Fractional Partial Differential Equation:
Mathematical and Numerical Analysis Issues

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$$\mathcal{L}_\theta^\beta u(x) := -D\big(K(x)\big(\theta\ {}_a^{C,l}D_x^{1-\beta}u - (1-\theta)\ {}_x^{C,r}D_b^{1-\beta}u\big)\big) = f(x), \quad x \in (a,b),$$
$$u(a) = u_l,\ u(b) = u_r, \quad 0 < \beta < 1,\ 0 \le \theta \le 1. \tag{1}$$

- derived from a local mass balance + a fractional Fick's law.
- θ is the weight of forward versus backward transition probability.
- The left- and right-fractional integrals, Caputo and Riemann-Liouville fractional derivatives are defined by

$$\begin{aligned}
{}_aI_x^\beta u(x) = {}_aD_x^{-\beta}u(x) &:= \frac{1}{\Gamma(\beta)}\int_a^x (x-s)^{\beta-1}u(s)ds,\\
{}_xI_b^\beta u(x) = {}_xD_b^{-\beta}u(x) &:= \frac{1}{\Gamma(\beta)}\int_x^b (s-x)^{\beta-1}u(s)ds,\\
{}_a^CD_x^{1-\beta}u := {}_aI_x^\beta Du, &\qquad {}_x^CD_b^{1-\beta}u := -{}_xI_b^\beta Du,\\
{}_a^{RL}D_x^{1-\beta}u := D\ {}_aI_x^\beta u, &\qquad {}_x^{RL}D_b^{1-\beta}u := -D\ {}_xI_b^\beta u.
\end{aligned} \tag{2}$$

- The left (right) Caputo and Riemann-Liouville fractional derivatives do not equal unless the zero boundary condition is imposed at $x = a$ ($x = b$).

- Galerkin formulation: given $f \in H^{-(1-\beta/2)}(a,b)$, seek $u \in H_0^{1-\beta/2}(a,b)$

$$\begin{aligned} B(u,v) &:= -\theta\big(K\ {}_aD_x^{1-\beta/2}u, {}_xD_b^{1-\beta/2}v\big) - (1-\theta)\big(K\ {}_xD_b^{1-\beta/2}u, {}_aD_x^{1-\beta/2}v\big) \\ &= \langle f, v\rangle, \qquad \forall\ v \in H_0^{1-\beta/2}(a,b). \end{aligned} \quad (3)$$

  - For constant K, ${}_aI_x^{\beta/2}$ on the trial function side can be switched to the test function side as ${}_xI_b^{\beta/2}$.

- Ervin & Roop proved the (β-dependent, not true for $\beta = 1$) equivalence between the fractional derivative norms and fractional Sobolev space norms, which gives the coercivity and boundedness of $B(\cdot,\cdot)$

$$B(u,u) = K\big({}_aI_x^{\beta/2}Du, {}_xI_b^{\beta/2}Du\big)_{L^2(a,b)} = \cos\big(\beta\pi/2\big)K|u|^2_{H^{1-\beta/2}(a,b)}.$$

Theorem

$B(\cdot,\cdot)$ is coercive and continuous on $H_0^{1-\beta/2}(a,b) \times H_0^{1-\beta/2}(a,b)$. Hence, the Galerkin weak formulation (3) has a unique solution. Moreover,

$$\|u\|_{H^{1-\beta/2}(a,b)} \le C\|f\|_{H^{-(1-\beta/2)}(a,b)}.$$

- For $\theta = 1/2$, $B(\cdot, \cdot)$ is symmetric. This problem reduces to the fractional Laplacian (in one space dimension). Acceleration techniques such as multigrid and domain decomposition have been developed and analyzed for the multiD analogue of (1) or fractional Laplacian (Ainsworth et al 17, 18; X. Xu et al 15, 17).
- Even for constant K, $B(\cdot, \cdot)$ is nonsymmetric for $\theta \neq 1/2$. The nonsymmetry in the leading order term in the FDE seems to introduce some technical difficulty in the analysis of multigrid and domain decomposition methods.

- Let $S_h(a,b) \subset H_0^{1-\beta/2}(a,b)$ be the finite element space of piecewise polynomials of degree $m-1$. Find $u_h \in S_h(a,b)$ such that

$$B(u_h, v_h) = \langle f, v_h \rangle, \qquad \forall v_h \in S_h(a,b).$$

- **Assume** that the true solution $u \in H^m(a,b) \cap H_0^{1-\beta/2}(a,b)$. Then the optimal-order error estimate in the energy norm holds

$$\|u_h - u\|_{H^{1-\beta/2}(a,b)} \le C h^{m-1+\beta/2} \|u\|_{H^m(a,b)}.$$

- **Assume** the dual problem has full regularity for $\forall g \in L^2$. The optimal-order error estimate in the L^2 norm holds for $u \in H^m(a,b) \cap H_0^{1-\beta/2}(a,b)$
- Extensions to spectral Galerkin methods and other methods were proved under the same assumptions.

- An optimal-order error estimate in the energy (and L^2) norm was proved for the numerical approximations to linear elliptic FPDE under the assumption that the solution (all the solutions to the dual problem) is smooth.
- Consider problem (1) with $K = f = 1$, $\theta = 1$, $u_l = u_r = 0$

$$D\big({}_0I_x^\beta Du\big) = 1,\ x \in (0,1),\quad u(0) = u(1) = 0 \quad \Longrightarrow \ {}_0I_x^\beta Du = x + C_0,$$

$$u(x) = {}_0I_x Du = {}_0I_x^{1-\beta}\,{}_0I_x^\beta u = {}_0I_x^{1-\beta}(x + C_0) = \frac{x^{2-\beta}}{\Gamma(3-\beta)} + \frac{C_0\ x^{1-\beta}}{\Gamma(2-\beta)}.$$

where we have used

$${}_0I_x^\gamma x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\gamma+\mu+1)} x^{\gamma+\mu},\quad 0 < \gamma < 1,\ \mu > -1.$$

Enforcing the boundary condition $u(1) = 0$ to obtain the unique solution

$$u(x) = \frac{x^{2-\beta} - x^{1-\beta}}{\Gamma(3-\beta)} \notin W^{1,1/\beta}(0,1).$$

In particular, $u \notin H^1(0,1)$ for $1/2 \le \beta \le 1$.

- Smooth data (& domain in multi-D) ensures smooth solutions for integer order linear elliptic PDEs, which is not true for FDEs.
- Solutions to FDEs with smooth data (& domain in multi-D) may have boundary layers and so low regularity, which need to be resolved numerically.
  - The Nitsche-lifting based proof of *optimal-order* L^2 error estimates in the literature does not hold even for constant $K > 0$.
  - Jin et al analyzed the Sobolev regularity of the solutions to one-sided constant coefficient FDEs by studying their analytical solutions, and used Nitsche-lifting to derive suboptimal-order L^2 error estimate.
- The solutions to FDEs (in 1D) were proved in some weighted Sobolev spaces and corresponding spectral methods were developed (Chen at al 16, Ervin et al 16, Mao & Karniadakis 18)

  \+ Optimal-order error estimates of numerical approximations in weighted Sobolev norms of data, not the true solution.

  − The accuracy of the approximations near boundary are compromised, which are often important in some applications.

Hypergeometric Function ${}_2F_1$

$$\begin{aligned} {}_2F_1(a,b;c;x) &:= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 z^{b-1}(1-z)^{c-b-1}(1-zx)^{-a}dz \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n x^n}{(c)_n n!} \end{aligned} \tag{4}$$

which converges only if $Re(c) > Re(b) > 0$. Here $(q)_n$ are defined by

$$(q)_n := \frac{\Gamma(q+n)}{\Gamma(q)} = q(q+1)\cdots(q+n-1). \tag{5}$$

Symmetry of ${}_2F_1$

For $Re(c) > Re(a) > 0$ and $Re(c) > Re(b) > 0$,

$${}_2F_1(a,b;c;x) = {}_2F_1(b,a;c;x). \tag{6}$$

In this part we assume $(a,b) = (0,1)$.

Theorem

A kernel function of the operator $D_0 I_x^\beta + D_x I_1^\beta$ *is* $k_{1/2}(x) := x^{-\beta/2}(1-x)^{-\beta/2}$.

Proof

$$
\begin{aligned}
{}_0I_x^\beta k_{1/2}(x) &= \frac{1}{\Gamma(\beta)} \int_0^x (x-y)^{\beta-1} y^{-\beta/2}(1-y)^{-\beta/2} dy \quad (z = y/x) \\
&= \frac{x^{\beta/2}}{\Gamma(\beta)} \int_0^1 z^{-\beta/2}(1-z)^{\beta-1}(1-xz)^{-\beta/2} dz \\
(a = \beta/2;\ & b-1 = -\beta/2, b = 1-\beta/2; c-b-1 = \beta-1, c = 1+\beta/2) \\
&= \frac{x^{\beta/2}}{\Gamma(\beta)} \frac{\Gamma(1-\beta/2)\Gamma(\beta)}{\Gamma(1+\beta/2)} {}_2F_1(\beta/2, 1-\beta/2; 1+\beta/2; x) \\
&= \frac{x^{\beta/2}}{\Gamma(\beta)} \frac{\Gamma(1-\beta/2)\Gamma(\beta)}{\Gamma(1+\beta/2)} {}_2F_1(1-\beta/2, \beta/2; 1+\beta/2; x) \\
&= \frac{\Gamma(1-\beta/2) x^{\beta/2}}{\Gamma(\beta/2)} \int_0^1 z^{\beta/2-1}(1-z)^{1+\beta/2-\beta/2-1}(1-xz)^{\beta/2-1} dz \\
&= \frac{\Gamma(1-\beta/2)}{\Gamma(\beta/2)} \int_0^x y^{\beta/2-1}(1-y)^{\beta/2-1} dy.
\end{aligned}
$$

$$D_0 I_x^\beta k_{1/2}(x) = \frac{\Gamma(1-\beta/2)}{\Gamma(\beta/2)} x^{\beta/2-1}(1-x)^{\beta/2-1}.$$

Similarly,

$$_xI_1^\beta k_{1/2}(x) = \frac{\Gamma(1-\beta/2)}{\Gamma(\beta/2)} \int_x^1 y^{\beta/2-1}(1-y)^{\beta/2-1}dy,$$

$$D_xI_1^\beta k_{1/2}(x) = -\frac{\Gamma(1-\beta/2)}{\Gamma(\beta/2)} x^{\beta/2-1}(1-x)^{\beta/2-1}, \quad D\big({_0I_x^\beta} + {_xI_1^\beta}\big)k_{1/2}(x) = 0.$$

Lemma

$ker(\mathcal{L}_{1/2}^\beta) = span\{1, K_{1/2}(x)\}$, *where* $K_{1/2}(x) := \int_0^x k_{1/2}(y)dy$.

The general case was proved in a similar manner.

Theorem

Let $k(x) := x^{-p}(1-x)^{-q}$. *Then* $K(x) := \int_0^x k(y)dy \in ker(\mathcal{L}_\theta^\beta)$ *if* $\beta = p+q$ *and* $\theta\sin(\pi q) = (1-\theta)\sin(\pi p)$. *Consequently,* $ker(\mathcal{L}_\theta^\beta) = span\{1, K(x)\}$.

- The general solution of a linear constant-coefficient (integer or fractional order) differential equation can be expressed as $u = u_f + u_c$, with u_c being the general solution of the homogeneous equation and u_f being a particular solution of the inhomogeneous problem.
  - In the integer-order case, u_c is infinitely many times differentiable. Hence, the regularity of u is limited by u_f that is determined by f.
  - In the fractional case, the regularity is limited by $K_{1/2}(x)$ that is not smooth and, thus, u_c is not smooth. Hence, u is not smooth no matter how smooth u_f is.
- This is the reason why raising the regularity of f cannot raise the regularity of u.

Lemma

$B(w,w) < 0$ *for some* $K(x)$ *of two positive constants and* $w \in H_0^{1-\frac{\beta}{2}}(0,1)$

Let $K(x)$ and $w \in H_0^1(0,1) \subset H_0^{1-\frac{\beta}{2}}(0,1)$ be defined by

$$K(x) := \begin{cases} K_l, & x \in (0,1/2), \\ 1, & x \in (1/2,1). \end{cases} \qquad w(x) := \begin{cases} 2x, & x \in (0,1/2], \\ 2(1-x), & x \in [1/2,1). \end{cases}$$

Direct calculation gives

$${}_0^{C,l}D_x^{1-\beta} w(x) = \begin{cases} 2x^\beta/\Gamma(\beta+1), & x \in (0,1/2), \\ 2\big(x^\beta - 2(x-1/2)^\beta\big)/\Gamma(\beta+1), & x \in (1/2,1). \end{cases}$$

Thus we have

$$B(w,w) = 2^{1-\beta}\big(K_l - \big(2^{\beta+1}-3\big)\big)/\Gamma(\beta+2).$$

As $0 < \log_2 3 - 1 < 1$, choose $\log_2 3 - 1 < \beta < 1$ so that $2^{\beta+1} - 3 > 0$. Select $K_l > 0$ such that $K_l - \big(2^{\beta+1}-3\big) < 0$. For such K and w, $B(w,w) < 0$. □

- Consider the one-sided version of the conservative FDE (1) with ($\theta = 1$)

$$-D\big(K \ {}_aI_x^{\beta} Du\big) = f(x), \quad x \in (a,b), \quad u(a) = u(b) = 0.$$

- For a variable diffusivity coefficient K

$$\begin{aligned} B(u,v) &= \theta\big\langle K \ {}_aI_x^{\beta} Du, Dv\big\rangle + (1-\theta)\big\langle K \ {}_xI_b^{\beta} Du, Dv\big\rangle \\ &\neq \theta\big\langle K Du, {}_xI_b^{\beta} Dv\big\rangle + (1-\theta)\big\langle K Du, {}_aI_x^{\beta} Dv\big\rangle \\ &\neq \big(K \ {}_aI_x^{\beta/2} Du, {}_xI_b^{\beta/2} Dv\big)_{L^2(a,b)} \end{aligned}$$

  - For a variable K, the three expressions are not equal in general.
  - The most likely expression to be coercive is the last one due to its symmetry with respect to ${}_aI_x^{\beta/2} Du$ and ${}_xI_b^{\beta/2} Dv$.

- For an integer-order analogue of elliptic FPDEs, the bilinear form reduces to $(K|\nabla u|^2)_{L^2(a,b)}$, which, combined with the homogeneous Dirichlet BC, guarantees the coercivity of the bilinear form.
- For FDE (1) with a constant $K > 0$, ${}_aI_x^{\beta/2}Du \neq {}_xI_b^{\beta/2}Du$. But

$$B(u,u) = K\big({}_aI_x^{\beta/2}Du, {}_xI_b^{\beta/2}Du\big)_{L^2(a,b)} = \cos\big(\beta\pi/2\big)K|u|^2_{H^{1-\beta/2}(a,b)},$$

  along with the homogeneous Dirichlet BC, ensures the coercivity of B.
- However, there are $u \in C_0^\infty(a,b)$ such that ${}_aI_x^{\beta/2}Du$ and ${}_xI_b^{\beta/2}Du$ have opposite sign for some $x \in (a,b)$. One can find a sufficiently smooth K with $0 < K_{\min} \leq K < \infty$, possibly with a large variation, such that

$$\big(K\ {}_aI_x^{\beta/2}Du, {}_xI_b^{\beta/2}Du\big)_{L^2(a,b)} < 0.$$

- Coercivity of the bilinear form B is a sufficient but not necessary condition for the wellposedness of FDE (1). What is the impact of losing coercivity?
- Numerical experiments showed that the corresponding finite element approximation may diverge (W., Yang & Zhu 14, 17).

- Galerkin formulation may lose coercivity on any product space $H \times H$ for variable K, so $H_0^{1-\beta/2}(a,b) \times H_0^{1-\beta/2}(a,b)$ is not a feasible choice.
- Consider the one-sided version of FDE (1), which is a local mass balance incorporated with a fractional Fick's law

$$-D\big(K\ {}_aI_x^{\beta} Du\big) = f(x), \quad x \in (a,b), \quad u(a) = u(b) = 0. \tag{7}$$

- It motivates a Petrov-Galerkin formulation: Seek $u \in H_0^{1-\beta}(a,b)$ such that

$$A(u,v) := \int_a^b K(x)\ \big({}_aI_x^{\beta} Du\big) Dv dx = \langle f, v \rangle, \quad \forall v \in H_0^1(a,b) \tag{8}$$

- Even for constant K, the Petrov-Galerkin formuation (8) differs from the Galerkin formuation (3)
  - (3) is defined on $H_0^{1-\beta/2}(a,b) \times H_0^{1-\beta/2}(a,b)$ for any $f \in H^{-(1-\beta/2)}(a,b)$ and $0 < \beta < 1$.
  - (8) is defined on $H_0^{1-\beta}(a,b) \times H_0^{1}(a,b)$ for any $f \in H^{-1}(a,b)$ and $0 < \beta < 1/2$, as the Dirichlet BC cannot be enforced for $1/2 < \beta < 1$.

Theorem

Assume $0 < \beta < 1/2$ *and* $0 < K_{min} \le K \le K_{max} < \infty$. *Then*

$$\begin{aligned} \inf_{w \in H_0^{1-\beta}(a,b)} \sup_{v \in H_0^1(a,b)} \frac{A(w,v)}{\|w\|_{H^{1-\beta}(a,b)} \|v\|_{H^1(a,b)}} \ge \gamma(\beta) > 0, \\ \sup_{w \in H_0^{1-\beta}(a,b)} A(w,v) > 0 \quad \forall\, v \in H_0^1(a,b) \setminus \{0\}. \end{aligned} \tag{9}$$

Hence, (8) has a unique solution $u \in H_0^{1-\beta}(a,b)$ *with the estimate*

$$\|u\|_{H^{1-\beta}(a,b)} \le (K_{max}/\gamma) \|f\|_{H^{-1}(a,b)}. \tag{10}$$

Theorem

u is the unique solution to (7) if and only if it can be expressed as

$$u(x) = {}_a^C D_x^\beta w_f(x) - {}_a^C D_b^\beta w_f(b) \big({}_a^C D_b^\beta w_b(b) \big)^{-1} {}_a^C D_x^\beta w_b(x), \tag{11}$$

where w_f and w_b are the solutions to the second-order differential equations

$$\begin{aligned} -D\big(K(x) D w_f\big) &= f, \quad x \in (a,b); \qquad w_f(a) = w_f(b) = 0, \\ -D\big(K(x) D w_b\big) &= 0, \quad x \in (a,b); \qquad w_b(a) = 0, \ w_b(b) = 1. \end{aligned} \tag{12}$$

- Let u be the solution to (7). Then $w := {}_aI_x^\beta u$ satisfies

$$-D\big(K(x)Dw\big) = f,\ x \in (a,b); \quad w(a) = 0,\ w(b) = {}_aI_b^\beta u.$$

- w can be expressed as a linear combination of w_f and w_b

$$w = w_f + Cw_b.$$

- We apply ${}_a^{RL}D_x^\beta = {}_a^CD_x^\beta$ (since $I_x^\beta u|_{x=0} = 0$) on both sides to get

$$u = {}_a^{RL}D_x^\beta\, {}_aI_x^\beta u = {}_a^CD_x^\beta\, {}_aI_x^\beta u = {}_a^CD_x^\beta w = {}_a^CD_x^\beta w_f + C\ {}_a^CD_x^\beta w_b. \tag{13}$$

- To find C we enforce the boundary condition $u(b) = 0$ to get

$${}_a^CD_b^\beta w_f(b) + C\ {}_a^CD_b^\beta w_b(b) = 0. \tag{14}$$

- Note that w_b can be solved explicitly as

$$w_b(x) = \Big(\int_a^b \frac{1}{K(s)}ds\Big)^{-1} \int_a^x \frac{1}{K(y)}dy.$$

- ${}_a^C D_x^\beta w_b$ can be evaluated as follows

$$\begin{aligned} {}_a^C D_x^\beta w_b(x) &= {}_a I_x^{1-\beta} D w_b(x) \\ &= \Big(\int_a^b \frac{1}{K(s)} ds \Big)^{-1} {}_a I_x^{1-\beta} \frac{1}{K(x)} \\ &= \Big(\int_a^b \frac{1}{K(s)} ds \Big)^{-1} \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{1}{K(s)(x-s)^\beta} ds > 0. \end{aligned}$$

- Thus, ${}_a^C D_b^\beta w_b(b) > 0$ and we can solve (14) for C as

$$C = -\big({}_a^C D_b^\beta w_b(b) \big)^{-1} {}_a^C D_b^\beta w_f(b).$$

- We insert C into (13) to finish the proof of the only if part of the theorem.
- Conversely, direct calculation verifies that any u given by (13) is a solution to problem (7).

$\square$

- For each $w \in H_0^{1-\beta}(a,b)$, ${}_a^{RL}D_x^{1-\beta} w \in L^2(a,b)$. Thus, $A(w,\phi)$ induces a bounded linear functional on $H_0^1(a,b)$.
- Riesz representation $\implies \exists$ a unique $v \in H_0^1(a,b)$ such that

$$\big(KDv, D\phi\big)_{L^2(a,b)} = A(w,\phi) \quad \forall \phi \in H_0^1(a,b). \tag{15}$$

- This in turn can be rewritten as

$$\big(KD(v - {}_aI_x^\beta w), D\phi\big)_{L^2(a,b)} = 0 \quad \forall \phi \in H_0^1(a,b).$$

$v - {}_aI_x^\beta w = 0$ at $x = a$ and $v - {}_aI_x^\beta w = -{}_aI_b^\beta w$ at $x = b$.

- This implies that

$$v - {}_aI_x^\beta w(x) = -\big({}_aI_b^\beta w(b)\big) w_b(x).$$

- We apply ${}_a^{RL}D_x^\beta$ to both sides of the equation to get

$$w(x) = {}_a^{RL}D_x^\beta v(x) + ({}_aI_b^\beta w(b))\, {}_a^{RL}D_x^\beta w_b(x). \tag{16}$$

- We enforce the condition $w(b) = 0$ and ${}^{RL}_{a}D^{\beta}_{b} w_b(b) > 0$ to (16) to obtain

$$ {}_{a}I^{\beta}_{b} w(b) = -{}^{RL}_{a}D^{\beta}_{b} v(b) \ \big({}^{RL}_{a}D^{\beta}_{b} w_b(b)\big)^{-1}. $$

- We apply ${}^{RL}_{a}D^{1-\beta}_{x}$ to (16) to get

$$ {}^{RL}_{a}D^{1-\beta}_{x} w(x) = Dv - \big({}^{RL}_{a}D^{\beta}_{b} v(b)\big)\big({}^{RL}_{a}D^{\beta}_{b} w_b(b)\big)^{-1} Dw_b(x). $$

- We use $\big|{}^{RL}_{a}D^{\beta}_{b} v(b)\big| \le C\|Dv\|_{L^2(a,b)}$ to bound ${}^{RL}_{a}D^{1-\beta}_{x} w(x)$

$$ \begin{aligned} &\|w\|_{H^{1-\beta}(a,b)} \\ &\quad \le C\Big(\|Dv\|_{L^2(a,b)} + \big|{}^{RL}_{a}D^{\beta}_{b} v(b)\big|\big({}^{RL}_{a}D^{\beta}_{b} w_b(b)\big)^{-1}\|Dw_b\|_{L^2(a,b)}\Big) \\ &\quad \le C\|Dv\|_{L^2(a,b)}. \end{aligned} \tag{17} $$

- We use (15) and (17) to bound $A(w,v)$ from below

$$ \begin{aligned} A(w,v) &= \big(KDv, Dv\big)_{L^2(a,b)} \ge K_{min}\|Dv\|^2_{L^2(a,b)} \\ &\ge \frac{K_{min}}{C}\|Dv\|_{L^2(a,b)}\|w\|_{H^{1-\beta}(a,b)}. \end{aligned} $$

- This proves the first estimate in the theorem with $\gamma := K_{min}/C$.

- To prove the second estimate, for each $v \in H_0^1(a,b) \setminus \{0\}$ we define

$$w(x) := {}_a^{RL}D_x^\beta v(x) - \left({}_a^{RL}D_b^\beta v(b)\right)\left({}_a^{RL}D_b^\beta w_b(b)\right)^{-1}\left({}_a^{RL}D_x^\beta w_b(x)\right).$$

- It is clear that $w \in H_0^{1-\beta}(a,b)$. Furthermore, we have

$${}_a^{RL}D_x^{1-\beta} w(x) = Dv(x) - \left({}_a^{RL}D_b^\beta v(b)\right)\left({}_a^{RL}D_b^\beta w_b(b)\right)^{-1} Dw_b(x).$$

- Here we have used the fact that

$$\begin{aligned} {}_a^{RL}D_x^{1-\beta}\,{}_a^{RL}D_x^\beta v(x) &= D_aI_x^\beta D_aI_x^{1-\beta} v(x) = D_aI_x^\beta\,{}_aI_x^{1-\beta} Dv(x) \\ &= D_aI_x Dv(x) = Dv(x). \end{aligned}$$

- Therefore, we arrive at

$$\begin{aligned} A(w,v) &= (KDv, Dv)_{L^2(a,b)} - ({}_aD_b^\beta v)({}_aD_b^\beta w_b)^{-1}(KDw_b, Dv)_{L^2(a,b)} \\ &= (KDv, Dv)_{L^2(a,b)} \geq K_{min}\|Dv\|_{L^2(a,b)}^2 > 0. \end{aligned}$$

- We have thus proved the estimate and so the theorem. □

- Consider the inhomogeneous Dirichlet boundary-value problems of
  - the Caputo flux FDE

$$-D\big(K(x)\,{}_0^C D_x^{1-\beta} u\big) = f(x),\ x \in (0,1),\quad u(0) = u_l,\ u(1) = u_r, \tag{18}$$

  - the Riemann-Liouville flux FDE

$$-D\big(K(x)\,{}_0^{RL} D_x^{1-\beta} u\big) = f(x),\ x \in (0,1),\quad u(0) = u_l,\ u(1) = u_r. \tag{19}$$

- (18) and (19) coincide for homogeneous Dirichlet boundary condition, but differ otherwise even for problems with a constant $K > 0$.
- A traditional homogenization of the inhomogeneous BC does not work, as the fractional derivative of an affine function introduces singularities.

Theorem

Assume $0 < \beta < 1/2$ *and* $0 < K_{min} \leq K \leq K_{max} < \infty$. *Then Petrov-Galerkin formulation for problem* (18) *admits a unique weak solution* $u \in H^{1-\beta}(0,1)$ *with the stability estimate*

$$\|u\|_{H^{1-\beta}(0,1)} \leq \frac{1}{\gamma}\|f\|_{H^{-1}(0,1)} + C\big(|u_l| + |u_r|\big).$$

But the Petrov-Galerkin weak formulation for problem (19) *admits no weak solution in* $H^{1-\beta}(0,1)$*!*

- Similar conclusions hold for two-sided problems with a constant diffusivity coefficient $K > 0$.

- Despite the rapidly increasing research on FPDEs in the literature, many fundamental issues remain, e.g., fractional flux boundary conditions (fBCs)
- Different (Riemann-Liouville, Caputo, Caputo flux) forms of FPDEs and fBCs were proposed in the literature.
- Extensive (stochastic and modeling) study has been conducted to seek the right form of FPDE and fBC in FPDE modeling and applications.

- We consider the Caputo, Caputo flux and Riemann-Liouville FDE

$$\begin{aligned} &\text{Caputo} && -{}_0^C D_x^{2-\beta} u(x) = f(x), && x \in (0,1), \\ &\text{Caputo flux} && -D\big({}_0^C D_x^{1-\beta} u(x)\big) = f(x), && x \in (0,1), \\ &\text{Riemann} - \text{Liouville} && -{}_0^R D_x^{2-\beta} u(x) = f(x), && x \in (0,1), \end{aligned} \tag{20}$$

- and the classical flux BC, the Caputo fBC and the Riemann-Liouville fBC

$$\begin{aligned} &\text{classical fBC} && Du\big|_{x=0} = a_0, \quad Du\big|_{x=1} = a_1, \\ &\text{Capute fBC} && {}_0^C D_x^{1-\beta} u\big|_{x=0} = a_0, \quad {}_x^C D_1^{1-\beta} u\big|_{x=1} = a_1, \\ &\text{Riemann} - \text{Liouville} && {}_0^{RL} D_x^{1-\beta} u|_{x=0} = a_0, \quad {}_x^{RL} D_1^{1-\beta} u\big|_{x=1} = a_1, \end{aligned} \tag{21}$$

- For the homogeneous Dirichlet BC, the Riemann-Liouville FDE and the Caputo flux FDE coincide, but they differ in the current context.

Table: Summary of the results

	Caputo FDE	Caputo flux FDE	R-L FDE
Classical fBC	✓	X	X
Caputo fBC	X	✓	✓
R-L fBC	✓	X	✓

- We proved the following results:
  - Five out of the nine combinations are well posed and the rest ill posed.
  - For each of the FDEs, there exist one of the three fBCs such that the combination is well posed and another of the three such that the combination is ill posed. The results are summarized in the table
- This suggests that the physical relevance of a specific combination of an FDE and a related fBC, rather than just an individual FDE model or fBC, should be investigated.

- Let $0 < \beta < 1$ and $0 < \varepsilon(\beta) < 1 - \beta$, we define $\kappa(\beta) = 2$ for $0 < \beta < 1/2$ and $1 + (1 - \beta - \varepsilon(\beta))/\beta$ for $1/2 \le \beta < 1$. In particular, $1 < \kappa(\beta) < 1/\beta$ but sufficiently close to $1/\beta$ for $1/2 \le \beta < 1$.
- For $0 < \mu < 1$ we define Riemann–Liouville fractional derivative spaces

$$H^{\mu}_{R,l} := \{v \in L^{\kappa} : {}^{RL}_{0}D^{\mu}_{x} v \in L^2\}, \ H^{\mu}_{R,r} := \{v \in L^{\kappa} : {}^{RL}_{x}D^{\mu}_{1} v \in L^2\}$$

$$H^{\mu,0}_{R,l} := \Big\{v \in H^{\mu}_{R,l} : \int_0^1 {}_0I^{1-\mu}_{x} v dx = 0\Big\},$$

$$H^{\mu,0}_{R,r} := \Big\{v \in H^{\mu}_{R,r} : \int_0^1 {}_xI^{1-\mu}_{1} v dx = 0\Big\}$$

equipped with the (semi) norms

$$|v|_{H^{\mu}_{R,l}} := \|{}^{RL}_{0}D^{\mu}_{x} v\|^2_{L^2}, \quad \|v\|_{H^{\mu}_{R,l}} := \big(\|v\|^2_{L^{\kappa}} + |v|^2_{H^{\mu}_{R,l}}\big)^{\frac{1}{2}},$$

$$|v|_{H^{\mu}_{R,r}} := \|{}^{RL}_{x}D^{\mu}_{1} v\|^2_{L^2}, \quad \|v\|_{H^{\mu}_{R,r}} := \big(\|v\|^2_{L^{\kappa}} + |v|^2_{H^{\mu}_{R,r}}\big)^{\frac{1}{2}}.$$

Theorem

(Riemann–Liouville fractional Friedrichs inequality for $0 < \beta < 1$)

$$\|v\|_{L^\kappa} \le C\Big(\Big|\int_0^1 {}_0I_x^\beta v dx\Big| + \big\|{}_0^{RL}D_x^{1-\beta} v\big\|_{L^2}\Big), \ \forall\ v \in H_{R,l}^{1-\beta},$$

$$\|v\|_{L^\kappa} \le C\Big(\Big|\int_0^1 {}_xI_1^\beta v dx\Big| + \big\|{}_x^{RL}D_1^{1-\beta} v\big\|_{L^2}\Big), \ \forall\ v \in H_{R,r}^{1-\beta}.$$

Consequently, $|v|_{H_{R,l}^{1-\beta}}$ *and* $|v|_{H_{R,r}^{1-\beta}}$ *define norms on* $H_{R,l}^{1-\beta,0}$ *and* $H_{R,r}^{1-\beta,0}$.

- We similarly define left and right Caputo fractional derivative spaces

$$H^{\mu}_{C,l} := \{v \in L^{\kappa} : \, {}^C_0 D^{\mu}_x v \in L^2\}, \; H^{\mu}_{C,r} := \{v \in L^{\kappa} : \, {}^C_x D^{\mu}_1 v \in L^2\}$$
$$H^{\mu,0}_{C,l} := \Big\{v \in H^{\mu}_{C,l} : \textstyle\int_0^1 v dx = 0\Big\}, \; H^{\mu,0}_{C,r} := \Big\{v \in H^{\mu}_{C,r} : \textstyle\int_0^1 v dx = 0\Big\}$$

equipped with the (semi) norms

$$|v|_{H^{\mu}_{C,l}} := \|{}^C_0 D^{\mu}_x v\|^2_{L^2}, \quad \|v\|_{H^{\mu}_{C,l}} := \big(\|v\|^2_{L^{\kappa}} + |v|^2_{H^{\mu}_{C,l}}\big)^{\frac{1}{2}},$$
$$|v|_{H^{\mu}_{C,r}} := \|{}^C_x D^{\mu}_1 v\|^2_{L^2}, \quad \|v\|_{H^{\mu}_{C,r}} := \big(\|v\|^2_{L^{\kappa}} + |v|^2_{H^{\mu}_{C,r}}\big)^{\frac{1}{2}}.$$

Theorem

(Caputo fractional Friedrichs inequality for $0 < \beta < 1$)

$$\|v\|_{L^{\kappa}} \le C\Big(\Big|\int_0^1 v dx\Big| + \big\|{}^C_0 D^{1-\beta}_x v\big\|_{L^2}\Big), \quad \forall \, v \in H^{1-\beta}_{C,l},$$
$$\|v\|_{L^{\kappa}} \le C\Big(\Big|\int_0^1 v dx\Big| + \big\|{}^C_x D^{1-\beta}_1 v\big\|_{L^2}\Big), \quad \forall \, v \in H^{1-\beta}_{C,r}.$$

$|v|_{H^{1-\beta}_{C,l}}$ *and* $|v|_{H^{1-\beta}_{C,r}}$ *define norms on* $H^{1-\beta,0}_{C,l}$ *and* $H^{1-\beta,0}_{C,r}$*, respectively.*

- $0 < \beta < 1/2 \Longrightarrow \kappa(\beta) = 2$. The Riemann-Liouville fractional spaces reduce to those in (Ervin & Roop 05). But they differ for $1/2 \le \beta < 1$.
- For the homogeneous Dirichlet BC, Riemann-Liouville and Caputo fractional spaces and fractional Sobolev space $H_0^{1-\beta}$ coincide with equivalent norms.
- Without the homogeneous Dirichlet BC, the Riemann-Liouville and Caputo fractional spaces differ from each other. For example,

$$ {}_0I_x^{\beta} x^{-\beta} = \Gamma(1-\beta), \quad {}_0^RD_x^{2-\beta} x^{-\beta} = {}_0^RD_x^{1-\beta} x^{-\beta} = 0 \Longrightarrow {}_0^RD_x^{1-\beta} x^{-\beta} \in L^2 $$

for $0 < \beta < 1$. In addition, $x^{-\beta} \in L^2$ for $0 < \beta < 1/2$ and $x^{-\beta} \in L^{\kappa}$ for $1/2 \le \beta < 1 \Longrightarrow x^{-\beta} \in H_{R,l}^{1-\beta}$. However,

$$ {}_0^CD_x^{1-\beta} x^{-\beta} = -\beta {}_0I_x^{\beta} x^{-\beta-1} = -\infty \Longrightarrow x^{-\beta} \notin H_{C,l}^{1-\beta}, \quad 0 < \beta < 1. $$

Theorem

For $0 < \beta < 1$, the fractional integral operators I_+^{β} (or I_-^{β}) defines an isomorphism from $H_{R,l}^{1-\beta}$ (or $H_{R,r}^{1-\beta}$) onto H^1 with equivalent norms. $H_{R,l}^{1-\beta}$ and $H_{R,r}^{1-\beta}$ are characterized by

$$ H_{R,l}^{1-\beta} = \Big\{ {}_0^{RL}D_x^{\beta} w(x) - w(0) x^{-\beta}/\Gamma(1-\beta) : \ w \in H^1 \Big\}, $$

$$ H_{R,r}^{1-\beta} = \Big\{ {}_x^{RL}D_1^{\beta} w(x) - w(1)(1-x)^{-\beta}/\Gamma(1-\beta) : \ w \in H^1 \Big\}. $$

- We multiply the Caputo FDE by any $v \in H^{1-\beta}_{R,r}$, integrate the resulting equation on $(0,1)$ and incorporate the Neumann BC to obtain

$$\begin{aligned}\langle f, v\rangle &= -\big({}_0I_x^{\beta} D^2 u, v\big) = -\big(D^2 u, {}_xI_1^{\beta} v\big)\\ &= \big(Du, D\,{}_xI_1^{\beta} v\big) - {}_xI_1^{\beta} v\ Du|_{x=1} + {}_0I_x^{\beta} v\ Du|_{x=0}\\ &= -\big(Du, {}_x^R D_1^{1-\beta} v\big) - a_1\ {}_xI_1^{\beta} v|_{x=1} + a_0\ {}_0I_x^{\beta} v|_{x=0}, \quad \forall v \in H^{1-\beta}_{R,r}.\end{aligned}$$

- This yields the following Petrov-Galerkin weak formulation: find $u \in H^1$ such that

$$\begin{aligned}A_C(u,v) &:= -\big(Du, {}_x^R D_1^{1-\beta} v\big) = l_C(v)\\ &:= \langle f, v\rangle + a_1\ {}_xI_1^{\beta} v|_{x=1} - a_0\ {}_xI_1^{\beta} v|_{x=0}, \quad \forall v \in H^{1-\beta}_{R,r}\end{aligned} \tag{22}$$

Theorem

Let $0 < \beta < 1$ *and* $f \in (H^{1-\beta}_{R,r})'$ *satisfy the constraint*

$$\langle f, (1-x)^{-\beta} \rangle + \Gamma(1-\beta)(a_1 - a_0) = 0. \tag{23}$$

Then the Petrov-Galerkin weak formulation (22) has a unique solution $u^* \in H^{1,0} := \{w \in H^1 : \int_0^1 w dx = 0\}$ *with a stability estimate*

$$\|u^*\|_{H^1} \leq C\big(\|f\|_{(H^{1-\beta,0}_{R,r})'} + |a_0| + |a_1|\big).$$

- We multiply the Riemann-Liouville FDE by ${}_0I_x^{\beta} v$ for any $v \in H_{R,l}^{1-\beta}$, integrate the resulting equation and incorporate the Riemann-Liouville fractional Neumann BC to obtain

$$\begin{aligned}\langle f, v\rangle &= -\big(D^2\, {}_0I_x^{\beta} u, {}_0I_x^{\beta} v\big) \\ &= \big(D\, {}_0I_x^{\beta} u, D\, {}_0I_x^{\beta} v\big) - D\, {}_0I_x^{\beta} u \ {}_0I_x^{\beta} v|_{x=1} + D\, {}_0I_x^{\beta} u \ {}_0I_x^{\beta} v|_{x=0} \\ &= \big({}_0^R D_x^{1-\beta} u, {}_0^R D_x^{1-\beta} v\big) - a_1 \ {}_0I_x^{\beta} v|_{x=1} + a_0 \ {}_0I_x^{\beta} v|_{x=0}, \quad \forall v \in H_{R,l}^{1-\beta}.\end{aligned}$$

- This yields the following Galerkin weak formulation: find $u \in H_{R,l}^{1-\beta}$ such that

$$\begin{aligned}A_{R,l}(u,v) &:= \big({}_0^R D_x^{1-\beta} u, {}_0^R D_x^{1-\beta} v\big) = l_{R,l}(v) \\ &:= \big(f, {}_0I_x^{\beta} v\big)_{L^2} + a_1 \ {}_0I_x^{\beta} v|_{x=1} - a_0 \ {}_0I_x^{\beta} v|_{x=0}, \quad \forall\ v \in H_{R,l}^{1-\beta}.\end{aligned} \tag{24}$$

Theorem

Let $0 < \beta < 1$ *and* $f \in (H^1)'$ *satisfy the constraint*

$$\langle f, 1 \rangle + a_1 - a_0 = 0. \tag{25}$$

Then the Galerkin formulation (24) has a unique solution $u^* \in H^{1-\beta,0}_{R,l}$ *with a stability estimate*

$$\|u^*\|_{H^{1-\beta}_{R,l}} \le C\big(\|f\|_{(H^1)'} + |a_0| + |a_1|\big). \tag{26}$$

Theorem

For $0 < \beta < 1/2$ the solution u to the inhomogeneous Dirichlet boundary-value problem of the FDE (7) can be decomposed as

$$u = u_l + \big(u_r - u_l - {}^{C}_{-1}D^{\beta}_{1} w_f\big)\big({}^{C}_{-1}D^{\beta}_{1} w_b\big)^{-1} {}^{C}_{-1}D^{\beta}_{x} w_b + {}^{C}_{-1}D^{\beta}_{x} w_f. \tag{27}$$

$$\begin{aligned} -D\big(K(x)Dw_f\big) &= f, \quad x \in (-1,1); \qquad w_f(-1) = w_f(1) = 0, \\ -D\big(K(x)Dw_b\big) &= 0, \quad x \in (-1,1); \qquad w_b(-1) = 0,\ w_b(1) = 1. \end{aligned} \tag{28}$$

- Use conventional FEMs to solve (28) for $w_{f,h}$

$$\big(K(x)Dw_h, Dv_h\big)_{L^2(-1,1)} = \big(f, v_h\big)_{L^2(-1,1)}, \quad \forall v_h \in S_h[-1,1].$$

- Use (27) to postprocess $w_{f,h}$ to obtain $u_{f,h}$

$$u_h = u_l + \big(u_r - u_l - {}^{C}_{-1}D^{\beta}_{1} w_f\big)\big({}^{C}_{-1}D^{\beta}_{1} w_b\big)^{-1} {}^{C}_{-1}D^{\beta}_{x} w_b + {}^{C}_{-1}D^{\beta}_{x} w_h. \tag{29}$$

- Evaluating ${}^{C}_{-1}D^{\beta}_{x} w_h$ requires numerical integration of a weakly singular integral, which may introduce some numerical issues.

Theorem

(W., Yang & Zhu 17) Let $0 < \beta < 1/2$, $K \in C^m[-1,1]$, *and* $f \in C^{m-2,\delta}[-1,1]$ *for some* $0 < \delta \leq 1$ *with* $m \geq 2$. *Then,*

$$\|u_h - u\|_{L^2(0,1)} \leq Ch^{m-\beta}$$

where $C = C\big(\beta, m, \|K\|_{C^m[-1,1]}, \|f\|_{C^{m-2,\delta}[-1,1]}\big)$.

- In summary, the indirect FEM
  - has a proved convergence rate, only under the assumptions of the regularity of the data (but not that of the true solution) of the FDE,
  - has a sub-optimal order convergence rate of order β less, due to the fractional post-processing,
  - requires careful evaluation of the fractional post-processing, as that involves the numerical evaluation of singular integrals.

$K = 1/(x+1)$, $u_l = 0$, $u_r = 2$, $\beta = 0.5$, and $u(x) = x^{1-\beta} + x^{9/2}$.

$$f(x) = -\frac{1}{(x+1)^2}\Big(\frac{2\Gamma(11/2)}{(7+2\beta)\Gamma(5/2+\beta)}x^{7/2+\beta} + \frac{\Gamma(11/2)}{\Gamma(7/2+\beta)}x^{5/2+\beta} - \Gamma(2-\beta)\Big).$$

Table: $\|u - u_h\|_{L^2}$ of the IFEM and the FEM, $\beta = 0.5$.

h	$m=2$		$m=3$		$m=4$	
	IFEM	FEM	IFEM	FEM	IFEM	FEM
$1/8$	4.384E-2	2.550E-2	1.855E-3	1.933E-2	3.509E-5	5.787E-3
$1/16$	1.655E-2	1.116E-2	3.365E-4	1.167E-2	3.127E-6	2.624E-3
$1/32$	6.071E-3	5.337E-3	6.022E-5	6.732E-3	2.774E-7	1.302E-3
$1/64$	2.193E-3	2.632E-3	1.071E-5	3.770E-3	2.457E-8	6.582E-4
$1/128$	7.857E-4	1.310E-3	1.899E-6	2.070E-3	2.191E-9	3.323E-4
κ	1.452	1.065	2.484	0.808	3.493	1.024

- The indirect FEMs exhibit the theoretically proved convergence rates.
- The conventional high-order FEMs only have at most the first-order convergence rate, due to the lack of regularity of the true solution.

- $P_N[-1,1]$: the space of polynomials of degree $\leq N$ on $[-1,1]$
- $L_n(x)$: the nth degree Legendre polynomial on $[-1,1]$

$$L_0(x)=1, \quad L_1(x)=x, \quad L_{n+1}(x)=\frac{2n+1}{n+1}xL_n(x)-\frac{n}{n+1}L_{n-1}(x), \; n\geq 1,$$
$$\int_{-1}^{1} L_n(x)L_m(x)dx=\frac{2}{2n+1}\delta_{m,n}, \qquad L_n(\pm 1)=(\pm 1)^n$$

- $\phi_n(x) := L_n(x)-L_{n+2}(x)$ are linearly independent with $\phi(\pm 1)=0$.

$$S_N[-1,1] := \{v\in P_N[-1,1] : v(-1)=v(1)=0\} = \mathrm{span}\{\phi_n\}_{n=0}^{N-2}.$$

- A spectral Galerkin method for problem (1): Seek $u_N \in S_N[-1,1]$ such that

$$\begin{aligned} B(u_N,v_N) &:= -\theta\big(K\ {}_{-1}D_x^{1-\frac{\beta}{2}}u_N, {}_xD_1^{1-\frac{\beta}{2}}v_N\big)-(1-\theta)\big(K\ {}_xD_1^{1-\frac{\beta}{2}}u_N, {}_{-1}D_x^{1-\frac{\beta}{2}}v_N\big) \\ &= \langle f, v_N\rangle, \quad \forall v_N\in S_N[-1,1]. \end{aligned}$$

Theorem

If $u \in H^r \cap H_0^{1-\beta/2}$ *and* $1 - \beta/2 \le s \le r$, *then*

$$\|u_N - u\|_{H^s} \le C N^{-(r-s)} \|u\|_{H^r}, \quad 1 - \beta/2 \le s \le r. \tag{30}$$

Assume full regularity of the dual problem for each right-hand side, then the estimate holds for $0 \le s \le r$.

Theorem

For $0 < \beta < 1/2$ the solution u to the inhomogeneous Dirichlet boundary-value problem of the one-sided version of the FDE (7) can be decomposed as

$$u = u_l + \big(u_r - u_l - {}^C_{-1}D^\beta_1 w_f\big)\big({}^C_{-1}D^\beta_1 w_b\big)^{-1}\, {}^C_{-1}D^\beta_x w_b + {}^C_{-1}D^\beta_x w_f. \tag{31}$$

$$\begin{aligned} -D\big(K(x)Dw_f\big) &= f, \quad x \in (-1,1); \qquad w_f(-1) = w_f(1) = 0, \\ -D\big(K(x)Dw_b\big) &= 0, \quad x \in (-1,1); \qquad w_b(-1) = 0,\ w_b(1) = 1. \end{aligned} \tag{32}$$

- Use SPG to solve (32) (Shen et al 11): Find $w_N \in S_N[-1,1]$ such that

$$\big(K(x)Dw_N, Dv_N\big)_{L^2(-1,1)} = \big(f, v_N\big)_{L^2(-1,1)}, \quad \forall v_N \in S_N[-1,1].$$

- Use (31) to postprocess w_N to obtain u_N

$$u_N := u_l + \big(u_r - u_l - {}^C_{-1}D^\beta_1 w_N\big)\big({}^C_{-1}D^\beta_1 w_b\big)^{-1}\, {}^C_{-1}D^\beta_x w_b + {}^C_{-1}D^\beta_x w_N. \tag{33}$$

- Does the ISPG have the same difficulty as IFEM in evaluating ${}^C_{-1}D^\beta_x w_N$?

- $J_n^{\mu,\nu}(x)$ – the nth order Jacobi polynomials that are orthogonal with respect to the Jacobi weight function $\omega^{\mu,\nu} := (1-x)^\mu(1+x)^\nu$

$$
\begin{aligned}
J_0^{\mu,\nu} &= 1, \quad J_1^{\mu,\nu} = \frac{1}{2}(\mu+\nu+2)x + \frac{1}{2}(\mu-\nu), \\
J_{n+1}^{\mu,\nu} &= \left(a_n^{\mu,\nu}x - b_n^{\mu,\nu}\right)J_n^{\mu,\nu} - c_n^{\mu,\nu}J_{n-1}^{\mu,\nu} \\
&= \frac{n+\mu+1}{n!\Gamma(n+\mu+\nu+1)}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n+k+\mu+\nu+1)}{\Gamma(k+\mu+1)}\left(\frac{x-1}{2}\right)^k, \\
& n \geq 1
\end{aligned}
$$

where $a_n^{\mu,\nu}$, $b_n^{\mu,\nu}$, and $c_n^{\mu,\nu}$ are constants having explicit expressions.

Theorem

(Huang et al 11; Shen et al 11) For $\mu > 0$,

$$
\begin{aligned}
{}_{-1}^{R}D_x^{\mu} L_n(x) &= \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}(1+x)^{-\mu} J_n^{\mu,-\mu}(x), \quad x \in [-1,1], \\
{}_{x}^{R}D_1^{\mu} L_n(x) &= \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}(1-x)^{-\mu} J_n^{-\mu,\mu}(x), \quad x \in [-1,1].
\end{aligned}
$$

- The SPG solution $w_N \in S_N[-1,1]$ can be expressed as

$$
w_N(x) = \sum_{n=0}^{N-2} d_n \phi_n(x) = \sum_{n=0}^{N-2} d_n (L_n(x) - L_{n+2}(x)).
$$

$$
\begin{aligned}
{}_{-1}^{C}D_x^{\beta} w_N = {}_{-1}^{R}D_x^{\beta} w_N &= \sum_{n=0}^{N-2} d_n (1+x)^{-\beta} \Big(\frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} J_n^{\beta,-\beta}(x) \\
&\quad - \frac{\Gamma(n+3)}{\Gamma(n+3-\beta)} J_{n+2}^{\beta,-\beta}(x) \Big).
\end{aligned}
$$

Theorem

(W. & Zhang 15) Let $0 < \beta < 1/2$, $K \in C^m[-1,1]$, and $f \in H^{m-1}(-1,1)$ for any $m \geq 1$. Then,

$$\|u_N - u\|_{L^2(-1,1)} \leq CN^{-m}.$$

where $C = C\big(\beta, m, \|K\|_{C^m[-1,1]}, \|f\|_{H^{m-1}(-1,1)}\big)$.

- Compared to the IFEM, the ISPG has the following salient features. The ISPG
  - has a proved convergence rate in the L^2 norm, only under the assumptions of the regularity of the data (but not that of the true solution) of the FDE,
  - has an optimal order convergence rate of order, which is independent of the post-process of the βth-order fractional differentiation,
  - does not have the subtlety in requiring the numerical integration of a singular integral, but rather, can evaluate the fractional derivative of w_N analytically.

- $K = 1$, $u_l = 0$, $u_r = 2$, and

$$f(x) \quad = -\frac{\Gamma(7)}{2^{2-\beta}\Gamma(5+\beta)}\Big(\frac{x+1}{2}\Big)^{4+\beta}.$$

- This gives the true solution $u(x) = \Big(\frac{x+1}{2}\Big)^{1-\beta} + \Big(\frac{x+1}{2}\Big)^{6}$.
- For SPG, $\|u_N - u\|_{L^2(-1,1)} \leq C_\kappa N^{-\kappa}$.
- For our improvements, $\|u_N - u\|_{L^2(-1,1)} \leq C_\kappa e^{-\kappa N}$.

Table: The comparison of the SPG and ISPG methods (W. & Zhang 15)

	$\|u_{SPG,N} - u\|_{L^2(0,1)}$			$\|u_{ISPG,N} - u\|_{L^2(0,1)}$		
N	$\beta = 0.1$	$\beta = 0.5$	$\beta = 0.9$	$\beta = 0.1$	$\beta = 0.5$	$\beta = 0.9$
4	2.139e-03	5.104e-02	1.677	9.377e-03	2.319e-02	7.737e-02
5	1.334e-03	4.195e-02	0.472	8.451e-04	2.823e-03	1.283e-02
6	9.014e-04	3.431e-02	1.331	6.482e-06	1.087e-04	9.541e-04
7	6.738e-04	2.676e-02	0.439	4.185e-07	3.892e-06	7.135e-06
8	5.204e-04	2.308e-02	1.119	5.348e-08	3.943e-07	5.563e-07
9	4.126e-04	1.913e-02	0.415	9.807e-09	6.239e-08	7.625e-08
10	3.342e-04	1.691e-02	0.986	2.280e-09	1.307e-08	1.468e-08
11	2.755e-04	1.454e-02	0.395	6.296e-10	3.324e-09	3.481e-09
12	2.306e-04	1.309e-02	0.893	1.984e-10	9.811e-10	9.807e-10
13	1.955e-04	1.154e-02	0.380	6.952e-11	3.248e-10	3.105e-10
14	1.676e-04	1.052e-02	0.824	2.656e-11	1.183e-10	1.097e-10
15	1.450e-04	9.439e-03	0.366	1.091e-11	4.659e-11	4.182e-11
κ	2.016	1.315	0.600	1.800	1.817	1.985

- The indirect SPG
  - exhibits the exponential convergence rate in the L^2 norm, under the assumptions of the regularity of the data (not the solution) of the FDE,
  - has the convergence rate independent of the order $0 < \beta < 1/2$.
  - The conventional SPG methods seem to have low-order (β-dependent) algebraic convergence rates, if measured in the standard L^2 norm.
- Spectral methods were developed and analyzed for two-sided constant coefficient FDEs (Chen at al 16, Ervin et al 16, Mao & Karniadakis 18)
  - which have proved high-order convergence rates in the appropriately weighted Sobolev spaces, only assuming the smoothness of data in some corresponding weighted Sobolev spaces.
- MultiD analogue of (1) with constant K was proved to be wellposed (Ervin & Roop 07). Regularity, numerical approximations under the smoothness of data, and variable-coefficient problems require further study!!!

Thank You

for Your Attention!