Proc Cambridge Philos Soc 60:533–538
Erdélyi A (1950–51) On some functional transformations. Univ Politec Torino Rend Sem Mat 10:217–234
Erdélyi A (1954) On a generalization of the Laplace transformation. Proc Edinburgh Math Soc 10(2):53–55
Erdélyi, A, Kober H (1940) Some remarks on Hankel transforms. Quart J Math Oxford Ser 11: 212–221
Erdélyi A, Magnus W, Oberhettinger F, Tricomi FG (1953) Higher transcendental functions Vol I, II. McGraw-Hill, New York
Erdélyi A, Magnus W, Oberhettinger F, Tricomi FG (1954) Tables of integral transforms Vol I, II. McGraw-Hill, New York
Erdélyi, A, Magnus W, Oberhettinger F, Tricomi FG (1955) Higher Transcendental Functions Vol III. McGraw-Hill, New York
Exton H (1972a) On two multiple hypergeomtric functions related to Lauricella's F_D. Jnanabha Sect A 2:59–73
Exton H (1972b) Certain hypergeometric function of four variables. Bull Soc Math Greece 13: 104–113
Exton H (1976) Multiple Hypergeometric Functions and Applications. (Ellis Horwood Chichester) Wiley, New York
Exton H (1978) Handbook of Hypergeometric Integrals: Theory, Applications, Tables, Computer Programs. (Ellis Horwood Chichester) Wiley, New York
Feller W (1952) On a generalization of Marcel Riesz' potentials and the semi-groups generated by them. Meddeladen Lund Universitets Matematisca Seminarium (Comm. Sém. Mathém. Université de Lund), Tome Supple. dédié a M. Riesz, Lund 73–81
Feller W (1966) An introduction to probability theory and its applications. Springer, New York
Fettis HE (1957) Lommel-type integrals involving three Bessel functions. J Math Phys 36:88–95
Fields JL (1973) Uniform asymptotic expansions of certain classes of Meijer G-functions for a large parameter. SIAM J Math Anal 4:482–507
Fisher RA (1937) The wave of advances of advantageous genes. Annals of Eugenics 7:353–369
Fourier JBJ (1822) Théorie Analytique de la Chaleur, Oeuvres de Fourier, Vol I. Didot, Paris, p 508
Fox C (1927) The expression of hypergeometric series in terms of similar series. Proc London Math Soc 26(2):201–210
Fox C (1928) The asymptotic expansion of generalized hypergeometric functions. Proc London Math Soc 27(2):389–400
Fox C (1961) The G and H-functions as symmetrical Fourier kernels. Trans Amer Math Soc 98:395–429
Fox C (1963) Integral transforms based upon fractional integration. Proc Cambridge Philos Soc 59:63–71
Fox C (1965a) A formal solution of certain dual integral equations. Trans Amer Math Soc 119: 389–398
Fox C (1965b) A family of distributions with the same ratio property. Canadian Math Bull 8: 631–635
Fox C (1971) Solving integral equation by L and L^{-1} operators. Proc Amer Math Soc 29:299–306
Fox C (1972) Application of Laplace transform and their inverses. Proc Amer Math Soc 35: 193–200
Frank TD (2005) Nonlinear Fokker-Planck Equations: Fundamentals and Applications. Springer, New York
Freed AD, Diethelm K (2007) Caputo derivatives in viscoelasticity: A non-fractional diffusion equations. Frac Calc Appl Anal 10(3):219-248
Gajic L, Stankovic B (1976) Some properties of Wright function. Publ Institut Math Beograd Nouvelle Ser 20(34):91–98

Gaishun IV Kilbas AA, Rosozin SV (eds) (1996) Boundary Value Problems, Special Functions and Fractional Calculus. Proceedings of the international conference dedicated to the ninetieth birthday of academician FD Gajkhov (1905–1980) held at Minsk, February 16–20 1996. Belarussian State University, Minsk

Galué L (1999) Generalized radiation integral involving product of Meijer G-functions. Hadronic J 22:391–405

Galué L (2000) Composition of hypergeometric fractional operators. Kuwait J Sci Eng 27:1–14

Galué L (2002) Differintegrals of Wright's generalized hypergeometric function. Internat J Appl Math 10:255–267

Galué L, Kalla SL (1994) Representation of operators of fractional integration by Laplace transformation. Anal Acad Nac Ca Ex Fis Nat Buenos Aires 46:99–104

Galué L, Kalla SL, Srivastava HM (1993) Further results on an H-function generalized fractional calculus. J Fract Calc 4:89–102

Galué L, Kiryakova VS, Kalla SL (1993) Solution of dual integral equations by fractional calculus. Mathematica Balkanica 7:53–72

Galué L, Kalla SL, Tuan YK (2000) Composition of Erdélyi–Kober fractional operators. Integral Transform Spec Funct 9(3):185–196

Garg RS (1982) On multidimensional Mellin convolutions and H-function transformations. Indian J Pure Appl Math 13:30–38

Gasper G, Rahman R (1990) Basic Hypergeometric Series Encyclopedia of Mathematics and Its Applications Vol 35. Cambridge University, Cambridge

Gelfand IM, Shilov GF (1964) Generalized functions, Vol I. Academic, London

George A, Mathai AM (1975) A generalized distribution for the inter-live-birth interval. Sankhya Ser B 37:332–340

Gilding BH, Kersner R (2004) Travelling Waves in Nonlinear Diffusion-Convection Reaction. Birkhaeuser-Verlag, Basel-Boston-Berlin

Glaeske H-J, Kilbas AA, Saigo M (2000) A modified Bessel-type integral transform and its compositions with fractional calculus operators on spaces $F_{p,\mu}$ and $F'_{p,\mu}$. J Comput Appl Math 118:151–168

Glaeske H-J, Kilbas AA, Saigo M, Shlapakov SA (1997) $L_{v,r}$-theory of integral transformation with the H-function in the kernel (Russian). Dokl Akad Nauk Belarusi 41(2):10–15

Glaeske H-J, Kilbas AA, Saigo M, Shlapakov SA (2000) Integral transforms with H-function kernels on $L_{v,r}$-spaces. Appl Anal 79:443–474

Glöckle WG, Nonnenmacher TF (1991) Fractional integral operators and Fox functions in the theory of viscoelasticity. Macromolecules American Chemical Society 24:6426–6434

Glöckle WG, Nonnenmacher TF (1993) Fox function representation of non-Debye relaxation processes. J Stat Phys 71:741–757

Gogovcheva E, Boyadijiev L (2005) Fractional extensions of Jacobi polynomials and Gauss hypergeometric function. Fract Calc Appl Anal 8(4):431–438

Gokhroo DC (1970) The Laplace transform of the product of Meijer's G-functions. Univ Nac Tucumán Rev Ser A 20:59–62

Gorenflo R, Mainardi F (1997) Fractional calculus, integral and differential equations of fractional order. In: Carpinteri A, Mainardi F (eds) Fractals and Fractional Calculus in Continuum Mechanics. Springer, Wiens, pp 223–276

Gorenflo R, Iskenderov A, Luchko Y (2000) Mapping between solutions of fractional diffusion wave equations. Fract Calc Appl Anal 3:75–86

Gorenflo R, Luchko Y, Mainardi F (1999) Analytic properties and applications of the Wright function. Fract Cal Appl Anal 2:383–414

Gorenflo R, Luchko Y, Mainardi F (1999) Analytical properties and applications of the Wright function. Frac Calc Appl Anal 2:383–414

Gorenflo R, Luchko Y, Mainardi F (2000) Wright functions as scale invariant solutions of the diffusion-wave equation. J Comput Appl Math 118:175–191

Gorenflo R, Loutchko J, Luchko Y (2002) Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivatives. Fract Calc Appl Anal 5:491–518

Golas PC (1968) Integration with respect to parameters. Vijnana Parishad Anusandhan Patrika 11:71–76
Golas PC (1969) On a generalized Stieltjes transform. Proc Natl Acad Sci India Sect A 39:42–48
Goyal AN (1969) Some infinite series of H-functions-I. Math Student 37:179–183
Goyal GK (1969) A finite integral involving H-function. Proc Natl Acad Sci India Sect A 39: 201–203
Goyal GK (1971) A generalized function of two variables I. Univ Studies Math 1:37–46
Goyal SP (1970) On some finite integrals involving generalized G-function. Proc Nat Acad Sci India Sect A 40:219–228
Goyal SP (1971a) On transformations of infinite series of Fox's H-function. Indian J Pure Appl Math 2(4):684–691
Goyal SP (1971b) On some finite integrals involving Fox's H-function. Proc Natl Acad Sci India Sect A 74:25–53
Goyal SP (1975) The H-function of two variables. Kyungpook Math J 15:117–131
Goyal SP, Agarwal RK (1982) Fox's H-function and electric circuit theory. Indian J Pure Appl Math 13:39–46
Goyal AN, Chaturvedi KK (1971) Integrals involving Fox H-function. Univ Studies 1:7–13
Goyal AN, Goyal GK (1967a) On the derivatives of the H-function. Proc Natl Acad Sci India Sect A 37:56–59
Goyal AN, Goyal GK (1967b) Expansion theorems of H-function. Vijnana Parishad Anusandhan Patrika 10:205–217
Goyal AN, Sharma S (1971a) Study of Meijer's G-function of two variables-I. Univ Studies 1: 82–89
Goyal AN, Sharma S (1971b) Series of Meijer's G-function of two variables-I. Univ Studies Math 1:29–35
Goyal SP, Jain RM, Gaur N (1991) Fractional integral operations involving a product of generalized hypergeometric functions and a general class of polynomials Indian J Pure Appl Math 11: 403–411
Grafiychuk V Datsko B, Maleshko V (2006) Mathematical modeling of pattern formation in sub and super-diffusive reaction-diffusion systems arXiv;nlin. A0/06110005v3
Grafiychuk V, Datsko B, Maleshko V (2007) Nonlinear oscillations and stability domains in fractional reaction-diffusive systems. arXiv:nlin.PS/0702013v1
Grosche C, Steiner F (1998) Handbook of Feynman Path Integrals. Springer Tract in Modern Physics Vol 145. Springer-Verlag, New York
Grünwald AK (1867) Über begrezte Derivationen und deren Anwendung Z Angew Math Phys 12:441–480
Gulati HC (1971a) Fourier series for G-function of two variables. Gaz Mat (Lisboa) 32(121–124):21–30
Gulati HC (1971b) Some contour integrals involving G-function of two variables. Defence Sci J 21:39–42
Gulati HC (1971c) Some formulae for G-function of two variables involving Legendre functions Vijnana Parishad Anusandhan Patrika 14:77–88
Gulati HC (1971d) Some recurrence formulae for G-function of two variables I II. Defence Sci J 21:101–106, 235–240
Gulati HC (1972) Derivatives of G-function of two variables. Math Education 6A:72A, 76
Gupta KC (1965) On the H-function. Ann Soc Sci Bruxelles Ser I 79:97–106
Gupta KC (1966) Integrals involving the H-function. Proc Natl Acad Sci India Sect A 36:504–509
Gupta SC (1969a) Integrals involving products of G-functions. Proc Natl Acad Sci India Sect A 39(2):193–200
Gupta SC (1969b) Reduction of G-function of two variables Vijnana Parishad Anusandhan Patrika 12:51–59
Gupta LC (1970) Some expansion formulae for Meijer's G-function. Univ Nac Tucumán Rev Ser A 20:109–115

Gupta SD (1973a) Some infinite series for the H-function of two variables. An Sti Univ Al I Cuza Iasi Sect la Mat (NS) 19:185–189
Gupta SD (1973b) Fourier series for the H-function of two variables. An Sti Univ Al I Cuza Iasi Sect la Mat (NS) 19:179–184
Gupta KC (2001) New relationships of the H-function with functions of practical utility in fractional calculus. Ganita Sandesh 15:63–66
Gupta IS, Debnath L (2007) Some properties of the Mittag-Leffler functions. Integral Transform Spec Funct 18:329–336
Gupta KC, Jain UC (1966) The H-function II. Proc Nat Acad Sci India Sect A 36:594–609
Gupta KC, Jain UC (1968) On the derivative of the H-function. Proc Nat Acad Sci India Sect A 38:189–192
Gupta KC, Jain UC (1969) The H-function IV. Vijnana Parishad Anusandhan Patrika 12:25–30
Gupta KC, Mittal PK (1970) The H-function transform. J Austral Math Soc 11:142–148
Gupta KC, Mittal PK (1971) The H-function transform II. J Austral Math Soc 12:444–450
Gupta KC, Olkha GS (1969) Integrals involving products of generalized hypergeometric functions and Fox's H-function. Univ Nac Tucumán Rev Ser A 19:205–212
Gupta KC, Saxena RK (1964a) Certain properties of generalized Stieltjes transform involving Meijer's G-function. Proc Natl Inst Sci India Sect A 30:707–714
Gupta KC, Saxena RK (1964b) On Laplace transform. Riv Mat Univ Parma Italie 5:159–164
Gupta KC, Soni RC (2001) New properties of generalization of hypergeometric series associated with Feynman integrals. Kyungpook Math J 41:97–104
Gupta KC Soni RC (2002a) A unified inverse Laplace transform formula, functions of practical importance and H-functions. J Rajasthan Acad Phys Sci 1(1):7–16
Gupta KC, Soni RC (2002b) On the inverse Lapalce transform. Ganita Sandesh 4:1
Gupta PM, Sharma CK (1972) On Fourier series for Meijer's G-function of two variables. Indian J Pure Appl Math 3:1073–1077
Gupta KC, Srivastava A (1970) On certain recurrence relations. Math Nachr 46:13–23
Gupta KC, Srivastava A (1971) On certain recurrence relations II. Math Nachr 49:187–197
Gupta KC, Srivastava A (1972) On finite expansions for the H-function. Indian J Pure Appl Math 3:322–328
Gupta KC, Srivastava A (1973) Certain results involving Kampé de Fériet's function. Indian J Math 15:99–102
Gupta KC, Goyal SP, Tariq OS (1998) On theorems connecting the Laplace transform and a generalized fractional integral operator. Tamkang J Math 29:323–333
Gupta KC Jain R, Agarwal R (2007) On existence conditions for a generalized Mellin-Barnes type integral. Nat Acad Sci Lett 30:169–172
Habibullah GM (1977) A note on a pair of integral operators involving Whittaker functions. Glasgow Math J 18:99–100
Hai NT, Yakubovich SB (1992) The double Mellin-Barnes type integrals and their applications to convolution theory. World Scientific, Singapore
Hai NT, Marichev OI, Buschman RG (1992) Theory of the general H-function of two variables. Rocky Mountain J Math 22(4):1317–1327
Hahn W (1949) Bertrage zur theorie der Heinischen Reihen die 24 integrale de hypergeometrischen q-differenzengleichung, des q-analogen der Laplace Transformation. Math Nachr 2:263–278
Haken H (2004) Synergetics introduction and advanced topics. Springer-Verlag, Berlin-Heidelberg
Haubold HJ (1998) Wavelet analysis of the new solar neutrino capture rate data for the Homestake experiment. Astrophysics & Space Science 258:201–218
Haubold HJ, John RW (1979) Spectral line Profiles, neutron cross sections new results concerning the analysis of Voigt functions. Astrophysics & Space Science 65:477–491
Haubold HJ, Mathai AM (1986) Analytic representation of thermonuclear reaction rates. Studies in Applied Mathematics 75:123–138
Haubold HJ, Mathai AM (1994a) The determination of the internal structure of the Sun by the density distribution. In: Basic space science. Proceedings No 320. American Institute of Physics, pp 89–101

Haubold HJ, Mathai AM (1994b) Solar nuclear energy generation, the chlorine solar neutrino experiment. In: Basic space science. Conference Proceedings No 320. American Institute of Physics, pp 102–116

Haubold HJ, Mathai AM (1995) A heuristic remark on the periodic variation in the number of solar neutrinos detected on Earth. Astrophysics & Space Science 228:113–124

Haubold HJ, Mathai AM (2000) The fractional kinetic equation and thermonuclear functions. Astrophysics & Space Science 273:53–63

Haubold HJ, Mathai AM, Saxena RK (2004) Boltzmann-Gibbs entropy versus Tsallis entropy: Recent contributions to resolving the argument of Einstein concerning "Neither Herr Boltzmann nor Herr Planck has given a definition of W". Astrophysics & Space Science 290:241–245

Haubold HJ, Mathai AM, Saxena RK (2007a) Solutions of the fractional reaction-diffusion equations in terms of the H-function. ArXiv 0704.0329V1 [math ST] 3 April 2007

Haubold HJ, Mathai AM, Saxena RK (2007b) Solution of fractional reaction-diffusion equations in terms of the H-function. Bull Astro Soc India 35(4):681–689

Henry BI, Wearne SL (2000) Fractional reaction diffusion. Physica A 276:448–455

Henry BI, Wearne SL (2002) Existence of Turing instabilities in a two species fractional reaction-diffusion system. SIAM Journal of Applied Mathematics 62:870–887

Henry BI, Langlands TAM, Wearne SL (2005) Turing pattern formation in fractional activator-inhibitor systems. Physical Review E 72:026101

Herz CS (1955) Bessel functions of matrix argument. Ann Math 61:474–523

Higgins TP (1964) A hypergeometric function transform. J Soc Indust Appl Math 12:601–612

Hilfer R (ed) (2000) Applications of fractional calculus in physics. World Scientific, Singapore

Hilfer R, Seybold HJ (2006) Computation of the generalized Mittag-Leffler function and its inverse in the complex plane. Integral Transform Spec Funct 17:637–652

Hille E, Tamarkin TD (1930) On the theory of linear integral equations Ann Math 31:479–528

Hua L-K (1959) Harmonic analysis of functions of several complex variables in classical domain. Moscow (in Russian)

Hundsdorfer, Werwer JG (2003) Numerical solution of time-dependent advection-diffusion-reaction equations. Springer–Verlag, Berlin, Heidelberg, New York

Inayat-Hussain AA (1987a) New properties of hypergeometric series derivable from Feynmann integrals: I. Transformation and reduction formulae. J Phys A: Math Gen 20:4109–4117

Inayat-Hussain AA (1987b) New properties of hypergeometric series derivable from Feynman integrals-II, a generalization of the H-function. J Phys A: Math Gen 20:4119–4128

Jaimini B, Saxena H (2007) Solutions of certain fractional differential equations. J Indian Acad Math 29(1):223–236

Jain RN (1965) Some infinite series of G-functions. Math Japon 10:101–105

Jain UC (1967) Certain recurrence relations for the H-function. Proc Nat Inst Sci India Part A 33:19–24

Jain UC (1968) On an integral involving H-function. J Austral Math Soc 8:373–376

Jain RN (1969) General series involving H-functions. Proc Cambridge Philos Soc 65:461–465

Jain NC (1971a) Integrals that contain hypergeometric functions and the H-function. Republ Venezuela Bol Acad Ci Fis Mat Natur 31(90):95–102

Jain NC (1971b) An integral involving the generalized function of two variables. Rev Roumanie Math Pures Appl 16:865–872

Jain PC, Sharma BL (1968a) An expansion for generalized function of two variables. Univ Nac Tucumán Rev Ser A 18:7–15

Jain PC, Sharma BL (1968b) Some new expansions of the generalized function of two variables. Univ Nac Tucumán Rev Ser A 18:25–33

Jaiswal NK (1968) Priority queues. Academic, New York

James AT (1954) Normal multivariate analysis and the orthogonal group. Ann Math Statist 25: 40–75

James AT (1960) The distribution of the latent roots of the covariance matrix. Ann Math Statist 31:151–158

James AT (1961a) The distribution of non-central means with known covariance. Ann Math Statist 32:874–882

James AT (1961b) Zonal polynomials of the real positive definite symmetric matrices. Ann Math 74:456–469

James AT (1964) Distributions of matrix variates and latent roots derived from normal samples. Ann Math Statist 35:475–501

James AT (1966) Inference on latent roots by calculations of hypergeometric functions of matrix argument. In: Krishnaiah PR (ed) Multivariate analysis. pp 209–235

James AT (1969) Tests of equality of latent roots of the covariance matrix. In: Krishnaiah PR (ed) Multivariate Analysis Vol 2, pp 205–218

James AT, Constantine AG (1974) Generalized Jacobi polynomials as spherical functions of the Grassman manifold. Proc London Math Soc 29(3):174–192

Jespersen S, Metzler R, Fogedby HC (1999) Lévy flights in external force fields: Langevin and fractional Fokker-Planck equations and their solutions. Physical Review E 59:2736–2745

Jones KRW (1993) Fractional integration and uniform densities in quantum mechanics. In: Kalia RN (ed), Recent Advances in Fractional Calculus. Global Publishing, Sauk rapids, MN, pp 203–218

Jorgenson J, Lang S (2001) The ubiquitous beat kernel. In: Engquist B, Schmid W (eds) Mathematics Unlimited – 2001 and Beyond. Springer-Verlag, Berlin, pp 655–683

Joshi N, Joshi JMC (1982) A real inversion theorem for H-transform. Ganita 33:67–73

Joshi VG, Saxena RK (1981) Abelian theorems for distributional H-transform. Math Ann 256: 311–321

Joshi VG, Saxena RK (1982) Structure theorems for H-transformable generalized functions. Indian J Pure Appl Math 13:25–29

Joshi VG, Saxena RK (1983) Complex inversion and uniqueness theorems for the generalized H-transform. Indian J Pure Appl Math 14:322–329

Kalia RN (ed) (1993) Recent advances on fractional calculus. Global Publishing, Sauk Rapids (Minnesota)

Kalla SL (1967) Some infinite integrals involving generalized hypergeometric functions ψ_2 and F_c. Proc Natl Acad Sci India Sect A 37:195–200

Kalla SL (1969a) Integral operators involving Fox's H-function. Acta Mexicana Ci Tech 3: 117–122

Kalla SL (1969b) Infinite integrals involving Fox's H-function and confluent hypergeometric functions. Proc Nat Acad Sci India Sect A 39:3–6

Kalla RN (1971) An application of a theorem on H-function. An Univ Timi Soara Ser Sti Mat 9:165–169

Kalla SL (1972) An integral involving Meijer's G-function and generalized function of two variables. Univ Nac Tucumán Rev Ser A 22:57–61

Kalla SL (1980) Operators of fractional integration, analytic functions. Kozubnik 1979, Proc Seventh Conf Kozubnik, 1979, (Lecture Notes in Mathematics), vol 798. Springer, Berlin, pp 258–280

Kalla SL (1987) Functional relations by means of Riemann-Liouville operator. Serdica 13:170–173

Kalla SL, Kiryakova VS (1990) An H-function generalized fractional calculus based upon composition of Erdélyi–Kober operators in L_p. Math Japon 35:1151–1171

Kalla SL, Kushwaha RS (1970) Production of heat in an infinite cylinder. Acta Mexicana Cien Tech 4:89–93

Kalla SL, Munot PC (1970) An expansion formula for the generalized Fox's function of two variables. Repub Venezuela Bol Acad Ci Fis Mat Natur 30(86):87–93

Kalla SL, Saxena RK (1969) Integral operators involving hypergeometric functions. Math Z 108:231–234

Kalla SL, Saxena RK (1971) Relations between Hankel and hypergeometric function operators. Univ Nac Tucumán Rev Ser A 211:231–234

Kalla SL, Yadav RK, Purohit SD (2005) On the Riemann-Liouville fractional q-integral operator involving a basic analogue of Fox's H-function. Frac Calc Appl Anal 8(3):313–322

Kalla SL, Al-Shammery AH, Khajah HG (2002) Development of the Hubbell rectangular source integral. Acta Applicandae Mathematicae 74:35–55

Kampé de Fériet J (1921) Les fonctions hypergéometriques d'ordre superieur a deux variables. CR Acad Sci Paris 173:401–404

Kant S, Koul CL (1991) On fractional integral operators. J Indian Math Soc 56:97–107

Kapoor VK, Gupta SK (1970) Fourier series for H-function. Indian J Pure Appl Math 1(4):433–437

Kapoor VK, Masood S (1968) On a generalized L-H transform. Proc Cambridge Philos Soc 64:399–406

Karlsson PW (1973) Reduction of certain generalized Kampé de Fériet functions. Math Scand 32:265–268

Karp D (2003) Hypergeometric reproducing kernels and analytic continuation from the half-line. J Integral Transforms Spec Funct 14(6):485–498

Kashyap BRK (1966) The double-ended queue with bulk service and limiting waiting space. Operations Research 14:822–834

Kaufman H, Mathai AM, Saxena RK (1969) Distributions of random variables with random parameters. South Afr Statist J 3:1–7

Khadia SS, Goyal AN (1975) On the generalized function of n variables-II. Vijnana Parishad Anusandhan Patrika 18:359–366

Khadia SS, Goyal AN (1970) On the generalized function of 'n' variables. Vijnana Parishad Anusandhan patrika 13:191–201

Khan S, Agarwal B, Pathan MA (2006) Some connections between generalized Voigt functions with different parameters. Appl Math Comput 181:57–64

Kilbas AA (2005) Fractional calculus of generalized Wright function. Frac Calc Appl Anal 8: 113–126

Kilbas AA, Kattuveettil A (2008) Representations of Dirichlet averages of generalized Mittag-Leffler function via fractional integrals and special functions. Frac Calc Appl Anal 11(4): 471–492

Kilbas AA, Saigo M (1994) On asymptotics of Fox's H-function at zero and infinity. Transforms Methods and Special Functions, Proc Intern Workshop 12–17 August 1994, Science Culture Techn, Singapore, 1995, pp 99–122

Kilbas AA, Saigo M (1996a) On Mittag-Leffler type function, fractional calculus operators and solutions of integral equations. Integral Transform Spec Funct 4:355–370

Kilbas AA, Saigo M (1996b) On generalized fractional integration operators with Fox's H-function on spaces $F_{\mu,p}$ and $F'_{\mu,p}$. Integral Transform Spec Funct 4:103–114

Kilbas AA, Saigo M (1998) Fractional calculus of the H-function. Fukuoka Univ Sci Rep 28:41–51

Kilbas AA, Saigo M (1999) On the H-function. J Appl Math Stochast Anal 12:191–204

Kilbas AA, Saigo M(2000) Modified H-transforms in $L_{\nu,r}$-spaces. Demonstratio Mathematica 33:603–625

Kilbas AA, Saigo M (2004) H-transforms, theory and applications. Chapman & Hall/CRC, Boca Raton, London, New York

Kilbas AA, Trujillo JJ (1999) On the Hankel type integral transform in $L_{\nu,r}$-spaces. Fract Calc Appl Anal 2:343–353

Kilbas AA, Trujillo JJ (2000) Computation of fractional integrals via functions of hypergeometric and Bessel type. J Comput Appl Math 118(1-2):223–239

Kilbas AA, Bonilla B, Trujillo JJ (2000) Existence and uniqueness theorems for nonlinear fractional differential equations. Demonstratio Math 33:583–602

Kilbas AA, Repin OA, Saigo M (2002) Generalized fractional integral transform with Gauss function kernels as G-transform. Integral Transform Spec Funct 13:285–307

Kilbas AA, Rodriguez L, Trujillo JJ (2002) Asymptotic representations for hypergeometric-Bessel type function and fractional integrals. J Comput Appl Math 149:469–487

Kilbas AA, Saigo M, Borovco AN (1999) On the Lommel-Maitland transform in $t_{\nu,r}$-space. Fract Calc Appl Anal 2(4):431–444

Kilbas AA, Saigo M, Saxena RK (2002) Solution of Volterra integro-differential equations with generalized Mittag-Leffler function in the kernels. J Integral Equations Appl 14:377–396

Kilbas AA, Saigo M, Saxena RK (2004) Generalized Mittag-Leffler function and generalized fractional calculus operators. Integral Transform Spec Funct 15:31–49

Kilbas AA, Saxena RK, Trujillo JJ (2006) Krätzel function as a function of hypergeometric type. Fract Calc Appl Anal 9:109–131

Kilbas AA, Saigo M, Shlapakov SA (1993) Integral transforms with Fox's H-function in spaces of summable functions. Integral Transform Spec Fucnt 1:87–103

Kilbas AA, Saigo M, Shlapakov SA (1993a) Integral transforms with Fox's H-function in $L_{\nu,r}$-spaces I. Fukuoka Univ Sci Rep 23:9–31

Kilbas AA, Saigo M, Shlapakov SA (1994) Integral transforms with Fox's H-function on $L_{\nu,r}$-spaces-II. Fukuoka Univ Sci Rep 24:13–38

Kilbas AA, Saigo M, Trujillo JJ (2002) On the generalized Wright function. Fract Calc Appl Anal 4:437–460

Kilbas AA, Srivastava HM, Trujillo JJ (2006) Theory and applications of fractional differential equations. Elsevier, Amsterdam

Kilbas AA, Pierantozzi T Trujillo JJ, Vázquez L (2004) On the solution of fractional evolution equations. J Phys A: Math Gen 37(9):3271–3282

Kilbas AA, Pierantozzi T Trujillo JJ, Vázquez L (2005) On generalized fractional evolution-diffusion equation. In: Le Meauté A, Tenrretro JA, Trigeassou JC, Sabatier J (eds) Fractional derivatives and their applications: mathematical tools, geometrical and physical aspects. UBOOKS, Germany, pp 135–150

Kilbas AA, Saigo M, Saxena RK, Trujillo JJ (2005) Asymptotic behavior of the Krätzel function and evaluation of integrals (preprint). Department of Mathematics and Mechanics, Belarusin State University

Kilbas AA, Saxena RK, Saigo M, Trujillo JJ (2006) Analytical methods of analysis and differential equations. AMADE 2003, Cambridge Scientific, Cambridge, pp 117–134

Kiryakova VS (1986) On operators of fractional integration involving Meijer's G-function. CR Acad Bulgare Sci 39:25–28

Kiryakova VS (1988a) A generalized fractional calculus and integral transforms. In: Generalized functions, convergence structures and their applications (Dubrovnik, 1987). Plenum, New York, pp 205–217

Kiryakova VS (1988b) Fractional integration operators involving Fox's $H_{m,m}^{m,0}$-function. CR Acad Bulgare Sci 41:11–14

Kiryakova VS (1988c) Generalized $H_{m,m}^{m,0}$-function fractional integration operators in some classes of analytic functions. Mat Vesnik 40:259–266

Kiryakova VS (1994) Generalized fractional calculus and applications. Pitman Res Notes Math 301, Longman Scientific & Technical; Harlow, Co-published with John Wiley, New York

Kiryakova VS (1997) All the special functions as fractional differintegrals of elementary functions. J Phys A Math Gen 30:5083–5103

Kiryakova VS (1999) Multi-index Mittag-Leffler functions related Gelfond-Leontiev operators and Laplace type integral transforms. Fract Calc Appl Anal 2:445–462

Kiryakova VS (2000) Multiple (multi-index) Mittag-Leffler functions and relations to generalized fractional calculus. J Comput Appl Math 118:241–259

Kiryakova VS (2006) On two Saigo's fractional integral opertors in the class of univalent functions. Fract Calc Appl Anal 9(2):159–176

Kiryakova VS Raina RK, Saigo M (1995) Representation of generalized fractional integrals in terms of Laplace transforms on spaces L_p. Math Nachr 176:149–158

Klusch D (1991) Astrophysical spectroscopy and neutron reactions: integral transforms and Voigt functions. Astrophysics & Space Science 175:229–240

Kober H (1940) On fractional integrals and derivatives. Quart J Math Oxford Ser 11:193–211

Kochubei AN (1990) Diffusion of fractional order. Differential Equations 26:485–492

Koh EL, Li CK (1994) On the inverse of the Hankel transform. Integral Transform Spec Funct 2:279–282

Koh EL, Li CK (1994a) The Hankel transformation M'_{μ} and its representation. Proc Amer Math Soc 122:1085–1094

Kolmogorov AN, Fomin SV (1984) Fundamentals of the theory of functions and functional analysis. Nauka, Moscow
Kolmogorov A, Petrovsky N, Piscounov S (1937) Etude de l'équations de la diffusion avec croissance de la quantité de matiére et son application a un probléme biologique. Moscow University, Bulletin of Mathematics 1:1–25
Koul CL (1972) Fourier series of a generalized function of two variables. Proc Indian Acad Sci Sect A 75:29–38
Koul CL (1973) Integrals involving a generalized function of two variables. Indian J Pure Appl Math 4(4):364–373
Koul CL (1974) On certain integral relations and their applications. Proc Indian Acad Sci Sect A 79:56–66
Krasnov KAI, Makarenko GI (1976) Integral Equations (Russian). Nauka, Moscow
Krätzel E (1979) Integral transformations of Bessel type. In Generalized functions & operational calculus. (Proc Conf Verna, 1975), Bulg Acad Sci, Sofia, pp 148–165
Krätzel E (1965) Eine Verallgemeineirung der Laplace and Meijer-transformation. Wiss Z Friedrich-Shiller-Univ Math-naturwiss Reihe 14:369–381
Kuipers L, Meulenbeld B (1957) On a generalization of Legendre's associated differential equation I and II. Neder Akad Wetensch Proc Ser A 60:436–450
Kulsrud RM (2005) Plasma physics for astrophysics. Princeton University Press, Princeton
Kumar R (1954) Some recurrence relations of the generalized Hankel transform-I. Ganita 5: 191–202
Kumar R (1955) Some recurrence relations of the generalized Hankel transform-II. Ganita 6:39–53
Kumar R (1957) Certain infinite series expansions connected with generalized Hankel transform. Ganita 8:1–7
Kumbhat RK, Saxena RK (1975) Theorems connectiing L, L^{-1} and fractional integration operators. Proc Nat Acad Sci India Sect A 45:205–209
Kumbhat RK (1976) An inversion formula for an integral transform. Indian J Pure Appl Math 7:368–375
Kuramoto Y (2003) Chemical oscillations, waves and turbulence. Dover, New York
Lacroix SF (1819) Traité du Calcul Difféntiel et du Calcul Intégral, 2nd edn. Courcier, Paris
Laurenzi BJ (1973) Derivatives of Whittaker function $W_{k,1/2}$ and $M_{k,1/2}$ with respect to order k. Math Comp 27:129–132
Lauricella G (1893) Sulle funzioni ipergeometriche a piú variabili. Rend Circ Mat Palermo 7: 111–158
Lawrynowicz J (1969) Remarks on the preceding paper of P. Anandani Ann Polon Math 21: 120–123
Lebedev NN (1965) Special functions, their applications (translated from Russian). Prentice-Hall, New Jersey
Letnikov AV (1872) An explanation of fundamental notions of the theory of differentiation of fractional order. Mat Sb 6:413–445
Lorenzo CF, Hartley TT (1998) Initialization, conceptualization, and applications in the generalized fractional calculus. NASA/TP-1998-208415:1–107
Lorenzo CF, Hartley TT (1999) Generalized functions for the fractional calculus. NASA/TP-1999-209424:1–17
Lorenzo CF, Hartley TT (2000) Initialized fractional calculus. Inter J Appl Math 3:249–265
Love ER (1967) Some integral equations involving hypergeometric functions. Proc Edinburgh Math Soc 15(2):169–198
Love ER, Prabhakar TR, Kashyap NK (1982) A confluent hypergeometric integral equation. Glasgow Math J 23:31–40
Lowndes JS (1964) Note on the generalized Mehler transform. Proc Cambridge Philos Soc 60: 57–59
Luchko YF (2001) On the distribution of the zeros of the Wright function. Integral Transform Spec Funct 11(2):195–200

Luchko YF (2000) Asymptotics of zeros of the Wright function. Z Anal Anwendungen 19(2): 583–595

Luchko YF, Gorenflo R (1998) Scale-variant solutions of a partial differential equation of fractional order. Fract Calc Appl Anal 1(1):63–78

Luchko YF, Kirayakova VS (2000) Hankel type integral transforms connected with the hyper-Bessel differential operators. In: Algebraic analysis and related topics (Warsaw, 2000), 155–165, Banach Center Publ, 53, Polish Acad Sci, Warsaw

Luchko YF, Srivastava HM (1995) The exact solution of certain differential equations of fractional order by using operational calculus. Appl Math Comput 29(8):73–85

Luke YL (1962) Integrals of Bessel functions. McGraw-Hill, New York

Luke YL (1969) The special functions and their approximations, Vol I, II. Academic Press, New York

Luque R, Galué L (1999) The application of a generalized Leibniz rule to infinite sums. Integral Transform Spec Funct 8(1-2):65–76

MacRobert TM (1959) Infinite series for E-functions. Math Z 71:143-145

MacRobert TM (1961) Fourier series for E-functions. Math Z 75:79–82

MacRobert TM (1962a) Functions of a complex variablen 5th edn. Macmillan, London

MacRobert TM (1962b) Evaluation of an E-function when three of its upper parameters differ by integral values. Pacific J Math 12:999–1002

MacRobert TM, Ragab FM (1962) E-function series whose sums are constants. Math Z 78: 231–234

Magnus W, Oberhettinger F, Soni RP (1966) Formulas and theorems for the special functions of mathematical physics 52. Springer-Verlag, New York

Mahato AK, Saxena KM (1992) A generalized Laplace transform of distributions. Anal Math 18:139–151

Mainardi F (1994) On the initial value problem for the fractional diffusion-wave equation. In: Waves and Stability in Continuum Media (Bologna, 1993), Ser Adv Math Appl Sci 23: 246–251

Mainardi F (1995) Fractional diffusive waves in viscoelastic solids. In: Wegner JL, Norwood FR (eds) Nonlinear waves in solids (ASME, 1995), pp 93–97

Mainardi F (1996) The fundamental solutions for the fractional diffusion-wave equation. Appl Math Lett 9(5):23–28

Mainardi F (1997) Fractional calculus: some basic problems in continuum and statistical mechanics. In: Carpintery A, Mainardi F (eds) Fractals and fractional calculus in continuum mechanics, Udine, 1996, CISM Courses and Lectures 378:291–348

Mainardi F, Pagnini G (2003a) The Wright functions as solutions of the time-fractional diffusion equation. Appl Math Comput 141:51–62

Mainardi F, Pagnini G (2003b) Salvatore Pincherle the pioneer of the Mellin-Barnes integrals. J Comput Appl Math 153:331–342

Mainardi F, Pagnini G (2008) Mellin-Barnes integrals for stable distributions and their convolutions. Frac Calc Appl Anal 11(4):443–456

Mainardi F, Tomirotti M (1997) Seismic pulse propagation with constant Q and stable probability distribution. Ann Geophysica 40(5):1311–1326

Mainardi F, Luchko Yu, Pagnini G (2001) The fundamental solution of the space-time fractional diffusion equation. Frac Calc Appl Anal 4:153–192

Mainardi F, Pagnini G, Saxena RK (2005) Fox H-functions in fractional diffusion. J Comput Appl Math 178:321–331

Makaka Ragy H, Simary MA (1971) Summations involving G-functions. Proc Math Phys Soc ARE No 35:1–7

Malgonde SP, Saxena Raj K (1981/82) An inversion formula for the distributional H-transformation. Math Ann 258:409–417

Malgonde SP, Saxena Raj K (1984) Some abelian theorems for the distributional H-transformation. Indian J Pure Appl Math 15:365–370

Malgonde SP, Saxena Raj K (1982) A representation of H-transformable generalized functions. Ranchi Univ Math J 12:1–8
Manne KK, Hurd AJ, Kenkre VM (2000) Nonlinear waves in reaction-diffusion systems: The effect of transport memory. Physical Review E 61:4177–4184
Marichev OI (1983) Handbook of integral transforms of higher transcendental functions: theory and algorithmic tables. Ellis Horwood, Chichester & Wiley, New York
Martic B (1973) A note on fractional integration. Publ Inst Math (Beograd), (NS) 16(30):111–115
Mathai AM (1970a) Applications of generalized special functions in statistics. Monograph, McGill University
Mathai AM (1970b) The exact distribution of a criterion for testing the hypothesis that several populations are identical. J Indian Statist Assoc 8:1–17
Mathai AM (1970c) The exact distribution of Bartlett's criterion for testing equality of covariance matrices. Publ L'Isup, Paris 19:1–15
Mathai AM (1970d) Statistical theory of distribution and Meijer's G-function. Metron 28:122-146
Mathai AM (1971a) An expansion of Meijer's G-function in the logarithmic case with applications. Math Nachr 48:129–139
Mathai AM (1971b) On the distribution of the likelihood ratio criterion for testing linear hypotheses on regression coefficients. Ann Inst Statist Math 23:181–197
Mathai AM (1971c) An expansion of Meijer's G-function and the distribution of the product of independent beta variates. S Afr Statist J 5:71–90
Mathai AM (1971d) The exact non-null distributions of a collection of multivariate test statistics. Publ L'Isup, Paris 20(1)
Mathai AM (1972a) Products and ratios of generalized gamma variates. Skandinavisk Aktuarietidskrift 55:193–198
Mathai AM (1972b) The exact distributions of three criteria associated with Wilks' concept of generalized variance. Sankhya Ser A 34:161–170
Mathai AM (1972c) The exact non-central distribution of the generalized variance. Ann Inst Statist Math 24:53–65
Mathai AM (1972d) The exact distribution of a criterion for testing that the covariance matrix is diagonal. Trab Estadistica 28:111–124
Mathai AM (1972e) The exact distribution of a criterion for testing the equality of diagonal elements given that the covariance matrix is diagonal. Trab Estadistica 23:67–83
Mathai AM (1973a) A few remarks on the exact distributions of likelihood ratio criteria-I. Ann Inst Statist Math, 25:557–566
Mathai AM (1973b) A review of the different methods of obtaining the exact distributions of multivariate test criteria. Sankhya Ser A 35:39–60
Mathai AM (1973c) A few remarks on the exact distributions of certain multivariate statistics-II. In: Multivariate Statistical Inference. North-Holland Publishing, Amsterdam, pp 169–181
Mathai AM (1979) Fox's H-function with matrix argument. Journal de Matematicae Estadistica 1:91–106
Mathai AM (1991) Special functions of matrix arguments and statistical distributions. Indian J Pure Appl Math 22:887–903
Mathai AM (1993a) Appell's and Humbert's functions of matrix argument. Linear Algebra and Its Applications 183:201–221
Mathai AM (1993b) Lauricella functions of real symmetric positive definite matrices. Indian J Pure Appl Math 24:513–531
Mathai AM (1993c) A Handbook of Generalized Special Functions for Statistical and Physical Sciences. Oxford University Press, Oxford
Mathai AM (1993d) The residual effect of a growth-decay mechanism and the distributions of covariance structures. The Canadian Journal of Statistics 21(1):277–283
Mathai AM (1995) Hypergeometric functions of many matrix variables and distributions of generalized quadratic forms. American J Math Management Sci 15:343–354
Mathai AM (1996) Whittaker and G-functions of matrix argument in the complex case. Int J Math & Math Sci 5:7–32

Mathai AM (1997a) Jacobians of matrix transformations and functions of matrix argument. World Scientific Publishing, New York
Mathai AM (1997b) Some properties of matrix-variate Laplace transforms and matrix-variate Whittaker functions. Linear Alg Appl 253:209–226
Mathai AM (1999) An introduction to geometrical probability: distributional aspects with applications. Gordon and Breach, New York
Mathai AM (2005) A pathway to matrix-variate gamma and normal densities. Linear Alg Appl 396:317–328
Mathai AM, Haubold HJ (1988) Modern problems in nuclear and neutrino astrophysics. Akademie-Verlag, Berlin
Mathai AM, Haubold HJ (2007) Pathway model, superstatistics, Tsallis statistics, and a generalized measure of entropy. Physica A 375:110–122
Mathai AM, Provost SB (1992) Quadratic Forms in random variables: theory and applications. Marcel Dekker, New York
Mathai AM, Provost SB (2006) Some complex matrix variate statistical distributions in rectangular matrices. Linear Algebra and Its Applications 410:198–216
Mathai AM, Provost SB, Hayakawa T (1995) Bilinear forms and zonal polynomials. Springer-Verlag, Lecture Notes in Statistics, No 102, New York
Mathai AM, Rathie PN (1970a) An expansion of Meijer's G-function and its application to statistical distributions. Acad Roy Belg Ci Sect(5) 56:1073–1084
Mathai AM, Rathie PN (1970b) The exact distribution of Votaw's criterion. Ann Inst Statist Math 22:89–116
Mathai AM, Rathie PN (1970c) The exact distribution for the sphericity test. J Statist Res 4: 140–159
Mathai AM, Rathie PN (1975) Basic Concepts in Information theory and statistics: axiomatic foundations and applications. Wiley Eastern, New Delhi and Wiley Halsted, New York
Mathai AM, Saxena RK (1969a) Distribution of a product and the structural setup of densities. Ann Math Statist 4:439–1448
Mathai AM, Saxena RK (1969b) Application of special functions in the characterization of probability distributions. S Afr Statist J 3:27–34
Mathai AM, Saxena RK (1971a) Extensions of an Euler's integral through statistical techniques. Math Nachr 51:1–10
Mathai AM, Saxena RK (1971b) Meijer's G-function with matrix argument. Acta Mexicana Ci Tech 5:85–92
Mathai AM, Saxena RK (1971c) A generalized probability distribution. Univ Nac Tucumán Rev Ser A 21:193–202
Mathai AM, Saxena RK (1972) Expansions of Meijer's G-function of two variables when the upper parameters differ by integers. Kyungpook Math J 12:61–68
Mathai AM, Saxena RK (1973a) On linear combinations of stochastic variables. Metrika 20(3):160–169
Mathai AM, Saxena RK (1973b) Generalized hypergeometric functions with applications in statistics and physical sciences. Lecture Notes Series No 348, Springer, Heidelberg
Mathai AM, Saxena RK (1978) The H-function with applications in statistics and other disciplines. Wiley Eastern, New Delhi and Wiley Halsted, New York
Mathai AM, Haubold HJ Mücket JP, Gottlöber S, Müller V (1988) Gravitational instability in a multicomponent cosmological medium. Journal of Mathematical Physics 29(9):2069–2077
Mathai AM, Saxena RK, Haubold HJ (2006) A certain class of Laplace transforms with applications in reaction and reaction-diffusion equations. Astrophysics & Space Science 305(3): 283–288
Mathur AB (1973) Integrals involving H-function. Math Student 41:162–166
Mathur SL (1970) Certain recurrence relations for the H-function. Math Education 4:132–136
Mathur SN (1970) Integrals involving H-functions. Univ Nac Tucumán Rev Ser A 20:145–148
McBride AC (1974/75) Solution of hypergeometric integral equations involving generalized functions. Proc Edinburgh Math Soc 19:265–285

McBride AC (1979) Fractional calculus and integral transforms of generalized functions. Research Notes in Math 31 Pitman, Boston
McBride AC, Roach, GF (1985) Fractional calculus, Notes in Mathematics, Vol 138. Pitman, Boston
McBride AC (1989) Connections between fractional calculus and some Mellin multiplier transform. In: Univalent functions, fractional calculus and their applications. (Koriyama, 1988), Ellis Horwood Chichester, pp 121–138
McBride AC (1985) Fractional calculus and integral transforms of generalized functions, Research Notes in Mathematics, Vol 31. Pitman Advanced Publishing Program, Boston
McLachlan NW (1963) Bessel functions for engineers, 2nd edn. Oxford University, London
McNolty F, Tomsky J (1972) Some properties of special functions bivariate distributions. Sankhya Ser B 34:251–264
Mehra AN (1971) On certain definite integrals involving the Fox's H-function. Univ Nac Tucumán Rev Ser A 21:43–47
Meijer CS (1940) Über eine Erweiterung der Laplace-Transformation. Neder Akad Wetensch Proc 43:599–608, 702–711 = Indag Math 2:229–238, 269–278
Meijer CS (1941a) Eine neus Erweiterung der Laplace-Transform. Neder Akad Wetensch Proc 44 727–737, = Indag Math 3: 338–348
Meijer CS (1941b) Multiplikations theoreme für die funktion $G_{p,q}^{m,n}(z)$. Nederl Akad Wetensch Proc 44:1062–1070
Meijer CS (1946) On the G-function I-VIII. Neder Akad Wetensch Proc 49:227–237; 344–356; 457–469; 632–641; 765–772; 936–943; 1063–1072; 1165–1175; = Indag Math 8:124–134; 213–225; 312–324; 391–400; 468–475; 595–602; 661–670; 713–723
Meijer CS (1952–1956) Expansion theorems for the G-function, I-XI. Neder Akad Wetensch, Proc Ser A 55= Indag Math 14:369–379; 483–487 (1952); Proc Ser A 56= Indag Math 15 43–49, 187–193, 349–357 (1953); Proc Ser A 57= Indag Math 16 77–82, 83–91, 273–279 (1954); Proc Ser A 58= Indag Math 17: 243–251, 309–314 (1955); Proc Ser A 59= Indag Math 17:70–82 (1956)
Mellin HJ (1910) Abrip einer einhaitlichen Theorie der Gamma und der Hypergeometrischen Funktionen Math Ann 68:305–337
Metzler R, Glöckle WG, Nonnenmacher TF (1994) Fractional model equation for anomalous diffusion. Physica A 211:13–24
Metzler R, Klafter J (2000) The random walk a guide to anomalous diffusion: a fractional dynamics approach. Phys Rep 339:1–77
Metzler R, Klafter J (2004) The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. J Phys A Math Gen 37: R161–R208
Meulenbeld B (1958) Generalized Legendre's associated functions for real values of the argument numerically less than unity. Neder Akad Wetensch Proc Ser A 61:557–563
Meulenbeld B, Robin L (1961) Noveaux resultants relatifs aux functions de Legendre generalisees. Neder Akad Wetensch Proc Ser A 64:333–347
Mikusinski J (1959) On the function whose Laplace transform is $\exp(-s^{\alpha}), 0 < \alpha < 1$. Studia Math J 18:191–198
Miller KS, Ross B (1993) An introduction to the fractional calculus and fractional differential equations. Wiley, New York
Milne-Thomson LM (1933) The calculus of finite differences. Macmillan, London
Mirervski SP, Boyadjiev L, Scherer R (2007) On the Riemann-Liouville fractional calculus, g-Jacobi functions and F-Gauss functions. Appl Math Comput 187:315–325
Misra OP (1972) Some abelian theorems for the generalized Meijer-Laplace transformation. Indian J Pure Appl Math 3:241–247
Misra OP (1981) Distributional G-transformation. Bull Calcutta Math Soc 73:247–255
Mittag-Leffler GM (1903) Sur la nouvelle fonction $E_{\alpha}(x)$. CR Acad Sci Paris 137:554–558
Mittag-Leffler GM (1905) Sur la representation analytique d'une fonction monogene (cinquieme note). Acta Math 29:101–181

Mittal PK (1971) Certain properties of Meijer's G-function transform involving the H-function. Vijnana Parishad Anusandhan Patrika 14:29–38

Mittal PK, Gupta KC (1972) An integral involving generalized function of two variables. Proc Indian Acad Sci Sect A 75:117–123

Mourya DP (1970a) Analytic continuations of generalized hypergeometric functions of two variables. Indian J Pure Appl Math 1(4):464–469

Mourya DP (1970b) The generalized hypergeometric functions of two variable, its analytic continuations and asymptotic expansions. Ph.D. Thesis, University of Indore, Indore, India

Muirhead RJ (1975) Expressions for some hypergeometric functions of matrix argument with applications. J Mult Anal 5:283–293

Munot PC (1972) Some formulae involving generalized Fox's H-functions of two variables Portugal Math 31(4):203–213

Munot PC, Kalla SL (1971) On an extension of generalized function of two variables. Univ Nac Tucumán Rev Ser A 21:67–84

Murray JD (2003) Mathematical Biology. Springer, New York

Nair VC (1971) On the Laplace transform-I Portugal Math 30:57–69

Nair VC (1972a) Differentiation formulae for the H-function-I. Math Student 40A:74–78

Nair VC (1972b) The Mellin transform of the product of Fox's H-function and Wright's generalized hypergeometric function. Univ Studies Math 2:1–9

Nair VC (1973a) Differentiation formulae for the H-function-II J Indian Math Soc (NS) 37: 329–334

Nair VS (1973b) Integrals involving the H-function where the integration is with respect to a parameter. Math Student 41:195–198

Nair VC, Nambudiripad KBM (1973) Integration of H-functions with respect to their parameters. Proc Natl Acad Sci India Sect A 43:321–324

Nair VC, Samar MS (1971a) An integral involving the product of three H-functions. Math Nachr 49:101–105

Nair VC, Samar MS (1971b) The product of two H-functions expressed as a finite integral of the sum of a series of H-functions. Math Education 5A, 45A, 48:101–105

Nagarsenker BN, Pillai KCS (1973) Distribution of the likelihood ratio criterion for testing hypothesis specifying a covariance matrix. Biometrika 60:359–364

Nagarsenker BN, Pillai KCS (1974) Distribution of the likelihood ratio criterion for testing $\Sigma = \Sigma_o, \mu = \mu_o$. J Multivariate Analysis 4:114–122

Narain R (1967) Fractional integration and certain dual integral equations. Math Zeitschr 98:83–88

Narain R (1965) A pair of unsymmetrical Fourier kernels. Trans Amer Math Soc 115:356–369

Nasim C (1983) Integral operators involving Whittaker functions. Glasgow Math J 24:139–148

Nasim C (1982) An integral equation involving Fox's H-function. Indian J Pure Appl Math 13:1149–1162

Nath R (1972) On an integral involving the product of three H-functions. CR Acad Bulgare Sci 25:1167–1169

Nguyen TH, Yakubovich SB (1992) The Double Mellin-Barnes type integrals and their applications to convolution theory. World Scientific, River Edge

Nicolis G, Prigogine I (1977) Self-organization in nonequilibrium systems: from dissipative structures to order through fluctuations. Wily, New York

Nielsen N (1906) Handbuch der Theorie der Gamma Funktion. B. G. Teubner, Leipzig

Nigam HN (1969) A note on Fox's H-function. Ganita 20(2):47–52

Nigam HN (1970) Integral involving Fox's H-function and integral function of two complex variables I Ganita 21(2):71–78

Nigam HN (1972) Integral involving Fox's H-function and integral function of two complex variables II. Bull Calcutta Math Soc 64:1–5

Nigmatullin RR (1986) The realization of the generalized transfer equation in a medium with fractal geometry. Phys Sta Sol(b) 133:425–430

Nigmatullin RR (1992) Fractional integral and its physical interpretation. Soviet J Theor and Math Phys 90(3):354–367

Nishimoto K (1984, 1987, 1989, 1991, 1996) Fractional calculus, Vol 1 (1984), Vol 2 (1987), Vol 3 (1989), Vol 4 (1991), Vol 5 (1996), Descartes Press, Koriyama, Japan
Nishimoto K (1989) Fractional calculus and its applications. Nishimoto K (ed). Proceedings of the International Conference at Nihon University. Nihon University, Koriyama, Japan
Nishimoto K (1991) An essence of Nishimoto's fractional calculus (Calculus of the 21st century) integration and differentiation of arbitrary order. Descartes, Koriyama, Japan
Nishimoto K (2006) Applications of N-fractional calculus to some triple infinite, finite and mixed sums. J Frac Calc 30:75–88
Nishimoto K, Saxena RK (1991) An application of Riemann-Liouville operators in the unification of certain functional relations. J College Engg Nihon University B32:133–139
Nishimoto K, Srivastava HM (1989) Certain classes of infinite series summable by means of fractional calculus. J College Engg Nihon Univ Ser B 30:97–106
Nonnenmacher TF (1990) Fractional integral and differential equations for a class of Lévy-type probability densities. J Phys A: Math Gen 23:L697–L700
Nonnenmacher TF, Nonnenmacher DF (1989) A fractal scaling law for protein gating kinetics Physics Letters A 140:323–326
Nonnenmacher TF, Metzler R (1995) On the Riemann-Liouville fractional calculus and some recent applications. Fractals 3(3):557–566
Nonnenmacher TF, Metzler R (2001) Applications of fractional calculus techniques to problems of biophysics. In: Hilfer R (ed) Applications of fractinal calculus in physics. World Scientific, Singapore, pp 377–427
Oldham KB, Spanier J (1974) The fractional calculus: theory and applications of differentiation and integration to arbitrary order. Academic, New York
Oliver ML, Kalla SL (1971) On the derivative of Fox's H-function. Acta Mexicana Ci Tech 5:3–5
Ortiz GL (1969) The Tau method. SIAM J Numer Anal 6:480–492
Olkha GS (1970) Some finite expansions for the H-function. Indian J Pure Appl Math 1(3): 425–429
Orsingher E, Beghin L (2004) Time-fractional telegraph equations and telegraph processes with Brownian time. Probability Theory and Related Fields 128:141–160
Orsingher E, Zhao X (2003) The space fractional telegraph equation and the related fractional telegraph process. Chinese Ann Math 24B(1):1–12
Panda R (1973) Some integrals associated with the generalized Lauricella functions. Publ Inst Math (Beograd) Nouvelle Ser 16(30):115–122
Pandey RN, Srivastava HM (1993) Fractional calculus and its applications involving certain classes of functional relations. Studies in Applied Math 89:153–165
Parashar BP (1967) Fourier series for H-functions. Proc Cambridge Philos Soc 63:1083–1085
Paris RB, Kaminski D (2001) Asymptotic and Mellin-Barnes Integrals. Cambridge University Press, Cambridge
Pathak RS (1970) Some results involving G- and H-functions. Bull Calcutta Math Soc 62:97–106
Pathak RS (1973) Finite integrals involving products of H-function and hypergeometric function. Progress Math Allahabad 7(1):45–72
Pathak RS (1979) On the Meijer transform of generalized functions. Pacific J Math 80:523–536
Pathak RS (1981) Abelian theorem for the G-transformation. J Indian Math Soc (NS) 45:243–249
Pathak RS (1985) On Hankel transformable spaces and Cauchy problem. Canadian J Math 37: 84–106
Pathak RS (1997) Integral transforms of generalized functions and their applications. Gordon and Breach Science, Amsterdam
Pathak RS, Prasad V (1972) The solution of dual integral equations involving H-functions by a multiplying factor method. Indian J Pure Appl Math 3:1099–1107
Pathan MA (1968) Certain recurrence relations. Proc Cambridge Philos Soc 64:1045–1048
Pathan MA, Kamarujjama M, Alam MK (2003) On multi-indices and multivariables presentation of the Voigt functions. J Comput Appl Math 160:251–257
Pendse A (1970) Integration of H-function with respect to its parameters. Vijnana Parishad Anusandhan Patrika 13:129–138

Phillips PC (1989) Fractional matrix calculus and the distribution of multivariate tests. Cowles Foundation Paper 767; Department of Economics, Yale University, New Haven, Connecticut

Phillips PC (1990) Operational calculus and regression t-tests. Cowles Foundation Paper 948; Department of Economics, Yale University, New Haven, Connecticut

Pillai KCS, Al-Ami S, Jouris GM (1969) On the distributions of the roots of a covariance matrix and Wilks' criterion for tests of three hypotheses. Ann Math Statist 40:2033–2040

Pincherle S (1888) Sulle funzioni ipergeometriche generalizzante. Note I, Atta della Reale Accademie dei Lincei, Rendiconti della classe di Scienza Fisiche Mathematiche e Naturali (Roma) 4:694–700

Podlubny I (1997) Riesz potential and Riemann-Liouville fractional integrals and derivatives of Jacobi polynomials. Appl Math Lett 10(1):103–108

Podlubny I (1999) Fractional differential equations. Academic, San Diego

Podlubny I (2002) Geometric and physical interpretation of fractional integration and fractional differentiation. Fract Calc Appl Anal 5(4):367–386

Post EL (1930) Generalized differentiation. Trans Amer Math Soc 32:723–781

Prabhakar TR (1969) Two singular integral equations involving confluent hypergeometric functions. Proc Cambridge Philos Soc 66:71–89

Prabhakar TR (1971) A singular integral equation with a generalized Mittag-Leffler function in kernel. Yokohama Math J 19:7–15

Prajapat JK, Raina RK, Srivastava HM (2007) Some inclusion properties for certain subclasses of strongly starlike and strongly convex functions involving a family of fractional integral operators. Integral Transforms and Special Functions 18(9):639–651

Prasad YN, Shyam Dhir Ram (1973) On some double integrals involving Fox's H-function. Progress Math Allahabad 7 (2):13–20

Prieto AI, Matera J, Srivastava HM (2006) Use of the generalized Lommel-Wright function in a unified presentation of the gamma-type functions occurring in diffraction theory and associated probability distributions. Integral Transform Spec Funct 17(5):365–378

Prieto AI, De Romero SS, Srivastava HM (2007) Some fractional calculus results involving the generalized Lommel-Wright and related functions. Appl Math Lett 20(1):17–22

Prudnikov AP, Brychkov YA, Marichev OI (1986) Integrals and series, Vol I, elementary functions. Gordon and Breach Science, New York

Prudnikov AP, Brychkov YA, Marichev OI (1986a): Integrals and series, Vol 2, special functions. Gordon and Breach Science, New York

Prudnikov AP, Brychkov YA, Marichev OI (1990) Integrals and series, Vol 3, more special functions. Gordon and Breach Science, New York

Ragab FM (1963) Expansions of Kampé de Fériet's double hypergeometric functions of higher order. J Reine Angew Math 2(212):113–119

Ragab FM (1967) Infinite series of Kampé de Fériet's double hypergeometric functions of higher order. Rend Circ Mat Palermo (2) 16:225–232

Ragab FM, Hamza AM (1970) Integrals involving E-functions and Kampé de Fériet's function of higher order. Ann Mat Pura Appl 87(4):11–24

Raina RK (1976) Some recurrence relations for the H-function. Math Educ 10:A45–A49

Raina RK (1979) On certain expansions involving H-function. Comment Math Univ St Pauli 28(2):115–119

Raina RK (1986) The H-function transform and the moments of probability distribution function of an arbitrary order. Simon Stevin 60:97–103

Raina RK, Bolia M (1986) On distortion theorems involving generalized fractional calculus operators. Tamkang J Math 27(3):233–241

Raina RK, Bolia M (1997) The decomposition structure of a generalized hypergeometric transformation of convolution type. Computers Math Appl 34(9):87–93

Raina RK, Kalia RN (1998) On convolution structures for H-function transformations. Analysis Mathematica 24:221–239

Raina RK, Koul CL (1977) Fractional derivatives of the H-function. Jnanabha 7:97–105

Raina RK, Koul CL (1979) The Weyl fractional calculus. Proc Amer Math Soc 73:188–192

Raina RK, Koul CL (1981) On Weyl fractional calculus and H-function transform. Kyungpook Math J 21(2):275–279

Raina RK, Saigo M (1993) A note on fractional calculus operators involving Fox's H-function on space $F_{p,\mu}$. In: Recent advances in fractional calculus, Global Research Notes Ser Math Global Sauk Rapids, pp 219–229

Raina RK, Saigo M (1997) On inter-connection properties associated with H-transform and certain fractional integrals on spaces of generalized functions. J Fract Calc 12:83–94

Raina RK, Srivastava HM (1993) Evaluation of certain class of Eulerian integrals. J Phys A: Math Gen 26:691–696

Rainville ED (1965) Special functions. MacMillan, New York

Rajković PM, Marinković, Sladana D, Stanković MS (2007) Fractional integrals and derivatives in q-calculus. Appl Anal Discrete Math 1(1):311–323

Rakesh SL (1973) Integrals involving products of generalized hypergeometric function and generalized H-function of two variables-I. Univ Nac Tucumán Rev Ser A 23:281–288

Rakesh SL (1973a) Recurrence relations. Defene Sci J 23:79–84

Rall LB (1969) Computational solution of nonlinear operator equations. Wiley, New York

Rao CR (1973) Linear statistical inference and its applications, 2nd edn. Wiley, New York

Rathie CB (1956) A theorem in operational calculus and some integrals involving Legendre, Bessel and E-functions. Proc Glasgow Math Assoc 2:173–179

Rathie CB (1960) Integrals involving E-functions. Proc Glasgow Math Assoc 4:186–187

Rathie PN (1967) Some finite integrals involving F_4 and H-functions. Proc Cambridge Philos Soc 63:1071–1081

Rathie PN (1967a) Some finite and infinite series for F_c, F_4, ψ_2 and G-function. Math Nachr 35:125–136

Rathie AK (1997) A new generalization of generalized hypergeometric functions. Le Matematische 52:287–310

Reed IS (1944) The Mellin type of double integral. Duke Math J 11:565–572

Riemann B (1847) Versuch einer Auffassung der Integration and Differentiation. In: Gesammelte Werke 1876 edn. Publ posthumously, pp 331–344; 1892 edn. pp 353–366, Teuber, Leipzig. Also in: Collected Works (Webber H, ed) Dover, New York, pp 354–360, 1953

Riesz M (1949) L'Integrale de Riemann-Liouville et le probleme de Cauchy. Acta Math 81:1–223

Rooney PG (1983/1984) On integral transformations with G-function kernels. Proc Royal Soc Edinburgh Sect A 93(1–4):265–297

Rooney PG (1994) On the range of an integral transformation. Canadian Math Bull 37:545–548

Ross B (ed)(1975) Fractional calculus and its applications. Lecture Notes in Mathematics Vol 457. Springer-Verlag, Belin

Ross B (1993) A formula for the fractional integration and differentiation of $(ax+b)^c$. J Frac Calc 5:87–89

Rusev P, Dimovski I, Kiryakova VS (eds) (1995 and 1997) Transform methods and special functions I,II. Science Culture Technology, Singapore

Rutnam RS (1994) On physical interpretations of fractional integration and differentiation. Theor and Math Phys 105(3):1509–1519

Sahai Gopalji (1972) An expansion formula for the generalized function of two variables. Bull Math Soc Sci Math RS Roumanie (NS) 16(64):83–92

Saichev A, Zaslavsky GM (1997) Fractional kinetic equations: solution and applications. Chaos 7(4):753–764

Saigo M (1978) A remark on integral operators involving the Gauss hypergeometric functions. Math Rep Kyushu University 11:135–143

Saigo M (1979) A certain boundary value problem for the Euler-Darboux equation. Math Japon 24:377–385

Saigo M (1980) A certain boundary value problem for the Euler-Darboux equation II. Math Japon 25:211–220

Saigo M (1981) A certain boundary value problem for the Euler-Darboux equation III. Math Japon 26:103–119

Saigo M, Glaeske H-J (1990) Fractional calculus operators involving the Gauss function in spaces $F_{p,\mu}$ and $F'_{p\mu}$. Math Nachr 147:285–306

Saigo M, Kilbas AA (1996) Compositions of generalized fractional calculus operators with Fox's H-function and a differential operator in axisymmetric potential theory (Russian) Dokl Akad Nauk Belarusi 40(6):12–17

Saigo M, Kilbas AA (1999) Generalized fractional calculus of the H-function. Fukuoka University Sci Rep 29:31–45

Saigo M, Maeda N (1998) More generalization of fractional calculus. Transform Methods & Special Functions 386–400, Varna 96 Proc 2nd Intern Workshop, Bulgar Acad Sci, Sofia

Saigo M, Raina RK (1991) On the fractional calculus operators involving Gauss' series and its application to certain statistical distributions. Rev Técn Fac Ingr Uni Zulia 14:53–62

Saigo M, Raina RK, Kilbas AA (1993) On generalized fractional calculus operators and their compositions with the axisymmetric differential operator of the potential theory on spaces $F_{p,\mu}$ and $F'_{p,\mu}$. Fukuoka University Sci Rep 23:133–154

Saigo M, Saxena RK (1998) Applications of generalized fractional calculus operators in the solution of an integral equation. J Frac Calc 14:53–63

Saigo M, Saxena RK (1999a) Unified fractional integral formulas for the multivariable H-function. J Frac Calc 15:91–107

Saigo M, Saxena RK (1999b) Unified fractional integral formulas for the multivariable H-function-II. J Frac Calc 16:99–110

Saigo M, Saxena RK (2001) Unified fractional integral formulas for the multivariable H-function-III. J Frac Calc 20:45–68

Saigo M, Saxena RK, Ram J (1992a) Certain properties of operators of fractional integration associated with Mellin and Laplace transformations. In: Srivastava HM, Owa S (eds) Current topics in analytic function theory. World Scientific Publishing, River Edge, pp 291–304

Saigo M, Saxena RK, Ram J (1992b) On the fractional calculus operator associated with H-function. Ganit Sandesh 6:36–47

Saigo M, Saxena RK, Ram J (2005) Fractional integration of the product of Appell function F_3 and multivariable H-function. J Frac Calc 27:31–42

Sakmann B, Nehar E (1983) Single Channel Recording. Plenum, New York

Saksena KM (1967) An inversion theory for the Laplace integral. Nieuw Arch Wisk (3)15:218–224

Samar MS (1973) Integrals involving the H-functions, the integration being with respect to a parameter. J Indian Math Soc 37:323–328

Samar MS (1974) Double integrals involving the product of Bessel, F and H-functions. Vijnana Parishad Anusandhan Patrika 14:89–95

Samko SG, Kilbas AA, Marichev OI (1993) Fractional integrals and derivatives: theory and applications. Gordon and Breach Science, Yverdon, 1993

Sansone G, Gerretsen JCH (1960) Lectures on the theory of functions of a complex variable I. Noordohof, Groningen

Saran S (1954) Hypergeometric functions of three variables. Ganita 5:77–99

Saran S (1965) A definite integral involving the G-function. Nieuw Arch Wisk (2)13:223–229

Saxena RK (1968) Definite integrals involving self-reciprocal functions. Proc Nat Inst Sci, India, Sect A 34:326–336

Saxena RK (1960) Some theorems on generalized Laplace transform-I. Proc Natl Inst Sci India Part A 26:400–413

Saxena RK (1960a) An integral involving G-function. Proc Nat Inst Sci India Part A 26:661–664

Saxena RK (1961) Some theorems in operational calculus and infinite integrals involving Bessel function and G-functions. Proc Nat Inst Sci India Part A 27:38–61

Saxena RK (1961a) A definite integral involving associated Legendre function of the first kind. Proc Cambridge Philos Soc 57:281–283

Saxena RK (1962) Definite integrals involving G-functions. Proc Cambridge Philos Soc 58: 489–491

Saxena RK (1963) Some formulae for the G-function. Proc Cambridge Philos Soc 59:347–350

Saxena RK (1963a) Some formulae for the G-function-II. Collect Math 15:273–283

Saxena RK (1964) Integrals involving G-functions. Ann Soc Sci Bruxelles Ser I 8:151–162
Saxena RK (1964a) Integrals involving products of Bessel functions. Proc Glasgow Math Assoc 6:130–132
Saxena RK (1964b) On some results involving Jacobi polynomials. J Indian Math Soc(NS) 28: 197–202
Saxena RK (1966) Integrals involving products of Bessel functions-II. Monatsh Math 70:161–163
Saxena RK (1966a) An inversion formula for a kernel involving a Mellin-Barnes type integral. Proc Amer Math Soc 17:771–779
Saxena RK (1966b) An integral involving products of G-functions. Proc Natl Acad Sci India Sect A 36:47–48
Saxena RK (1966c) On the reducibility of Appell's function F_4. Canadian Math Bull 9:215–222
Saxena RK (1967) On the formal solution of certain dual integral equations involving H-functions. Proc Cambridge Philos Soc 63:171–178
Saxena RK (1967a) On the formal solution of dual integral equations. Proc Amer Math Soc 18:1–8
Saxena RK (1967b) On fractional integration operators. Math Z 96:288–291
Saxena RK (of Kolhapur) (1968) Definite integrals involving self-reciprocal functions. Proc Nat Inst Sci India Sect A 34:326–336
Saxena RK (1970) Integrals involving Kampé de Fériet function and Gauss' hypergeometric functions. Ricerca (Napoli) 2:21–27
Saxena VP (1970) Inversion formulae to certain integral equations involving H-function. Portugal Math 29(1):31–42
Saxena RK (1971) Integrals of products of H-functions. Univ Nac Tucumán Rev SerA 21:185–191
Saxena RK (1971a) Definite integrals involving Fox's H-function. Acta Mexicana Ci, Tech 5(1): 6–11
Saxena RK (1971b) An integral associated with generalized H-function and Whittaker functions. Acta Mexicana Ci Tech 5(3):149–154
Saxena RK (1973) Integration of certain product associated with Bessel and confluent hepergeometric functions. Bull Math Soc RS Roumanie 16(64):93–96
Saxena RK (1973a) Abelian theorems for the distributional H-transform. Acta Mexicana Ci Tech 7:66–76
Saxena RK (1974) On a generalized function of n variables. Kyungpook Math J 14:255–259
Saxena RK (1977) On the H-function of n variables. Kyungpook Math J 17:221–226
Saxena RK (1980) On the H-function of n variables II. Kyungpook Math J 20(2):273–278
Saxena VP (1982) Formal solution of certain new pair of dual integral equations involving H-functions. Proc Natl Acad Sci India Sect A 52:366–375
Saxena RK (1998) Functional relations involving generalized H-function. Le Matematiche LIII:123–131
Saxena RK (2003) Alternative derivation of the solution of certain integro-differential equations of Volterra-type. Ganita Sandesh 17:51–56
Saxena RK, Gupta N (1994) Some abelian theorems for distributional $\bar{H}$-function transformation. Indian J Pure Appl Math 25:869–879
Saxena RK, Gupta N (1995) A complex inversion theorem for a modified H-transformation of distributions. Indian J Pure Appl Math 26:1111–1117
Saxena RK, Gupta N (1995a) Some characterizations of the H-transform for distributions. The Math Student 64(1–4):79–86
Saxena RK, Gupta N (1995b) On the asymptotic expansion of generalized Stieltjes transform. The Mathematics Student 64(1–4):51–56
Saxena RK, Gupta N (1997) On distributional generalized H-transformation. The Math Student 66(1–4):249–259
Saxena RK, Kalla SL (2000) A new method for evaluating Epstein-Hubbell generalized elliptic-type integral. Int J Appl Math 2:732–742
Saxena RK, Kalla SL (2003) On a fractional generalization of the free electron laser equation. Appl Math Comput 143:89–97

Saxena RK, Kalla SL (2004) Asymptotic formulas for unified elliptic-type integrals. Integral Transform Spec Funct 15:359–368

Saxena RK, Kalla SL (2005) Solutions of Volterra-type integro-differential equations with a generalized Lauricella confluent hypergeometric function in the kernels. Internat J Math Math Sci 8:1155–1170

Saxena RK, Kalla SL (2006) On a unified mixture distribution. Appl Math Comput 182:325–332

Saxena RK, Kalla SL (2007) On a generalization of Krätzel function and associated inverse Gaussian probability distributions. Algebras, Groups and Geometries 24(3):303–324

Saxena RK, Kalla SL (2008) On the solutions of certain fractional kinetic equations. Appl Math Comput 199:504–511

Saxena RK, Kumar R (1995) A basic analogue of the generalized H-function. Le Matematiche Vol L:263–271

Saxena RK, Kumbhat RK (1975) Some properties of generalized Kober operators. Vijnana Parishad Anusandhan Patrika 18:139–150

Saxena RK, Kumbhat RK (1974) Integral operators involving H-function. Indian J Pure Appl Math 5(1):1–6

Saxena RK, Kumbhat RK (1974a) Dual integral equations associated with H-function. Proc Nat Acad Sci Allahabad, Sect A 44:106–112

Saxena RK, Kumbhat RK (1974b) A formal solution of certain triple integral equations involving H-function. Proc Nat Acad Sci Allahabad Part II, Sect A 44:153–160

Saxena RK, Kumbhat RK (1973) Fractional integration operators of two variables. Proc Indian Acad Sci Bangalore 78(4):177–186

Saxena RK, Kumbhat RK (1973a) A generalization of Kober operators. Vijnana Parishad Anusandhan Patrika 16:31–36

Saxena RK, Kushwaha RS (1972) Certain dual integral equations associated with a kernel of Fox. Proc Nat Acad Sci India Sect A 42:39–45

Saxena RK, Kushwaha RS (1972a) An inetegral transform associated with a kernel of Fox Math Student 40:201–206

Saxena RK, Mathur SN (1971) A finite series of the H-functions. Univ Nac Tucumán Rev Ser A 21:49–52

Saxena RK, Modi GC (1974) Some expansions involving H-function of two variables. Comptes Rendus de l'Academie Bulgare des Sci 27(2):165–168

Saxena RK, Modi GC (1980) Multidimensional fractional integration operators associated with hypergeometric functions. Nat Acad Sci Lett 3:153–157

Saxena RK, Modi GC (1985) Multidimensional fractional integration operators associated with hypergeometric functions-II. Vijnana Parishad Anusandhan Patrika 28:87–97

Saxena RK, Nishimoto K (1994) Fractional integral formula for the H-function. J Fract Calc 6: 65–75

Saxena RK, Nishimoto K (2002) On a fractional integral formula for Saigo operator. J Fract Calc 22:57–58

Saxena RK, Nishimoto K (2006) N-fractional calculus of power functions. J Fract Calc 29:57–64

Saxena RK, Nishimoto K (2007) N-fractional calculus of the multivariable H-functions. J Fract Calc 31:43–52

Saxena RK, Nonnenmacher TF (2004) Application of H-function in Markovian and non-Markovian chain models. Fract Calc Appl Anal 7:135–148

Saxena RK, Pathan MA (2003) Asymptotic formulas for unified elliptic-type integrals. Demonstratio Mathematica 36:579–589

Saxena RK, Ram J (1990) On certain multidimensional generalized Kober operators. Collect Math 41(1):27–34

Saxena RK, Ram C (2006) I-function and equation of internal blood pressure. Acta Ciencia Indica 32(2):539–541

Saxena RK, Saigo M (1998) Fractional integral formula for the H-function-II. J Frac Calc 13:37–41

Saxena RK, Saigo M (2001) Generalized fractional calculus of the H-function associated with the Appell function F_3. J Fract Calc 19:89–104

Saxena RK, Saigo M (2005) Certain properties of fractional calculus operators associated with generalized Mittag-Leffler function. Frac Calc Appl Anal 8:141–154

Saxena RK, Sethi PL (1973) Relations between generalized Hankel and modified hypergeometric function operators. Proc Indian Acad Sci 78(6):267–273

Saxena RK, Sethi PL (1973a) Certain properties of bivariate distributions associated with generalized hypergeometric functions. Canadian J Statist 1(2):171–180

Saxena RK, Sethi PL (1973b) A formal solution of dual integral equations associated with H-function of two variables. Univ Nac Tucumán Rev Ser A 23:121–130

Saxena RK, Sethi PL (1975) Applications of fractional integration operators to triple integral equations. Indian J Pure Appl Math 6(5):512–521

Saxena RK, Singh Y (1993) Integral operators involving generalized H-function. Indian J Math 35:177–188

Saxena RK, Soni MK (1997) On unified fractional integration operators. Math Balkanica 11:69–77

Saxena RK, Yadav RK (1995) A basic analogue of the generalized H-function. Le Mathematische 50:263–271

Saxena RK, Koranne VD, Molgonde SP (1985) On a distributional generalized Stieltjes transformation. J Indian Acad Math 7:105–110

Saxena RK, Kalla SL, Bora SL (1971) Addendum to a paper on integral transform. Univ Nac Tucumán rev Ser A 21:289

Saxena RK, Kalla SL, Hubbell JH (2001) Asymptotic expansion of a unified elliptic-type integral Math Balkanica 15:387–396

Saxena RK, Kalla SL, Kiryakova VS (2003) Relations connecting multi-index Mittag-Leffler functions and Riemann-Liouville fractional calculus. Algebras Groups and Geometries 20:363–386

Saxena RK, Kiryakova VS, Dave OP (1994) Unified approach to certain fractional integration operators. Math Balkanica, New Series 8:211–219

Saxena RK, Mathai AM, Haubold HJ (2006) Solutions of certain fractional kinetic equations and a fractional diffusion equation. J Scientific Res 15:1–17

Saxena RK, Mathai AM, Haubold HJ (2008a) Solutions of fractional reaction-diffusion equations in terms of the H-function-II. (preprint)

Saxena RK, Mathai AM, Haubold HJ (2008b) An alternative method for solving a certain class of fractional kinetic equations. (preprint)

Saxena RK, Mathai AM, Haubold HJ (2002) On fractional kinetic equations. Astrophysics & Space Science 282:281–287

Saxena RK, Mathai AM, Haubold HJ (2004a) Unified fractional kinetic equation and a fractional diffusion equation. Astrophysics & Space Science 290:299–310

Saxena RK, Mathai AM, Haubold HJ (2004b) Astrophysical thermonuclear functions for Boltzmann-Gibbs statistics and Tsallis statistics. Physica A 344:649–656

Saxena RK, Mathai AM, Haubold HJ (2004c) On generalized fractional kinetic equations. Physica A 344:657–664

Saxena RK, Mathai AM, Haubold HJ (2006a) Fractional reaction-diffusion equations. Astrophysics & Space Science 305:289–296

Saxena RK, Mathai AM, Haubold HJ (2006b) Reaction-diffusion systems and nonlinear waves. Astrophysics & Space Science 305:297–303

Saxena RK, Mathai AM, Haubold HJ (2006c) Solution of generalized fractional reaction-diffusion equations. Astrophysics & Space Science 305:305–313

Saxena RK, Mathai AM, Haubold HJ (2006d) Solutions of fractional reaction-diffusion equations in terms of Mittag-Leffler functions. Int J Sci Res 15:1–17

Saxena RK, Mathai AM, Haubold HJ (2007) Solution of certain fractional kinetic equations and a fractional diffusion equation. Int J Sci Res 17:1–8

Saxena RK, Ram C, Kalla SL (2002) Applications of generalized H-function in bivariate distributions. Rev Acad Canar Cienc 14(1–2):111–120

Saxena RK, Ram J, Chandak S (2005) Integral formulas for the H-function generalized fractional calculus associated with Erdélyi–Kober operator of Weyl type. Acta ciencia Indica 31:761–766

Saxena RK, Ram J, Chandak S (2007a) Integral formulas for the generalized Erdélyi-Kober operator of Weyl type. J Indian Acad Math 29(2):495–504

Saxena RK, Ram J, Chandak S (2007b) Unified fractional integral formulas involving the I-function associated with the modified Saigo operator. Acta Ciencia Indica 33(3):693–704

Saxena RK, Ram J, Chauhan AR (2002) Fractional integration of the product of I-function and Appell function F_3. Vijnana Parishad Anusandhan Patrika 45(4):345–371

Saxena RK, Ram J, Kalla SL (2002) Unified fractional integral formulas for the generalized H-function. Rev Acad Canar 14:97–109

Saxena RK, Ram J, Suthar DL (2004) Integral formulas for the H-function generalized fractional calculus. South East Asian J Math & Math Sci 3:69–74

Saxena RK, Ram J, Suthar DL (2006) Representation of H-function fractional integration operators in terms of $L\&L^{-1}$ operators. Acta Ciencia Indica 32:643–650

Saxena RK, Ram J, Suthar DL (2007) Integral formulas for the H-function generalized fractional calculus II. South East Asian J Math& Math Sci 5:23–31

Saxena RK, Ram J, Suthar DL, Kalla SL (2006) On a generalized Wright transform. Algebras, Groups and Geometries 23:25–42

Saxena RK, Yadav RK, Purohit SD, Kalla SL (2005) Kober fractional q-integral operator of the basic analogue of the H-function. Rev Tech Ing Univ 28(2):154–158

Schissel H, Metzler R, Blumen A, Nonnenmacher TF (1995) Generalized viscoelastic models: their equations with solutions. J Phyhs A: Math Gen 28:6567–6584

Schneider WR (1986) Stable distributions, Fox function representation and generalization. In: Albeverio S, Casati G, Merilini D (eds) Stochastic processes in classical and quantum systems, Lecture Notes in Physics, Vol 262. Springer, Berlin, pp 497–511

Schneider WR, Wyss W (1989) Fractional diffusion and wave equations. J Math Phys 30:134–144

Sear (1964) Astrophys J 140:477

Shah M (1969) Some results on Fourier series for H-functions. J Natur Sci Math 9(1):121–131

Shah M (1969a) Some results on the H-functions involving the generalized Leguerre polynomial. Proc Cambridge Philos Soc 65:713–720

Shah M (1969b) On some results involving H-functions and associated Legendre functions. Proc Nat Acad Sci India Sect A 39:503–507

Shah M (1969c) On application of Mellin's and Laplace's inversion formulae to H-functions. Lebadev J Sci Tech Part A 7:10–17

Shah M (1969d) On some relation of H-functions and Chebyshev polynomials of the first kind. Vijnanaparishad Anusandhan Patrika 12:61–67

Shah M (1970) Generalized function of two variables and potential about spherical surface. J Natur Sci Math 10:247–268

Shah M (1970a) Some results involving generalized function of two variables. J Natur Sci Math 10:109–124

Shah M (1971) Expansion formulas for Meijer's G-function of two variables in series of circular functions. Jnanabha Sect A 1:35–44

Shah M (1971a) A result on generalized hypergeometric function and generalized Meijer function of two variables. An Sti Univ Al I Cuza Iasi Sect I a Mat (NS) 17:331–338

Shah M (1971b) Some results on generalized functions and their applications. Proc Nat Acad Sci India Sect A 41:241–255

Shah M (1971c) Some results involving generalized Meijer functions associated with Gegenbauer (ultraspherical) polynomials. Indian J Pure Appl Math 2(3):387–400

Shah M (1971d) On Fourier series for generalized Meijer functions of two variables and their applications. Indian J Pure Appl Math 2(3):464–478

Shah M (1971e) Some results involving a generalized Meijer function. Mat Vesnik 8(23):3–16

Shah M (1972) On some problems of Fox' H-function of two variables and Gegenbauer polynomials. Istanbul Tek Univ Bull 25(2):111–120

Shah M (1972a) A note on a generalization of Edelstein's theorem on G-functions. Glasnik Mat Ser III 7(27):201–205

Shah M (1972b) On generalized Meijer's and generalized associated Legendre functions. Portugal Math 31:57–66

Shah M (1972c) Generalized Meijer function and temperature in a non-homogeneous bar. An Univ Timisoara Ser Sti Mat 10:95–101

Shah M (1972d) Expansion formulae for H-functions in series of trigonometrical functions with their applications. Math Student 40A:56–66

Shah M (1972e) On some problems leading to certain results involving generalized Meijer functions of two variables and associated Legendre functions. Math Student 40A:124–133

Shah M (1972f) On some results on H-functions associated with orthogonal polynomials. Math Scand 30:331–336

Shah M (1972g) Some results on generalization of Fox's H-functions. Bull Soc Math Phys Macedoine 23:13–24

Shah M (1973) Several properties of generalized Fox's H-functions and their applications. Protugal Math 32:179–199

Shah M (1973a) On some applications related to Fox's H-function of two variables. Publ Inst Math (Beograd), NS 16(30):123–133

Shah M (1973b) On a generalized Fox's H-function. Indian J Pure Appl Math 4(4):422–427

Shah M (1973c) A theorem on generalized Meijer function of two variables. Istanbul Tek Univ Bull 26:30–38

Shah M (1973d) A new generalized theorem on Fox's H-function. Gac Mat (Madrid) 26(1): 158–165

Sharma KC (1964) Integrals involving products of G-function and Gauss hypergeometric function. Proc Cambridge Philos Soc 60:539–542

Sharma KC (1965) On an integral transform. Math Z 89:94–97

Sharma OP (1965) Some finite and infinite integrals involving H-function and Gauss' hypergeometric functions. Collect Math 17:197–209

Sharma BL (Sharma, Bhagirath Lal; also see Abiodun, RFA)(1965) On a generalized function of two variables-I. Ann Soc Sci Bruxelles Ser 1 79:26–40

Sharma BL (1966) Integrals involving hypergeometric function of two variables. Proc Nat Acad Sci India Sect A 36:713–718

Sharma OP (1966) Certain infinite and finite integrals involving H-function and confluent hypergeometric function. Proc Nat Acad Sci India Sect A 36:1023–1032

Sharma BL (1967) Integrals associated with generalized function of two variables. Mathematica (Cluj) 9(32):361–374

Sharma BL (1967a) Integrals involving generalized function of two variables-II. Proc Nat Acad Sci India Sect A 37:137–148

Sharma BL (1968) Some formulae for generalized function of two variables. Math Vesnik 5(20):43–52

Sharma OP (1968) On the Hankel transformations of H-functions. J Math Sci 3:17–26

Sharma BL (1968a) An integral involving products of G-function and generalized function of two variables. Univ Nac Tucumán Rev Ser A 18:17–23

Sharma OP (1969) On H-function and heat production in a cylinder. Proc Nat Acad Sci India Sect A 39:355–360

Sharma BL (1971a) Sum of a series involving Laguerre polynomials and generalized function of two variables. An Sti Univ Al I Cuza Iasi, n, Ser Sect. 1 17:117–122

Sharma BL (1971b) Expansion formulae for generalized function of two variables. Bull Math Soc Sci Math RS Roumanie (NS) 15(63):237–245

Sharma BL (1972) An integral involving products of G-function and generalized function of two variables. Rev Mat Hisp Amer 32(4):188–196

Sharma OP (1972) Certain infinite integrals involving H-function and MacRobert's E-function. Lebdev J Sci Tech Part A 10:9–13

Sharma CK (1972) Fourier series for Fox's H-function of two variables. Defence Sci J 22:227–230

Sharma CK (1973) On certain finite and infinite summation formulae of generalized Fox H-functions. Indian J Pure Appl Mat 4(3):278–286

Sharma BL, Abiodun RFA (1973) New generating functions for the G-function. Ann Polon Math 27:159–162

Sharma CK, Gupta PM (1972) On certain integrals involving Fox's H-function. Indian J Pure Appl Math 3:992–995

Shlapakov SA (1994) An integral transformation with the Fox H-function in the space of summable functions (Russian). Dukl Akad Nauk Belarusi 38(2):14–18

Shlapakov SA Saigo M, Kilbas AA (1998) On inversion of H-transform in $L_{v,r}$-space. Internat J Math Math Sci 21:713–722

Shukla AK, Prajapati JC (2007) On a generalization of Mittag-Leffler function and its properties. J Math Anal Appl 336:797–811

Shukla AK, Prajapati JC (2008) A general class of polynomials associated with generalized Mittag-Leffler function. Integral Transform Spec Funct 19(1):23–34

Simary MA (1973) On hypergeometric functions of matrix argument. Bull Math Soc Sci Math RS Roumanie (NS) 16(64):111–118

Singh RP (1964) A note on Gegenbauer and Laguerre polynomials. Math Japon 9:1–4

Singh R (1970) An inversion formula for Fox's H-transform. Proc Nat Acad Sci India Sect A 40:57–64

Singh F (1972) Application of E-operator to evaluate a definite integral. J Indian Math Soc (NS) 35:217–225

Singh F (1972a) On some results associated with a generalized Meijer function. Math Student 40A:291–296

Singh F (1972b) Integration of certain products involving H-function and double hypergeometric function II. Math Student 40A:42–55

Singh F (1972c) Application of E-operator in evaluating certain finite integrals. Defence Sci J 22:105–112

Singh NP (1973) A definite integral involving generalized Fox's H-function with applications. Kyungpook Math J 13:253–264

Singh F, Varma RC (1972) Application of E-operator to ealuate a definite integral and its application in heat conduction. J Indian Math Soc (NS) 36:325–332

Skibinski P (1970) Some expansion theorems for the H-function. Ann Polon Math 23:125–138

Slater LJ (1960) Confluent hypergeometric functions. Cambridge University, Cambridge

Slater LJ (1961) Generalized hypergeometric series. Cambridge University, Cambridge

Smoller J (1983) Shock waves and reaction-diffusion equations. Springer, New York

Sneddon IN (1966) Mixed boundary value problems in potential theory. North-Holland Publishing, Amsterdam

Sneddon IN (1974) The use of integral transforms, THM Edition. McGraw-Hill, New Delhi

Sneddon IN (1975) The use in mathematical physics of Erdélyi-Kober operators, and some of their applications. In: Ross B (ed) Lecture Notes in Mathematics, Vol 457, pp 37–79

Somorjai RL, Bishop DM (1970) : Integral transformation trial functions of the fractional integral class. Phys Rev A1:1013

Soni SL (1970) Fourier series of H-function involving orthogonal polynomials. Math Edu 4: A80–A84

Srivastava HM (1964) Hypergeometric function of three variables. Ganita 15:97–108

Srivastava KN (1964) Some polynomials related to Laguerre polynomials. J Indian Math Soc (NS) 28:43–50

Srivastava HM (1968) On an extension of the Mittag-Leffler function. Yokohama J Math 16(2): 77–88

Srivastava MM (1969) Infinite series of H-functions. Istanbul Univ Fen Fak. Meem Ser A 34:79–81

Srivastava SK (1972) Fourier series for H-function of two vaiables. Math Bulkanica 2:219–225

Srivastava SK (1972a): On the H-function of two variables. Bull Math Soc Sci Math RS R n Ser 16(64):119–123

Srivastava GP (1971) Some new transformations and reducible cases of Appell's double series and their generalizations. Math Student 39:319–326

Srivastava HM (1972a) A contour integral involving Fox's H-function. Indian J Math 14:1–6

Srivastava HM (1972b) A class of integral equations involving H-function as kernel. Neder Akad Wetensch Proc Ser A 75 = Indag Math 34:212–220
Srivastava HM (1973) On the reducibility of Appell's function F_4. Canadian Math Bull 16: 295–298
Srivastava TN (1976) Certain properties of bivariate distributions involving the H-function of Fox. Canadian J Statist 4:227–236
Srivastava HM (1991) A simplified overview of certain relations among infinite series that arose in the context of fractional calculus. J Math Anal Appl 162:152–158
Srivastava HM (1992) A simple algorithm for the evaluation of a class of generalized hypergeometric series. Studies in Applied Mathematics 86:79–86
Srivastava HM (1994) A certain family of sub-exponential series. Int J Math Educ Technol 25(2):211–216
Srivastava HM (2003) Fractional calculus and its applications. Cubo Matematica Educacional 5(1):33–48
Srivastava HM, Buschman RG (1973) Composition of fractional integral operators involving Fox's H-function. Acta Mexicana de Ciencia y Technologia 7(1-2-3):21–28
Srivastava HM, Buschman RG (1974) Some convolution integral equations. Neder Akad Wetensch Proc Ser A 77(3)= Indiag Math 36(3):211–216
Srivastava HM, Buschman RG (1975) Some polynomial defined by generating relations. Trans Amer Math Soc 205:360–370
Srivastava HM, Buschman RG (1976) Mellin convolutions and H-function transformations. Rocky Mountain J Math 6(2):341–343
Srivastava HM, Buschman, RG (1992) Theory and applications of convolution integral equations Math Appl. 79 Kluwer Academic Publishing, Dordrecht
Srivastava HM, Chen, MP (1992) Some unified presentation of Voigt functions. Astrophysics & Space Science 192:63–74
Srivastava HM, Daoust MC (1969) On Eulerian integrals associated with Kampé de Fériet's function. Publ Inst Math Nouvelle Serie 9(23):199–202
Srivastava HM, Daoust MC (1969a) Certain generalized Neumann expansions associated with the Kampé de Fériet function. Neder Akad Wetensch Proc Ser A 72(5) = Indag Math 31(5): 449–457
Srivastava HM, Daoust MC (1972) A note on the convergence of Kampé de Fériet's double hypergeometric series. Math Nachr 53:151–159
Srivastava A, Gupta KC (1970) On certain recurrence relations. Math Nachr 46:13–23
Srivastava A, Gupta KC (1971) On certain recurrence relations-II. Math Nachr 49:187–197
Srivastava HM, Hussain MA (1995) Fractional integration of the H-function of several variables. Computers Math Appl 30:73–85
Srivastava HM, Joshi CM (1968) Certain integrals involving a generalized Meijer function. Glasnik Mat Ser III 3(23):183–191
Srivastava HM, Joshi CM (1969) Integration of certain products associated with a generalized Meijer function. Proc Cambridge Philos Soc 65:471–477
Srivastava HM, Karlsson PW (1985) Multiple Gaussian Hypergeometric Series. Wiley Halsted, New York and Ellis Horwood, Chichester
Srivastava HM, Manocha HL (1989) A treatise on generating functions. Ellis Horwood, Wiley, New York
Srivastava HM, Miller EA (1987) A unified presentation of the Voigt functions. Astrophysics & Space Science 135:111–118
Srivastava HM, Owa S (eds)(1992) Current topics in analytic function theory. World Scientific, Singapore
Srivastava HM, Owa S (1989) Univalent functions, fractional calculus and their applications. Halsted Press, New York (Ellis Horwood, Chichester)
Srivastava HM, Panda R (1973) Some operational techniques in the theory of special functions. Neder Akad Wetsensch Prc Ser A 76 = Indag Math 35:308–319

Srivastava HM, Panda R (1975) Some analytic or asymptotic confluent expansions for functions of several variables. Math Comput 132:1115–1128
Srivastava HM, Panda R (1975a) Some expansion theorems and generating relations for the H-function of several complex variables I. Comment Math Univ St Paul 24(2):119–137
Srivastava HM, Panda R (1976) Some bilateral generating functions for a class of hypergeometric polynomials. J Reine Agnew Math 283/284:265–274
Srivastava HM, Panda R (1976a) Expansion theorems for the H-function of several complex variables. J Reine Angew Math 288:129–145
Srivastava HM, Panda R (1976b): Some expansion theorems and generating relations for the H-function of several complex variables II. Comment Math Univ St Paul 25(2):167–197
Srivastava HM, Panda R (1976c) An integral representation for the product of two Jacobi polynomials. J London Math Soc 12(2):419–425
Srivastava HM, Raina RK (1992) On certain methods of solving a class of integral equation of Fredholm type. J Australian Math Soc 52:1–10
Srivastava HM, Saigo M (1987) Multiplication of fractional calculus operators and boundary value problems involving the Euler-Darboux equation. J Math Anal Appl 128:325–369
Srivastava GP, Saran S (1967) Integrals involving Kampé de Fériet function. Math Z 98:119–125
Srivastava HM, Saxena RK (2001) Operators of fractional integration and their applications. Appl Math Comput 118:1–52
Srivastava HM, Saxena RK (2005) Some Volterra-type fractional integro-differential equations with a multivariable confluent hypergeometric function in their kernel. J Integral Eq Appl 17:199–217
Srivastava HM, Siddiqi RN (1995) A unified presentation of certain elliptic-type integrals related to radiation field problems. Radiat Phys Chem 46(3):303–315
Srivastava HM, Singhal JP (1968) Double Meijer transformations of certain hypergeometric functions. Proc Cambridge Philos Soc 64:425–430
Srivastava HM, Singhal JP (1969) Certain integrals involving Meijer's G-function of two variables. Proc Nat Inst Sci India Part A 35:64–69
Srivastava HM, Verma RU (1970) On summation of Meijer's G-function of two variables. Indian J Math 12:137–140
Srivastava TN, Singh YP (1968) On Maitland's generalized Bessel function. Canadian Math, Bull 2:739–741
Srivastava HM, Gupta KC, Goyal SP (1982) The H-functions of one and two variables with applications. South Asian Publishers, New Delhi
Srivastava HM, Gupta KC, Handa S (1975) A certain double integral transformation. Neder Akad Wetensch Proc Ser A 78= Indag Math 37:402–406
Srivastava HM, Lin Shy-Der, Wang P-Y (2006) Some fractional calculus results for the $\bar{H}$-function associated with a class of Feynmann integrals. Russian J Math Phys 15(1):94–100
Srivastava HM Owa S, Sakina T (2007) Analytic function theory, fractional calculus and their applications. Appl Math Comput 187(1):1–2
Srivastava HM Saigo M, Raina RK (1993) Some existence and connection theorems associated with the Laplace transform and certain class of integral operators. J Math Anal Appl 172:1–10
Srivastava HM, Saxena RK, Ram J (1995) Some multidimensional fractional integral operators involving a general class of polynomials. J Math Anal Appl 193(2):373–389
Srivastava HM Saxena RK, Ram C (2005) A unified presentation of the gamma-type functions occurring in diffraction theory and associated probability distributions. Appl Math Comput 162:931–947
Srivastava HM Yakubovich SB, Luchko YF (1993) The convolution method for the development of new Leibniz rules involving fractional derivatives and of their integral analogues. Integral Transform Spec Funct 1:119–134
Stanislavsky AA (2004) Probability interpretation of the integral of fractional order. Theor Math Phys 138(3):418–431
Stankovic B (1970) On the function of EM Wright. Publ, de λ, Institut Mathematique, Nouvelle ser 10(24):113–124

Strier D, Zanette DH, Wio HS (1995) Wave fronts in a bistable reaction-diffusion system with density-dependent diffusivity. Physica A 226:310
Südland N, Baumann N, Nonnenmacher TF (1998) Open problem: who knows about the aleph ℵ-functions?. Frac Calc Appl Anal 1(4):401–402
Südland N, Baumann G, Nonnenmcher TF (2001) Fractional driftless Fokker-Planck equation with power law diffusion coefficients. In: Ganzha VG, Mayr EW, Vorozhtsov EV (eds) Computer algebra in scientific computing. Springer CASC 2001, Berlin, pp 513–528
Subrahmaniam K (1973) On some functions of matrix argument. Utilitas Math 3:83–106
Subrahmaniam K (1974) Recent trends in multivariate normal distribution theory: On the zonal polynomials and other functions of matrix argument. Technical Report No 69, University of Manitoba (Department of Statistics)
Sud K, Wright LE (1976) A new analytic continuation of Appell's hypergeometric series F_2. J Math Phys 17(9):1719–1721
Sundararajan PK (1966) On the derivative of a G-function whose argument is a power of the variable. Compositio Math 7:286–290
Swaroop R (1965) A general expansion involving Meijer's G-function. Ann Soc Sci Bruxelles Ser 1 79:47–57
Szegö G. (1939) Orthogonal polynomials. Amer Math Soc Colloquium Publ, No 23
Taxak RL (1970) Some results involving Fox's H-function and associated Legendre functions. Vijnana Parishad Anusandhan Patrika 13:161–168
Taxak RL (1971) Integration of some H-functions with respect to their parameters. Defence Sci J 21:111–118
Taxak RL (1971a) A contour integral involving Fox's H-function and Whittaker function. An Vac Ci Univ Porta 54:353–362
Taxak RL (1971b) Fourier series for Fox's H-function. Defence Sci J 21:43–48
Taxak RL (1972) Some integrals involving Bessel's functions and Fox's H-function. Defence Sci J 22:15–20
Taxak RL (1973) Some series for the Fox's H-function. Defence Sci J 23:33–36
Titchmarsh EC (1937) Introduction to the Theory of Fourier Integrals. Clarendon Press, Oxford
Titchmarsh EC (1986) Introduction to the Theory of Fourier Transforms. Chelsea Publishing, New York, 1986; first edition by Oxford University Press, Oxford
Tomovski Z (2007) Integral representations of generalized Mathieu series via Mittag-Leffler type functions. Fract Calc Appl Anal 10(2):127–138
Tonchev NS (2007) Finite size-scaling in anisotropic systems. Physical Review E 75:031110
Tonchev NS (2005) Finite-size scaling in systems with strong anisotropy: an analytic example. Communications of the Joint Institute for Nuclear Research E17-2005-148, Dubna
Toscano L (1944) Transformata di Laplace di Prodotti di fanzioni di Bessel polinomi di Laguerre FA di Louricella. Part Acad Sci (commen) 5:471–500
Toscano L (1972) Sui polinomi ipergeometriche a piu variabili del tipo F_D di Laricella. Le Matematische 27:219–250
Tranter CJ (1956) Integral transforms in mathematical physics, 2nd edn. Methuen, London
Tranter CJ (1969) Bessel function with some physical applications. Hart Publishing, New York
Tremblay R, Lavertu ML (1972) P Humbert's confluent hypergeometric function $\phi(\alpha, \beta; \gamma; x, y)$. Jnanabha Sect A 2:11–18
Tsallis C (2004) What should a statistical mechanics satisfy to reflect nature? Physica D 193:3–34
Tsallis C (1988) Possible generalization of Boltzmann-Gibbs statistics. J Stat Phys 52:479–487
Tsallis C, Bukmann DJ (1996) Anomalous diffusion in the presence of external forces: Exact time-dependent solutions and their thermostatistical basis. Physical Review E 54:R2197–R2200
Varma RS (1951) On a generalization of Laplace integral. Proc Nat Acad Sci India Sect A 20: 209–216
Varma VK (1963) On another representation of an H-function. Proc Nat Acad Sci India Sect A 33(2):275–278

Varma VK (1965) On a multiple integral representation of a kernel of Fox. Proc Cambridge Philos Soc 61:469–474

Varma VK (1966) On a new kernel and its relation with H-function of Fox. Proc Nat Acad Sci India Sect A 36(2):389–394

Vasishta S (1974) Some integrals involving the H-function of two variables. Math Education 8A:65–71

Vasishta SK, Goyal SP (1975) Certain relations between generalized Kontorovich-Lebedev transform and the H-transform. Ranchi Univ Math J 8:95–102

Verma A (1966) A note on an expansion of hypergeometric functions of two variables. Math Comp 20:413–417

Verma A (1966a) Expansions involving hypergeometric functions of two variables. Math Comp 20:590–596

Verma CL (1966) On H-function of Fox. Proc Nat Acad Sci India Sect A 36(3):637–642

Verma RU (1966) Certain integrals involving G-function of two variables. Ganita 17:43–50

Verma RU (1966a) On some integrals involving Meijer's G-function of two variables. Proc Nat Inst Sci India Sect A 32:509–515

Verma RU (1967) On some infinite series of the G-function of two variables. Mat Vesnik 19(4):265–271

Verma RU (1969/70) Reduction formula for Meijer's G-function of two variables. Univ Lisboa Rev Fac Ci A(2)13:131–133

Verma RU (1970) Addition theorem on G-function of two variables. Math Vesnik 22(7):165–168

Verma RU (1970a) Expansion formula for the G-function of two variables. An Sti Univ Al I Cuza, Iasi n Ser Sect 1a 16:289–291

Verma RU (1971) On the H-function of two variables-II. An Sti Univ Al I Cuza, Iasi Sect Ia, Mat (NS) 17:103–109

Verma RU (1971a) Integrals involving G-function of two variables-II. CR Acad Bulgare Sci 24:427–430

Verma RU (1971b) On the H-function of two variables V. An Univ Timisoara Ser Sti Mat 9: 205–209

Verma RU (1972) H-function of two variables VI. Defence Sci J 22:241–244

Verma RU (1972a) A generalization of integrals involving Meijer's G-function of two variables. Math Student 40A:40–46

Verma RU (1974) Solution of an integral equation by L and L^{-1} operators. An Sti Univ Al I Cuza, Iasi 20:381–387

Vyas RC, Saxena RK (1973) Integrals involving G-function of two variables. Univ Nac Tucumán Rev Ser A 23:17–23

Vyas RC, Saxena RK (1974) On Kummer's transform of two variables involving Meijer's G-function. Rev Mat Hisp Amer 14(4):335–338

Wilhelmsson H, Lazzaro E (1996) Reaction-diffusion problems in the physics of hot plasmas. Institute of Physics, Bristol

Wilhelmsson H, Lazzaro E (2001) Reaction-diffusion problems in the physics and hot plasmas. Institute of Physics, Bristol and Philadelphia

Wiman A (1905) Über den Fundamental salz im der Theorie der Funktionen $E_\alpha(x)$. Acta Math 29:191–201

Wiman A (1905a) Über die Nullstellun der Funktionen $E_\alpha(x)$. Acta Math 29:217–234

Wright EM (1933) On the coefficients of power series having exponential singularities. J London Math Soc 8:71–79

Wright EM (1934) The asymptotic expansion of the generalized Bessel function. Proc London Math Soc 38(2):257–270

Wright EM (1935) The asymptotic expansion of the generalized hypergeometric function. J London Math Soc 10:286–293

Wright EM (1940) The asymptotic expansion of the generalized hypergeometric function. Proc London Math Soc 46(2):389–408

Wright EM (1940a) The generalized Bessel function of order greater than one. Quart J Math Oxford Ser 11:36–48

Wright EM (1940b) The asymptotic expansion of integral functions defined by Taylor series. Philos Trans Roy Soc London, Ser A 239:217–222

Yang A (1994) A unification of the Voigt functions. Int. J Math Edu Sci Tech 25(6):845–851

Yakubovich SB, Luchko YF (1994) The hypergeometric approach to integral transforms and convolutions, Math Appl 287. Kluwer Academic, Dordrecht

Yakubovich SB, Hai NT, Buschman RG (1992) Convolutions for H-function transformations. Indian J Pure Appl Math 23(10):743–752

Yu R, Zhang H (2006) New function of Mittag-Leffler type and its application in the fractional diffusion-wave equation. Chaos, Solitons and Fractals 30:946–955

Zaslavsky GM (1994) Fractional kinetic equation for Hamiltonian chaos. Physica D 76(1-3): 110–122

Zayed AI (1996) Handbook of functions and generalized function transformations. CRC Press, Boca Baton

Zemanian AH (1968) Generalized integral transformations, Pure Applied Mathematics 18. Interscience Publishing [Wiley, New York]

Zhang S-Q (2007) Solution of semi-boundless mixed problem for time-fractional telegraph equation. Acta Mathematicae Applicatae Sinica 33(4):511–618

Zhang S, Jin J (1996) Computation of special functions. Wiley, New York

Zhang S, Shu-qin (2007) Solution of semi-boundless mixed problem for time-fractional telegraph equation. Acta Mathemticae Applicatae Sinica 23(4):611–618

Zu-Guo Yu, Fu-Yao, Zhou J (1997) Fractional integral associated to generalized cookie-cutter set and its physical interpretation. J Phys A: Math Gen 30:5569–5577

Glossary of Symbols

$H^{m,n}_{p,q}[z]$	H-function	Definition 1.1	2
$L_{+\infty}, L_{-\infty}, L_{i\gamma\infty}$	contours for H-function	Definition 1.1	2
$(a)_k$	Pochhammer symbol	Example 1.2	7
$E_\alpha(z)$	Mittag-Leffler function	Definition 1.2	8
$E_{\alpha,\beta}(z)$	Mittag-Leffler function	Definition 1.3	8
$E^{\gamma}_{\alpha,\beta}(z)$	Mittag-Leffler function	Definition 1.4	9
$G^{m,n}_{p,q}[z]$	Meijer's G-function	Section 1.8	21
${}_pF_q(z)$	hypergeometric function	Definition 1.6	22
$E(.,\ldots,.;.,\ldots,.:z)$	MacRobert's function	Definition 1.7	22
$J^{\mu}_{\nu}(z)$	Bessel-Maitland function	Definition 1.8	22
$Z^{\nu}_{\rho}(z)$	Krätzel function	Definition 1.10	22
${}_p\psi_q(z)$	Wright function	Definition 1.12	23
${}_2R_1$	Dotsenko function	Definition 1.14	31
$M\{f(t):s\}, f^*(s)$	Mellin transform	Section 2.2	45
$M^{-1}\{f^*(s);x\}$	inverse Mellin transform	Section 2.2	45
$L\{f(t):s\}, (Lf)(s)$	Laplace transform	Section 2.2.6	48
$L^{-1}[F(s);t]$	inverse Laplace transform	Section 2.2.6	48
$R_\nu\{f(x);p\}$	K-transform	Section 2.2.11	53
$V\{f,k,m;s\}$	Varma transform	Section 2.2.13	55
$H_\nu\{f(x):\rho\}$	Hankel transform	Section 2.2.15	56
${}_aI^\alpha_x, {}_aD^{-\alpha}_x, I^\alpha_{a+}$	fractional integrals	Section 3.3.1	79
${}_xI^\alpha_b, {}_xD^{-\alpha}_b, I^\alpha_{b-}$	fractional integrals	Section 3.3.1	79
$({}_xW^\alpha_\infty, {}_xI^\alpha_\infty, I^\alpha_-$	Weyl integrals	Section 3.5	91
${}_xD^\alpha_\infty$	Weyl derivative	Definition 3.10	91
${}^c{}_aD_x{}^\alpha$	Caputo derivative	Section 3.6.3	95

$I(\alpha, \eta; f)$	Erdélyi-Kober operator	Section 3.8.1	98
$E_{0,x}^{\eta,\alpha}(f)$	Erdélyi-Kober operator	Section 3.8.1	98
$I_{\eta,\alpha}^{+} f$	Erdélyi-Kober operator	Section 3.8.1	98
$K_{x,\infty}^{\alpha,\zeta}, K_{x}^{\zeta,\alpha}$	Erdélyi-Kober operator	Section 3.8.1	98
$K_{\zeta,\alpha}^{-}$	Erdélyi-Kober operator	Section 3.8.1	98
$K(\alpha, \zeta, f)$	Erdélyi-Kober operator	Section 3.8.1	98
$I(\alpha, \beta, \gamma; m, \mu, \alpha; f)$	generalized Kober operator	Section 3.9	101
$J(.,.,.;.,.; f)$	generalized Kober operator	Section 3.9	101
$K[f(x)], K\begin{bmatrix}\alpha, \beta, \gamma : \\ \delta, \rho, a : f\end{bmatrix}$	Kober operators	Section 3.9	101
$I_{0+}^{\alpha,\beta,\gamma}, I_{-}^{\alpha,\beta,\gamma}$	Saigo integral operators	Section 3.10	103
$D_{0+}^{\alpha,\beta,\gamma}, D_{-}^{\alpha,\beta,\gamma}$	Saigo differential operators	Section 3.10	103
$I_{.,.,.,m}^{.,.}$	multiple Erdélyi-Kober operator	Section 3.11	113
$E(\cdot)$	expected value	Section 4.2	119
$B(\alpha, \beta)$	beta function	Section 4.2	119
$\mathrm{tr}(\cdot)$	trace of the matrix $(\cdot)$	Section 5.1	139
$\int_A^B f(X)\mathrm{d}X$	integral over matrices	Section 5.1	139
$\mathrm{d}x \wedge \mathrm{d}y$	wedge product of differentials	Section 5.1	139
$\Gamma_p(\alpha)$	real matrix-variate gamma	Section 5.1	139
J	Jacobian	Section 5.1	139
$B_p(\alpha, \beta)$	real matrix-variate beta	Section 5.4	146
$(\mathrm{d}X)$	matrix of differentials	Section 5.4	146
$M(r), P(r), T(r)$	mass, pressure, temperature	Section 6.2	159
$< . >$	expected value	Section 6.11.5	189
$H_{.,.;...}^{.,.;...}$	H-function of many variables	Appendix A.1	205
$F_{.;.;.}^{.;.;.}$	Kampé de Fériet function	Appendix A.2	207
F_1, F_2, F_3, F_4	Appell functions	Appendix A.3	211
F_A, F_B, F_C, F_D	Lauricella functions	Appendix A.4	213
$\bar{H}$	H-bar function	Appendix A.5	215
$I_{p_i,q_i}^{m,n}$	I-function	Appendix A.7	219

Author Index

M

N

O

P

Subject Index