Algebraic Combinatorics

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1 Introduction

Algebraic Combinatorics studies the relationship between algebra (especially linear algebra and group theory) and combinatorial problems. The study of combinatorics traces back to the roots of mathematics. The Rhind Mathematical Papyrus, a 16th century BCE Egyptian text, contains early combinatorial problems [1]. Combinatorics maintains prevalency throughout the development of mathematics. In the 18th century, Euler breaks new ground with the development of graph theory, which he applies to the famous Seven Bridges of Königsberg problem. Euler shows that there was no feasible walk through Königsberg that passes over each of its seven bridges only one time. However, during the 19th and early 20th century, as other mathematical fields such as analysis and algebra are extensively unified and and standardized, combinatorics remains an expansive and disjointed field. G.C. Rota notes that people viewed combinatorics as a field with "too many theorems, matched with very few theories" [2]. However, with the rise of computer science in the mid 20th century, combinatorics gains increased significance and urgency. Rapid development occurs under mathematicians such as John von Neumann, G.C. Rota, and Richard Stanley [2]. As combinatorics develops, so does its relationships with other fields of math, especially algebra.

2 Background

Given the finite set S and the integer $k \geq 0$, $\binom{S}{k}$ denotes the set of k-element subsets of S. A **multiset** is a set with repeated elements. For instance, while 1 and $1, 1$ are the same set, they are not the same multiset. We have that $\left(\binom{S}{k}\right)$ denotes the set of k-element multisets of S. If we abbreviate a set/multiset such as $\{1, 2\}$ as 12, then for $S = \{1, 2, 3\}$, we have

$$\binom{S}{k} = \{12, 13, 23\}, \quad \left(\binom{S}{k}\right) = \{11, 22, 33, 12, 13, 23\}.$$

We can now formally define a graph. A **graph** is a triple (V, E, φ) with a **vertex set** $V = \{v_1, v_2, \ldots, v_p\}$, an **edge set** $E = \{e_1, e_2, \ldots, e_q\}$ and a function $\varphi : E \to \left(\binom{V}{2}\right)$. For example, if we have $V = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2, e_3\}$, and φ such that $\varphi(e_1) = v_2v_2$, $\varphi(e_2) = v_1v_2$, $\varphi(e_3) = v_1v_3$, then the associated graph is shown below in Figure 1. We now introduce some linear algebra to help us study these graphs. An **adjacency matrix** is an $p \times p$ matrix $\mathbf{A} = \mathbf{A}(G)$ over the field of complex numbers such that the (i, j)-entry $a_{i,j}$ is

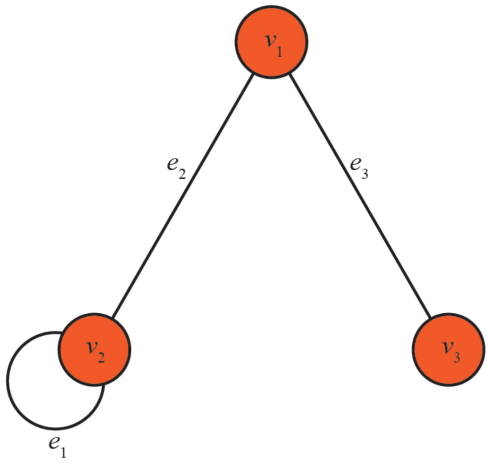

Figure 1

the number of edges between v_i and v_j. Thus for Figure 1, we have that

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The use of a matrix does more than just provide a way to represent the graph. The matrix helps us compute different properties of the graph. Let G be an arbitrary graph and take some integer $l \geq 1$. We define a **walk** of length l from vertices u to v to be a sequence $v_1, e_1, v_2, e_2, \ldots, v_l, e_l, v_{l+1}$ where each v_i and e_j are vertices and edges of G respectively, $v_1 = u$, $v_{l+1} = v$, and the vertices of e_i are v_i and v_{i+1}. It follows from the definition of matrix multiplication that for any integer $l \geq 1$, the (i, j)-entry of $\mathbf{A}(G)^l$ is the number of walks of length l from v_i to v_j.

The tools that matrices provide go deeper than surface level applications such as matrix multiplication. Take another arbitrary graph G with p vertices and a $p \times p$ adjacency matrix $\mathbf{A}$. We notice that the (i, j)-entry and the (j, i)-entry of $\mathbf{A}$ is the same real number. Therefore, $\mathbf{A}$ is a real symmetric matrix. Thus $\mathbf{A}$ has p linearly independent real eigenvectors which can be choosen to be orthonormal. Let $u_1, u_2, \ldots, u_p$ be these vectors with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$. If $U = (u_{ij}) = [u_1, u_2, \ldots, u_p]$, then we have that U diagonalizes $\mathbf{A}$, that is

$$U^{-1}\mathbf{A}U = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p).$$

It follows that for any two vertices v_i and v_j of G, there exist real numbers $c_1, c_2, \ldots, c_p$ such that for $l \geq 1$, we have

$$(\mathbf{A}^l)_{ij} = c_1\lambda_1^l + c_2\lambda_2^l + \cdots + c_p\lambda_p^l.$$

Furthermore, we have that

$$c_k = u_{ik}u_{jk}.$$

If we define a **closed walk** of G as a walk that ends at the vertex it starts at, then a corlloary is that the total number $f_G(l)$ of closed walks of length l in G is given by

$$f_G(l) = \lambda_1^l + \lambda_2^l + \cdots + \lambda_p^l = \operatorname{tr}(\mathbf{A}^l).$$

3 Proposed Methodology

Our study of algebraic combinatorics does not work towards a single result or application but instead builds a cumulative understanding of the subject. We use [3] to guide this study. Each week we read a new chapter of the book and work through the exercises provided at the end chapter. The exercises not only reinforce the material covered but extends and abstracts the material in different directions. Chapter 1 of [3] covers the essential information about adjacency matrices that is foundational for the rest of the book. It also studies different examples of graphs such as complete graphs and bipartite graphs. We note that the exercises study and abstract these examples in depth.

From here, each chapter of [3] studies a different connection between algebra and graph theory with the adjancency matrix and path counts appearing as a common theme. These chapters touch upon a wide array of subjects that include random walks, group actions, and Young diagrams. This also shows how the book goes beyond standard linear algebra into more difficult topics from algebra such as group theory.

The chapters are not entirely disjoint and chapters often refer to examples and results from previous chapters. For instance, in chapter 2 we study the n-cube, C_n. The vertex set of C_n is $\mathbb{Z}_2^n$ and there is an edge between any two vertices u, v if they differ in only one component, that is $u + v$ has only one component equal to 1 modulo 2. The n-cube is an important example which provides intuition for subsequent chapters. Furthermore, chapter 2 introduces the discrete Radon transform, a type of linear transformation of the vector space $\mathcal{V}$ of all functions $f : \mathbb{Z}_2^n \to \mathbb{C}$ to itself [3]. The discrete Radon transformation provides an essential tool to study C_n and count the walks between any two vertices of C_n. While chapter 2 confines itself to $\mathbb{Z}_2^n$, the end of the chapter alludes to how the material will be built upon in subsequent. The results for the n-cube, such as the finite Radon transform, can be abstracted to more general settings, with $\mathbb{Z}_2^n$ replaced by some arbitrary group (abelian or nonabelian) G. In one exercise, $\mathbb{Z}_2^n$ is replaced with the cyclic group $\mathbb{Z}_n$, and all the results from the chapter are then reproved for this setting.

Each chapter also progresses in difficulty. One example of a difficult result that we are working up to the Matrix-Tree theorem, which is Theorem 9.8 in [3]. Let G be a finite connected graph without loops, with laplacian matrix $\mathbf{L} = \mathbf{L}(G)$. Let $\mathbf{L}_0$ denote $\mathbf{L}$ with the last row and column removed (or with the ith row and column removed for any i). Then $\det \mathbf{L}_0 = \kappa(G)$, where $\kappa(G)$ represents the number of spanning trees of G. The theory developed in these chapters with graphs and trees also have concrete applications. In chapter 11 of [3], Theorem 11.14 directly applies algebraic combinatorics to the study of electrical networks.

References

[1] R. Graham, *Handbook of Combinatorics*, Handbook of Combinatorics, Elsevier Science, 1995.

[2] R. P. Stanley, *Enumerative combinatorics volume 1 second edition*, Cambridge studies in advanced mathematics, (2011).

[3] ——, *Algebraic Combinatorics: Walks, Trees, Tableaux, and More*, Springer, 2018.