The Form of Swarms: A Primer of Swarm Equilibria

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Abstract:

Biological aggregations (swarms) exhibit morphologies governed by social interactions and responses to environment. Starting from a particle model we derive a nonlocal PDE, known as the aggregation equation, which describes evolving population density. The solutions to the aggregation equation can exhibit a variety of behaviors including spreading without bound, concentrating into δ-functions and formation of compactly supported equilibria. We describe some tools for investigating the asymptotic behavior of solutions. We also study equilibria and their stability via the calculus of variations which yields analytical solutions. Finally we present a case study about how these methods can be used to construct a model of locust swarms.

Useful References:

- A. J. Bernoff & C. M. Topaz, "Nonlocal aggregation models: A primer of swarm equilibria." SIAM Review 55 (2013) 709-747*. Note the introduction and epilogue to this paper contain a potentially useful literature review.*
- A. J. Leverentz, C. M. Topaz & A. J. Bernoff. "Asymptotic Dynamics of Attractive-Repulsive Swarms," SIAM J. Appl. Dyn. Sys. 8 (2009) 880-908.
- C. M. Topaz, A. J. Bernoff, S. Logan & W. Toolson, "Aggregations, Interactions, and Boundaries: A Minimal Model for Rolling Swarms of Locusts," Eur. Phys. J. - Spec. Top. 157 (2008) 93-109.

Many aggregations have sharp boundaries.


Plate 3. Wildebeest massing in a grazing front on the Serengeti Plains. March 1973.

Sinclair, 1977

Biological question

How are individual behavior and group behavior connected?

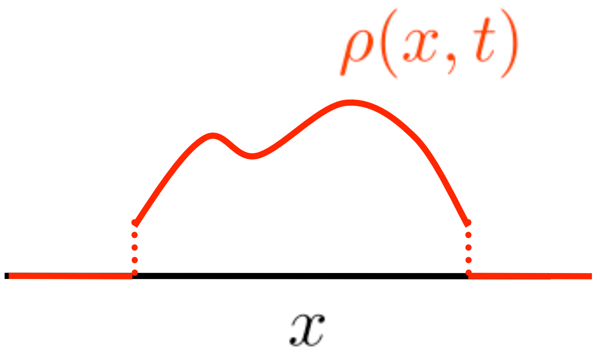
Mathematical question

How do different social interaction kernels affect the (asymptotic) macroscopic behavior of a solution?

A Continuum Model of Swarming

Three Key Ideas:

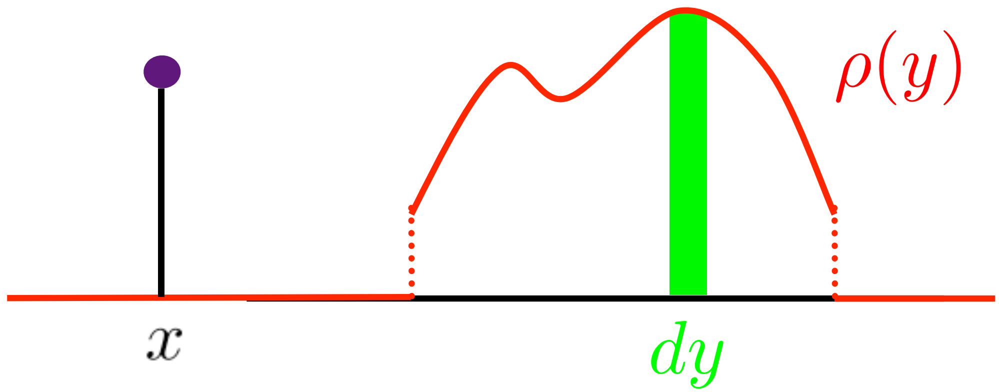
- Continuum model of swarm density, $\rho(x,t)$.
- Individuals interact pairwise and social forces are additive.
  - Repel at short distances (collision avoidance)
  - Attract at longer distances (swarming behavior)
- Kinematic Model (velocity $\propto$ social forces)



Previous Work:

- *Bodnar & Velazquez* (2006) considered well-posedness and blow-up in 1D.
- *Bertozzi & Laurent* (2007) proved axisymmetric blow-up in N-dimensions.
- *Bertozzi & collaborators* (2008 . . .) working on general blow-up in N-dimensions.

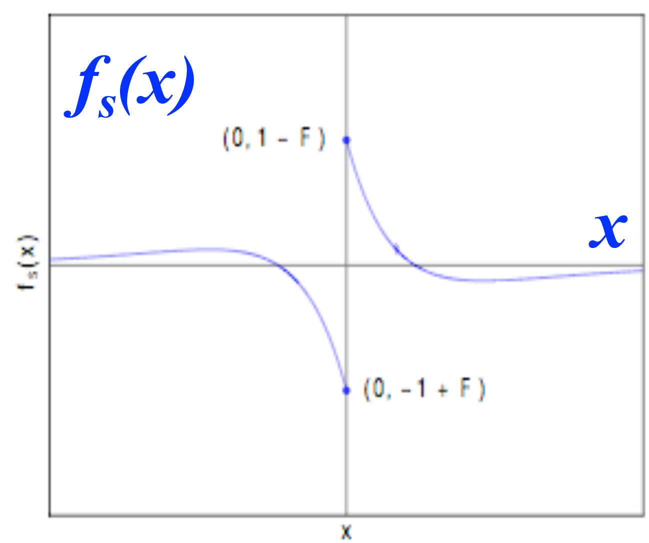
Computing the Velocity



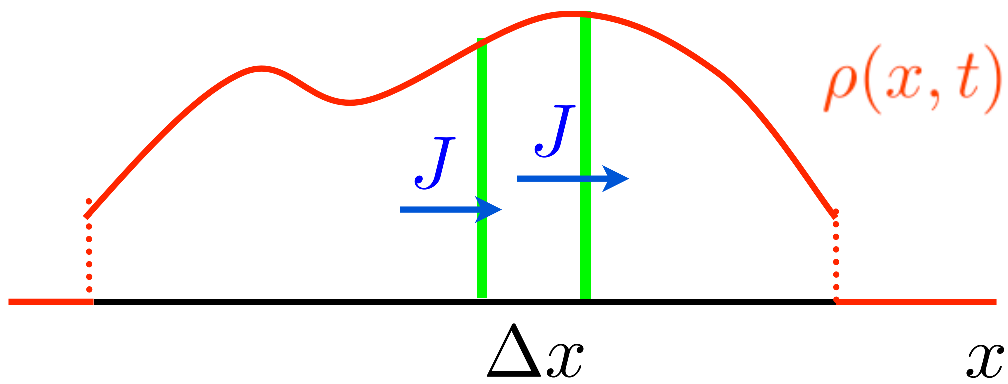
Velocity for a test mass at x from the density $\rho(y)$

$$v(x) = \int_{y=-\infty}^{\infty} f_s(x-y)\rho(y)\ dy \ \equiv f_s * \rho$$

Where $f_s(x)$ is the *social force* induced by the mass distribution with density $\rho(y)$.



Conservation of Mass



The flux, J, is given by:

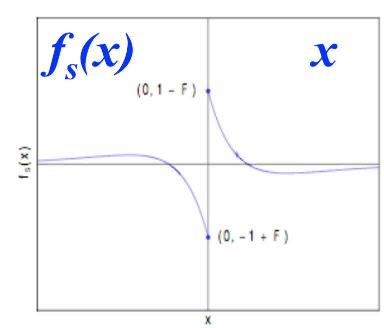
$$J = v\rho$$

So the equation of conservation of mass is:

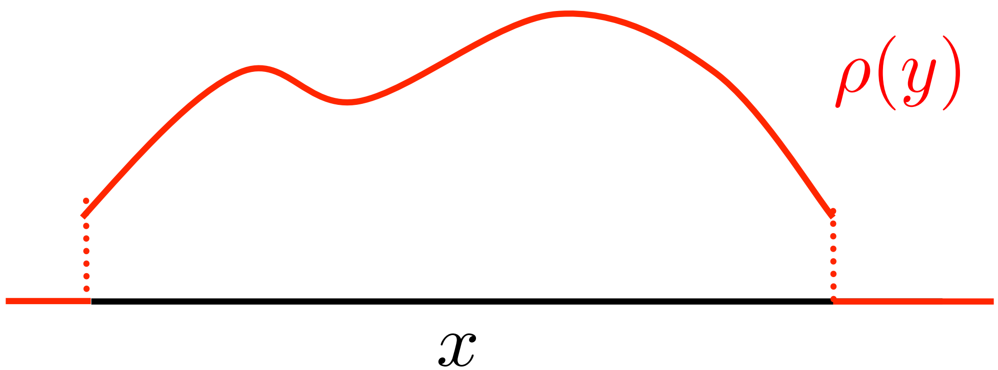
$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0 \quad \Rightarrow \quad \rho_t + (v\rho)_x = 0$$

And the velocity is given by

$$v = \int_{-\infty}^{\infty} f_s(x-y)\rho(y)\, dy \equiv f_s * \rho$$



Conservation of Mass



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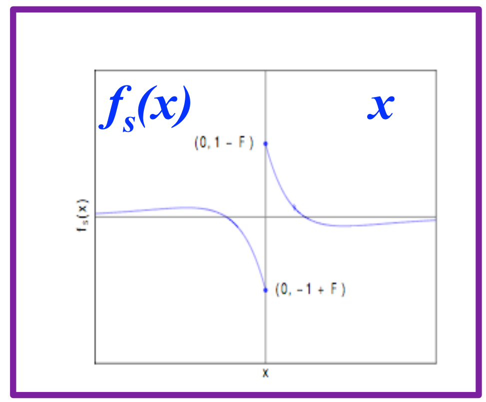
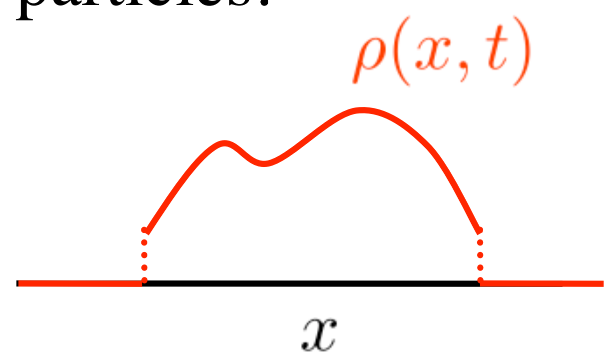


Continuum vs. Discrete

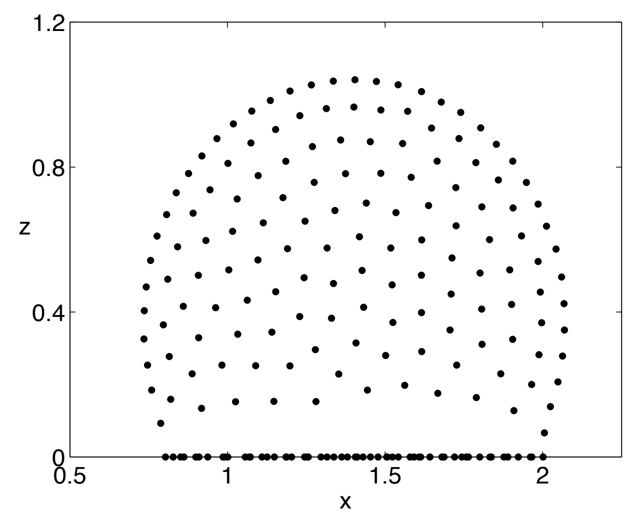
Continuum Model:

- 😡 Difficult to simulate, particularly, in higher dimensions
- 😃 Simulation difficulty independent of number of particles.
- 😕 Relationship to biology not always clear.
- 😃 Much better tools for analysis



Discrete Model:

- 😃 Easy to simulate a small number of particles.
- 😃 Easy to generalize to higher dimensions.
- 😃 Relationship to biology well understood.
- 😡 Difficult to analyze or find exact solutions.



Meet Andy.

SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 8, No. 3

Asymptotic Dynamics of Attractive-Repulsive Swarms*

Andrew J. Leverentz†, Chad M. Topaz‡, and Andrew J. Bernoff†

Andy Leverentz
HMC 2008

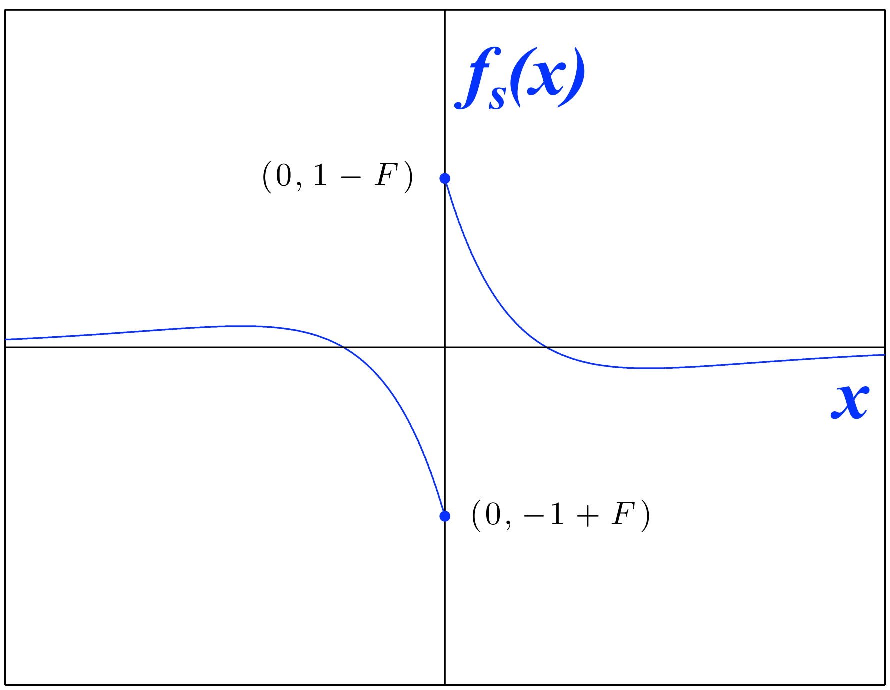
The Continuum Model (1D)

$$
\begin{aligned}
\rho_t &+ (\rho v)_x = 0 \\
v &= \int_{-\infty}^{\infty} f_s(x-y)\rho(y)\,dy \\
&\equiv \rho * f_s
\end{aligned}
$$

Conservation of Mass

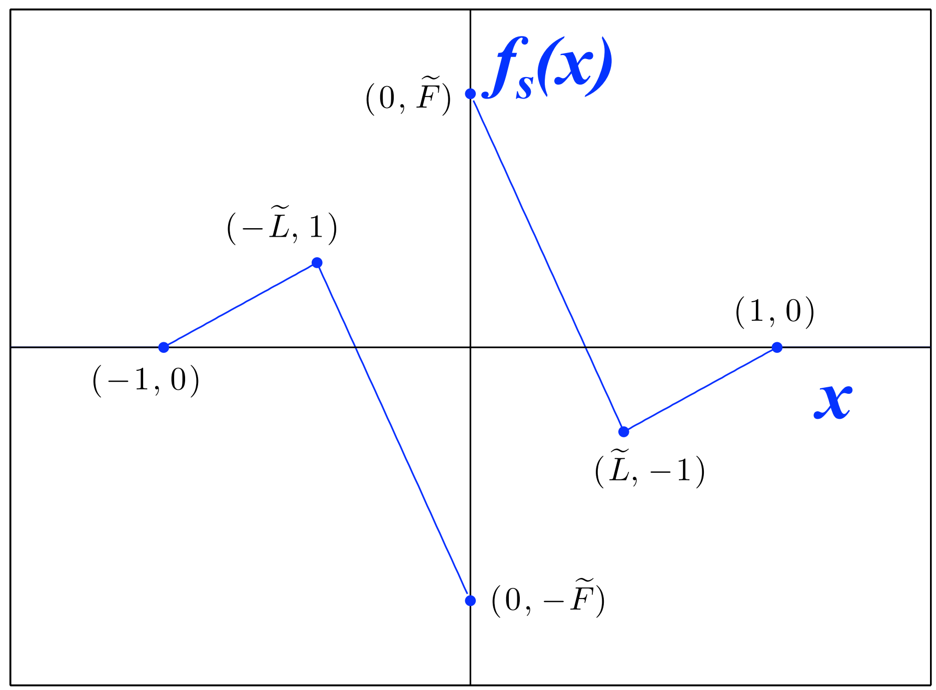
- Odd
- Finite first moment
- Cont. & piecewise diff. (x ≠ 0)
- Jump discontinuity (x = 0)
- Crosses zero for exactly one |x|

Morse Social Force

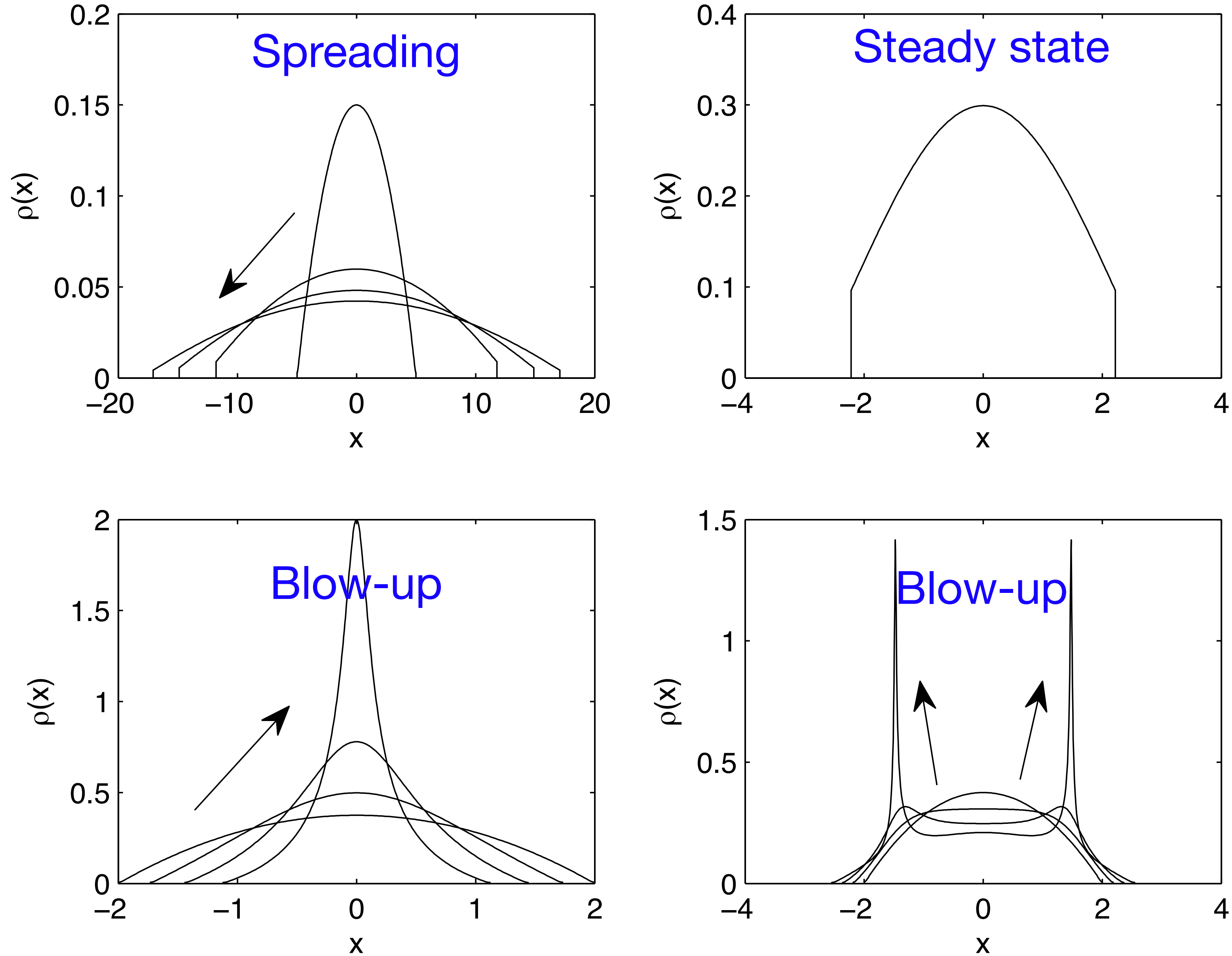


$$f_s(x) = \mathrm{sgn}(x)\left(e^{-x} - Fe^{-x/L}\right)$$

Piecewise Linear Social Force

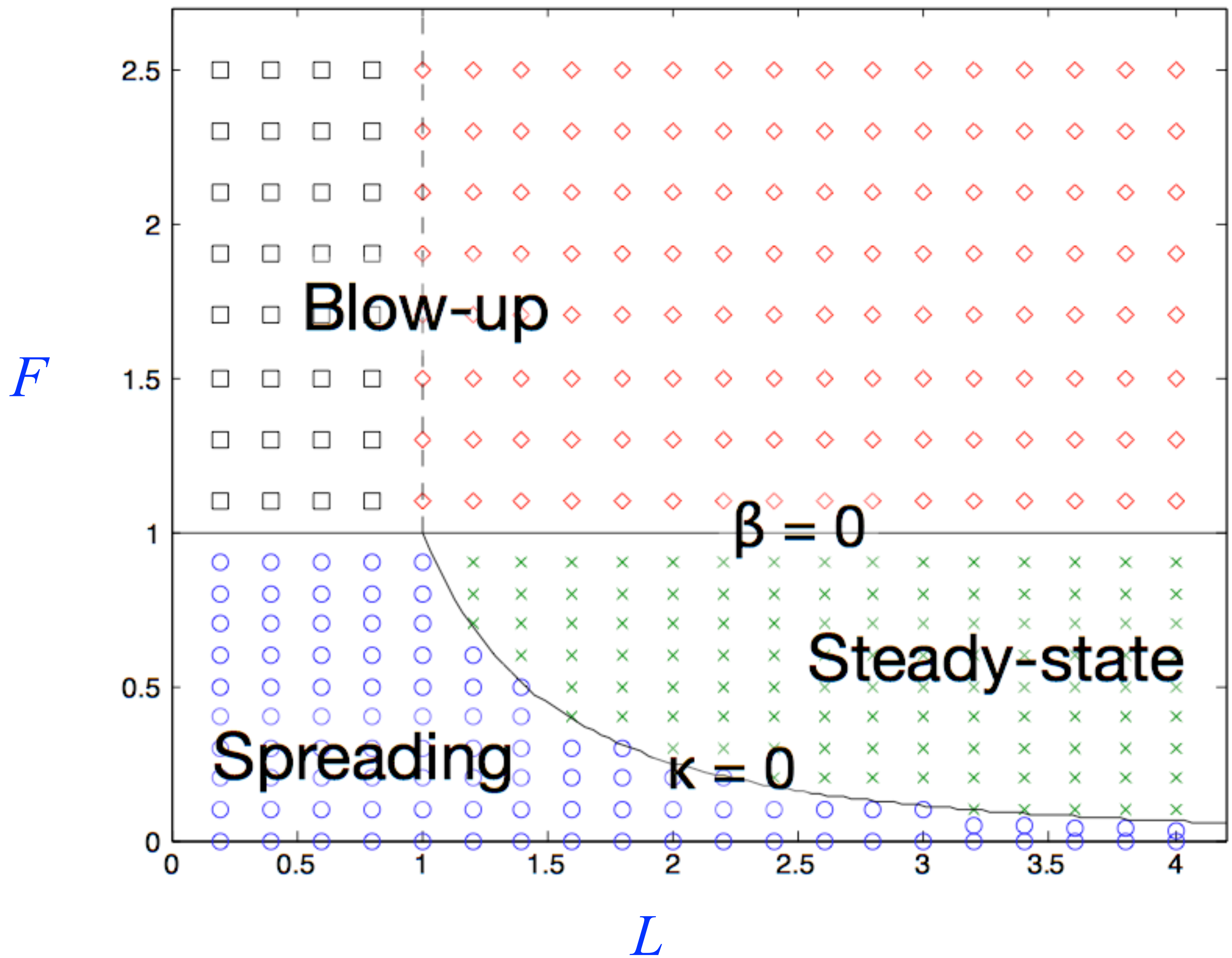


A Menagerie of Behaviors

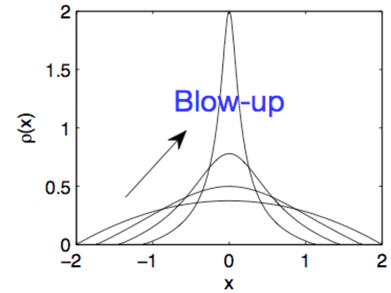


A Phase Diagram

Morse Social Force: $f_s(x) = \text{sgn}(x)\left(e^{-x} - Fe^{-x/L}\right)$

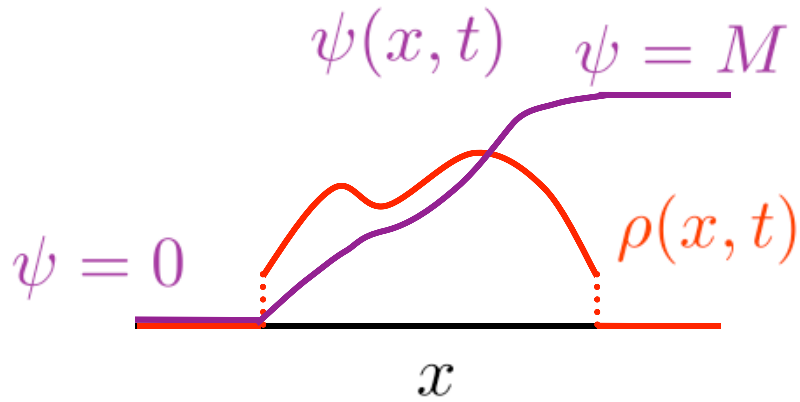


Cumulative Mass Function



Define the cumulative mass function:

$$\psi(x,t) = \int_{-\infty}^{x} \rho(z,t)\ dz$$

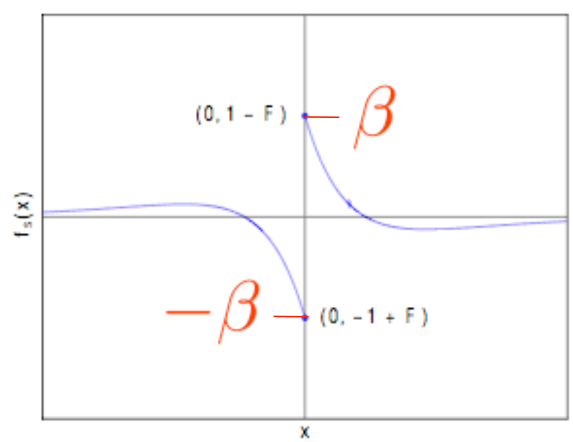


Where ψ satisfies a wave equation

$$\psi_t + v\psi_x = 0$$

And the velocity can be approximated as

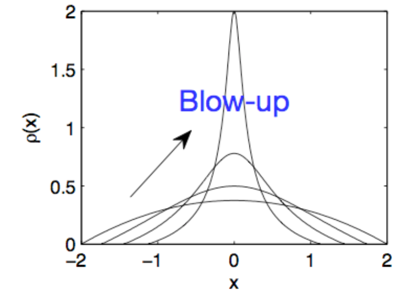
$$\begin{aligned} v &= \int_{-\infty}^{\infty} f_s(x-z)\rho(z)\ dz \\ &\approx \beta\psi + \text{“regular”} \end{aligned}$$



Where $\beta = f_s(0^+)$

Note that as the density is positive, ψ , is increasing.

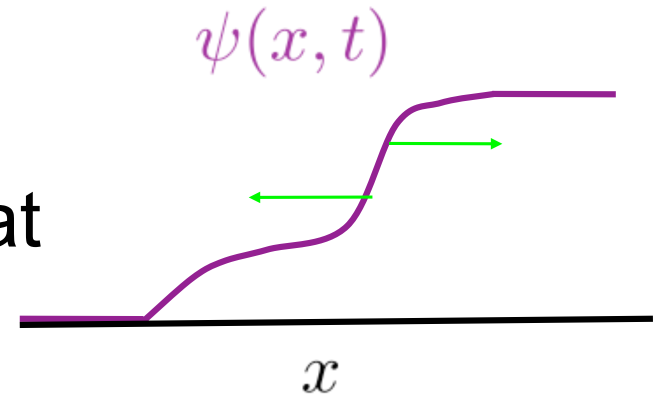
Burger's Equation



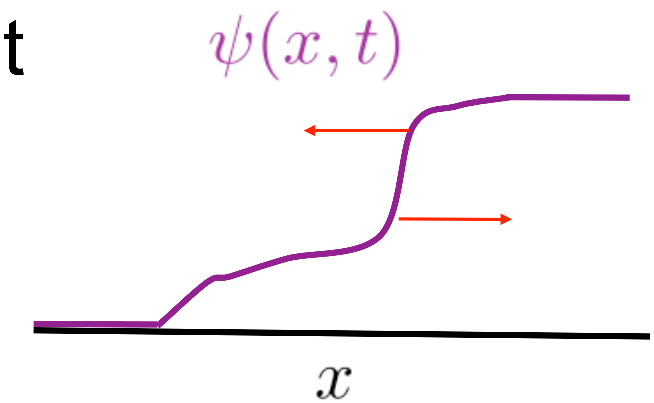
The dynamics are dominated by

$$\psi_t + \beta\psi\psi_x \approx 0 \qquad \beta = f_s(0^+)$$

- If $\beta > 0$ social forces are repulsive at small distances and density is bounded for all t.



- If $\beta < 0$ social forces are attractive at small distances and ψ will form a shock.



But as $\rho = \psi_x$

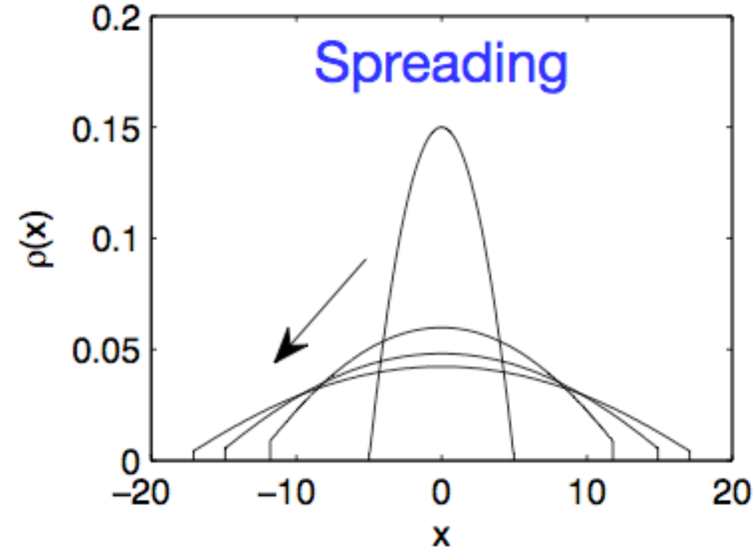
<u>Shocks = Density δ-functions!!</u>

See Bodnar & Velazquez (2006)

Spreading Swarms

Expand velocity at long lengthscales:

$$v = \int_{-\infty}^{\infty} f_s(x-y)\rho(y)\,dy$$
$$\approx -\kappa\rho_x$$



Where κ is the first moment

$$\kappa = \int_0^{\infty} z f_s(z)\,dz$$

Substitute into C of M,

$$\rho_t + (\rho v)_x = 0$$
,

to yield the porous media equation,

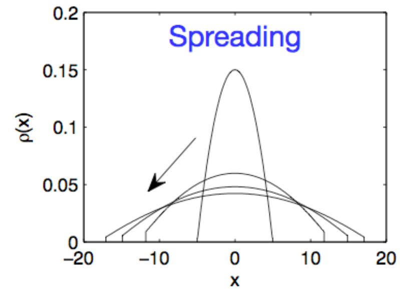
$$\boxed{\rho_t = \kappa(\rho\rho_x)_x}$$

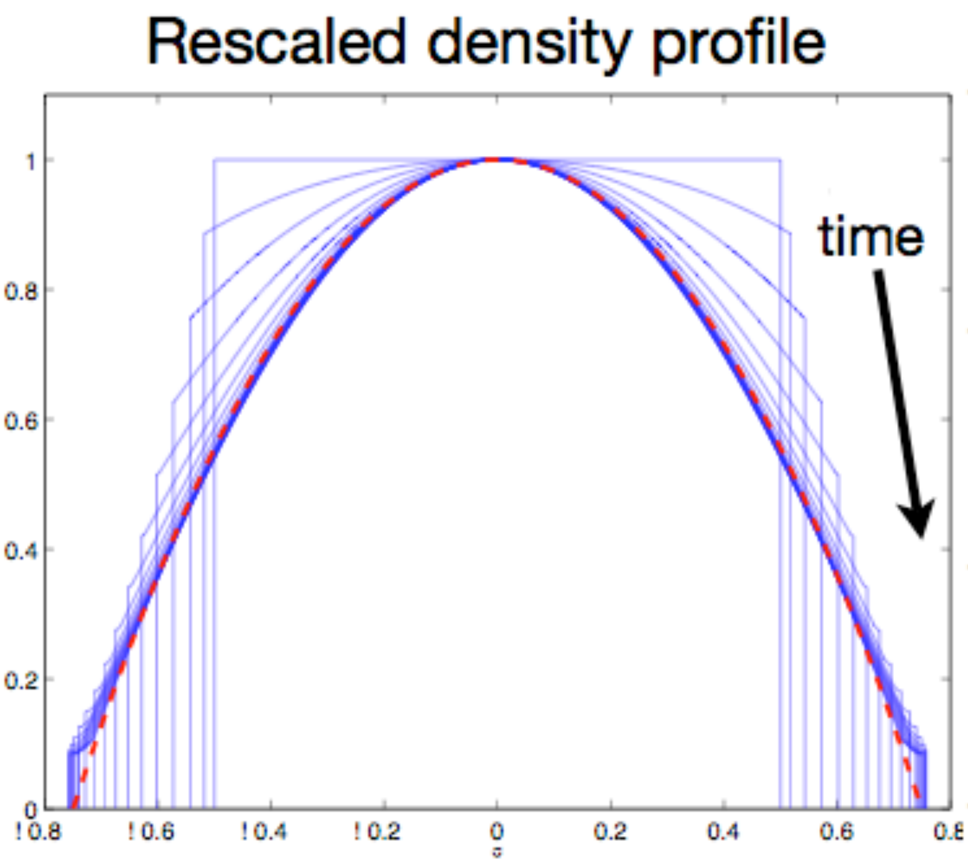
which has spreading solutions for $\kappa < 0$.

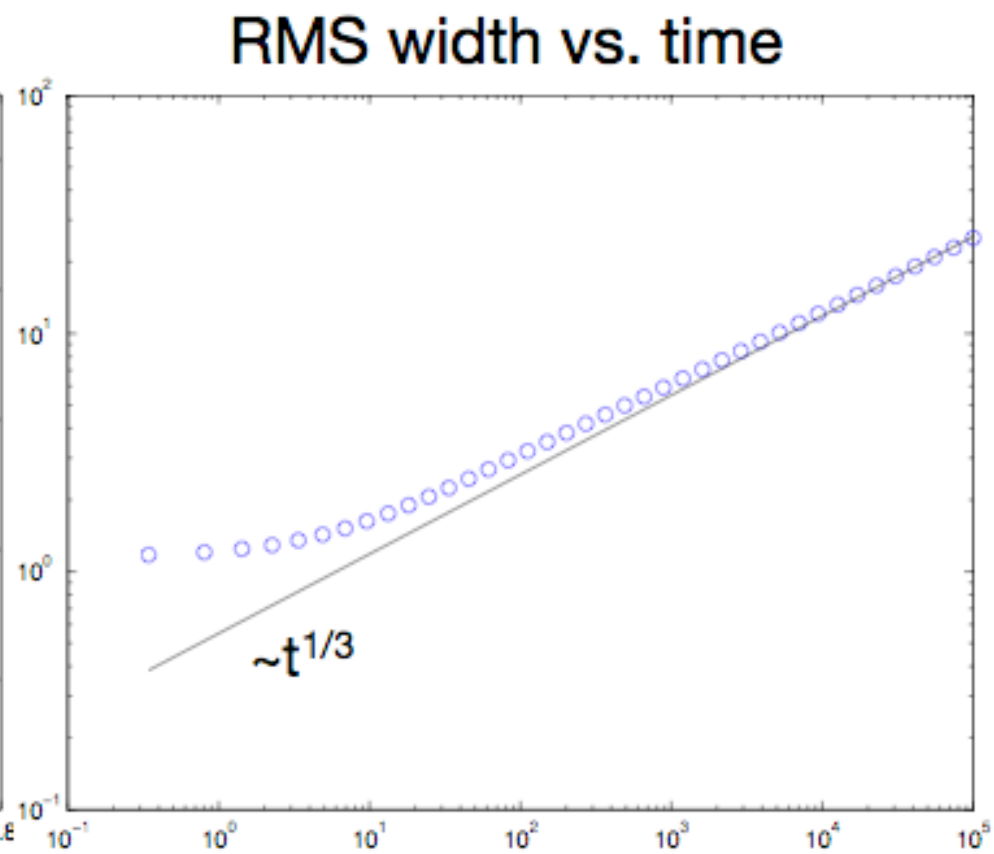
$$\rho(x,t) = \frac{3^{1/3}M^{2/3}}{4[\kappa(t-t_0)]^{1/3}}\left[1-\left(\frac{x-x_0}{[9M\kappa(t-t_0)]^{1/3}}\right)^2\right]_+$$

Barenblatt Solution

Barenblatt Solution



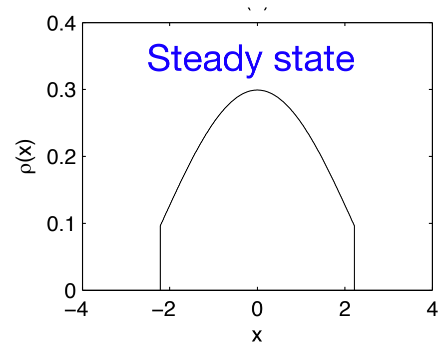




Rescaled Density Approaches Barenblatt Solution

RMS width has proper time dependence

Finding Steady States



When $\beta > 0,\ \kappa < 0$ we expect steady solutions.

Energy $\xrightarrow{\text{first variation}}$ Fredholm integral equation

Details Coming Later

Fredholm Integral Equation

$$\int_{\alpha}^{\beta} V(|z - z'|)\rho(z')\,dz' = \lambda$$

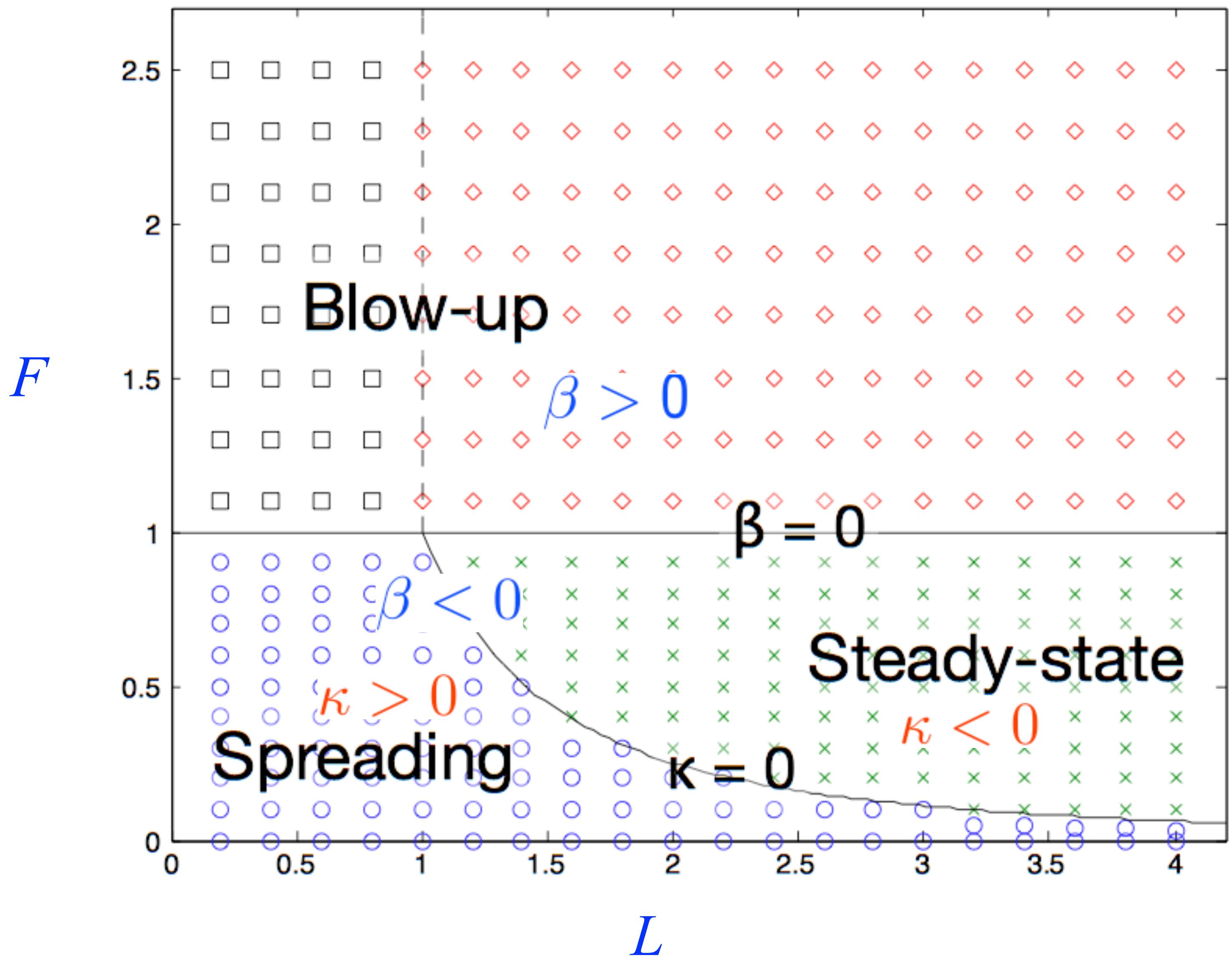
Mass Constraint

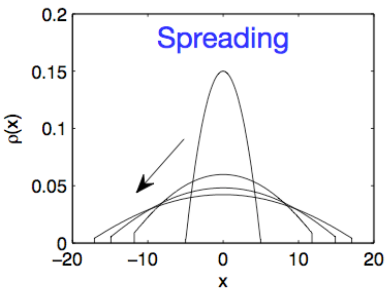
$$\int_{\alpha}^{\beta} \rho(z')\,dz' = M$$

Where $\alpha < z < \beta$ is the support of the solution

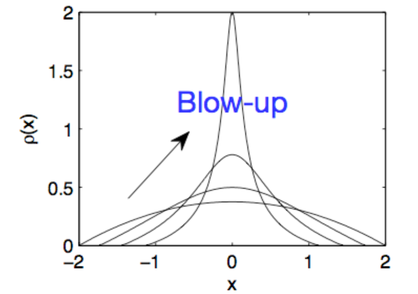
Phase Diagram Revisited

Morse Social Force: $f_s(x) = \mathrm{sgn}(x)\left(e^{-x} - Fe^{-x/L}\right)$





Some conclusions



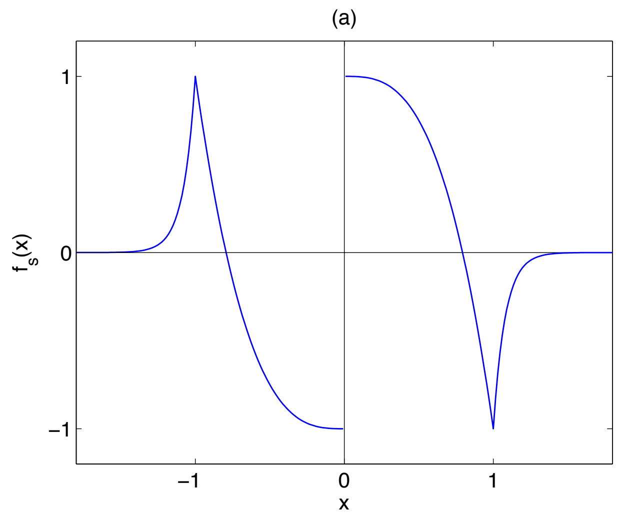
- Repulsion (attraction) at the origin suppresses (causes) blow-up and density concentrations.
- First moment governs spreading at long lengthscales. Spreading governed by porous media equation.
- Steady states can be found via integral equations.

Open Questions

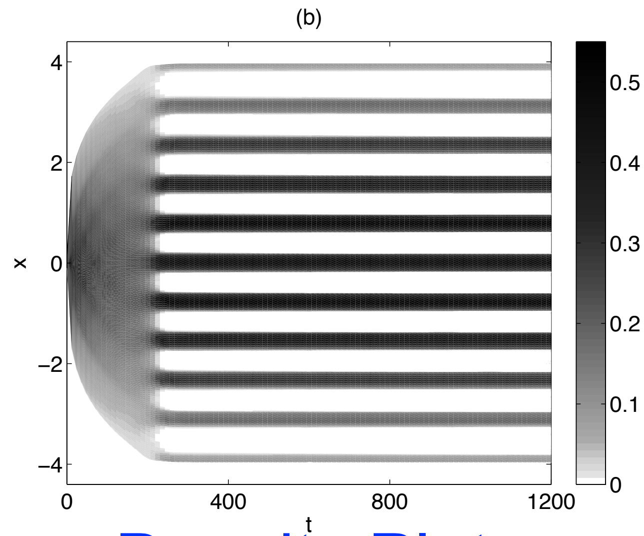
- Necessary and sufficient conditions for spreading and convergence to Barenblatt solution.

A fly in the ointment . . .

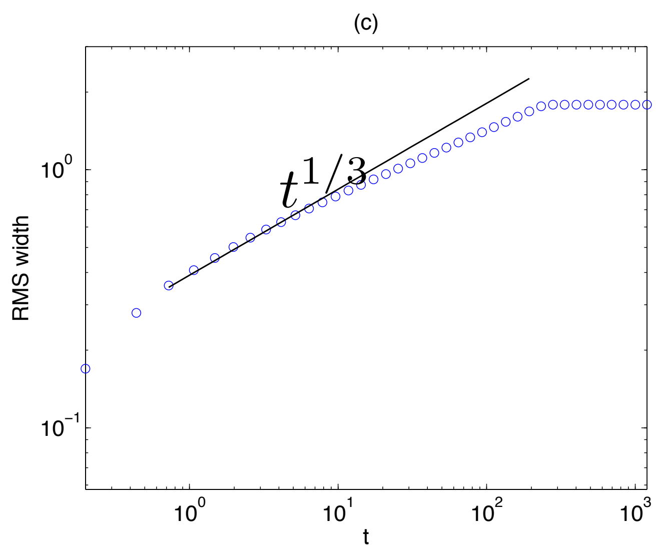
This potential has $\beta > 0$, $\kappa > 0$ which suggests spreading



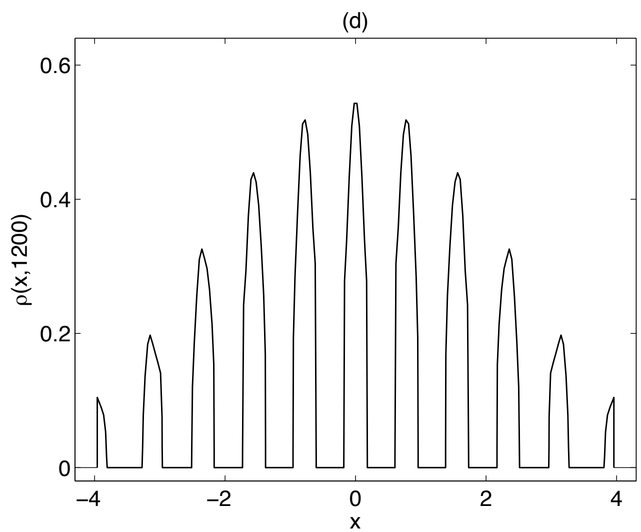
Social Force



Density Plot



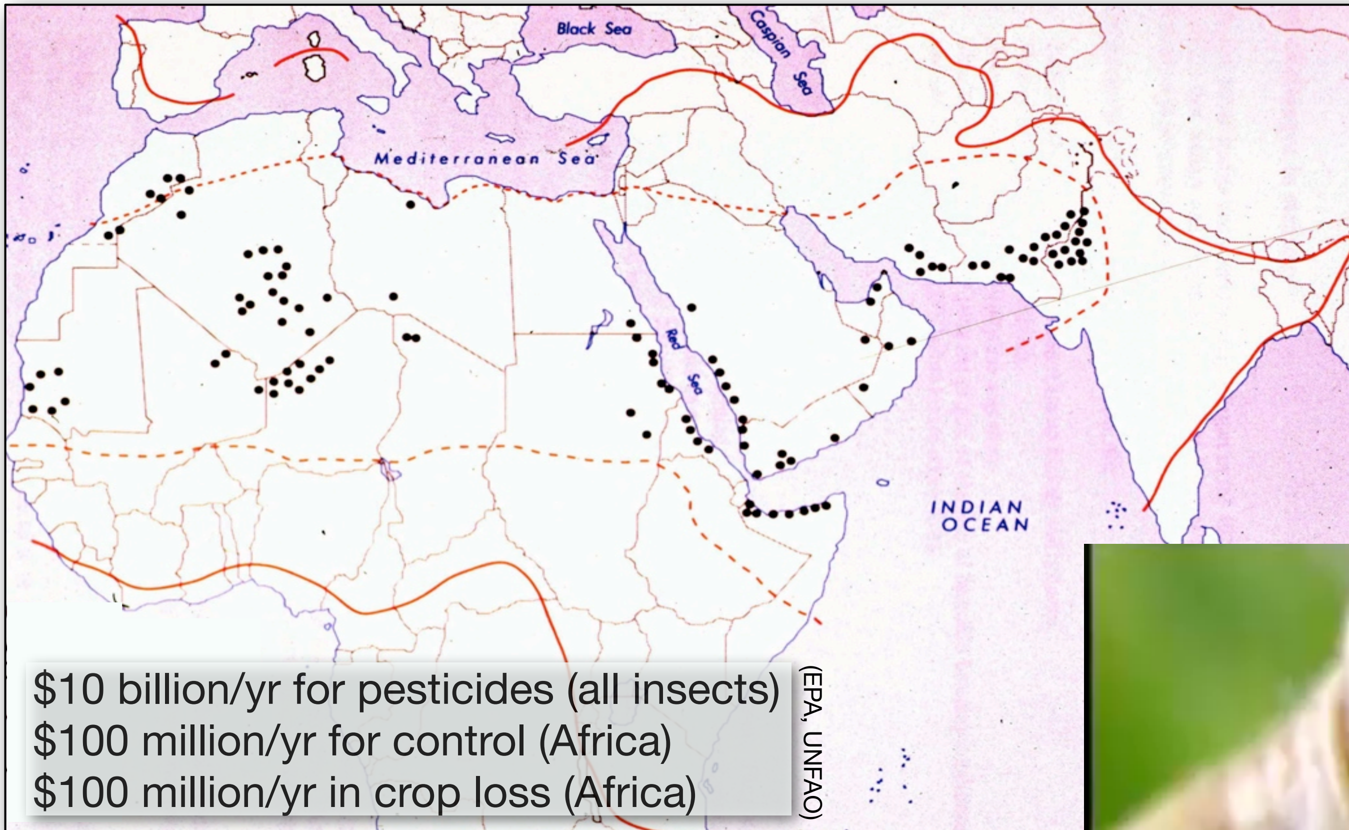
RMS Width



Final Steady State

Let’s focus on locusts . . .

Locusts form giant, destructive swarms.

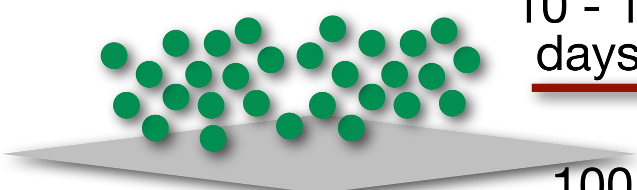


$10 billion/yr for pesticides (all insects)
$100 million/yr for control (Africa)
$100 million/yr in crop loss (Africa)

(EPA, UNFAO)

National Geographic, "A Perfect Swarm"

10^{10} locusts



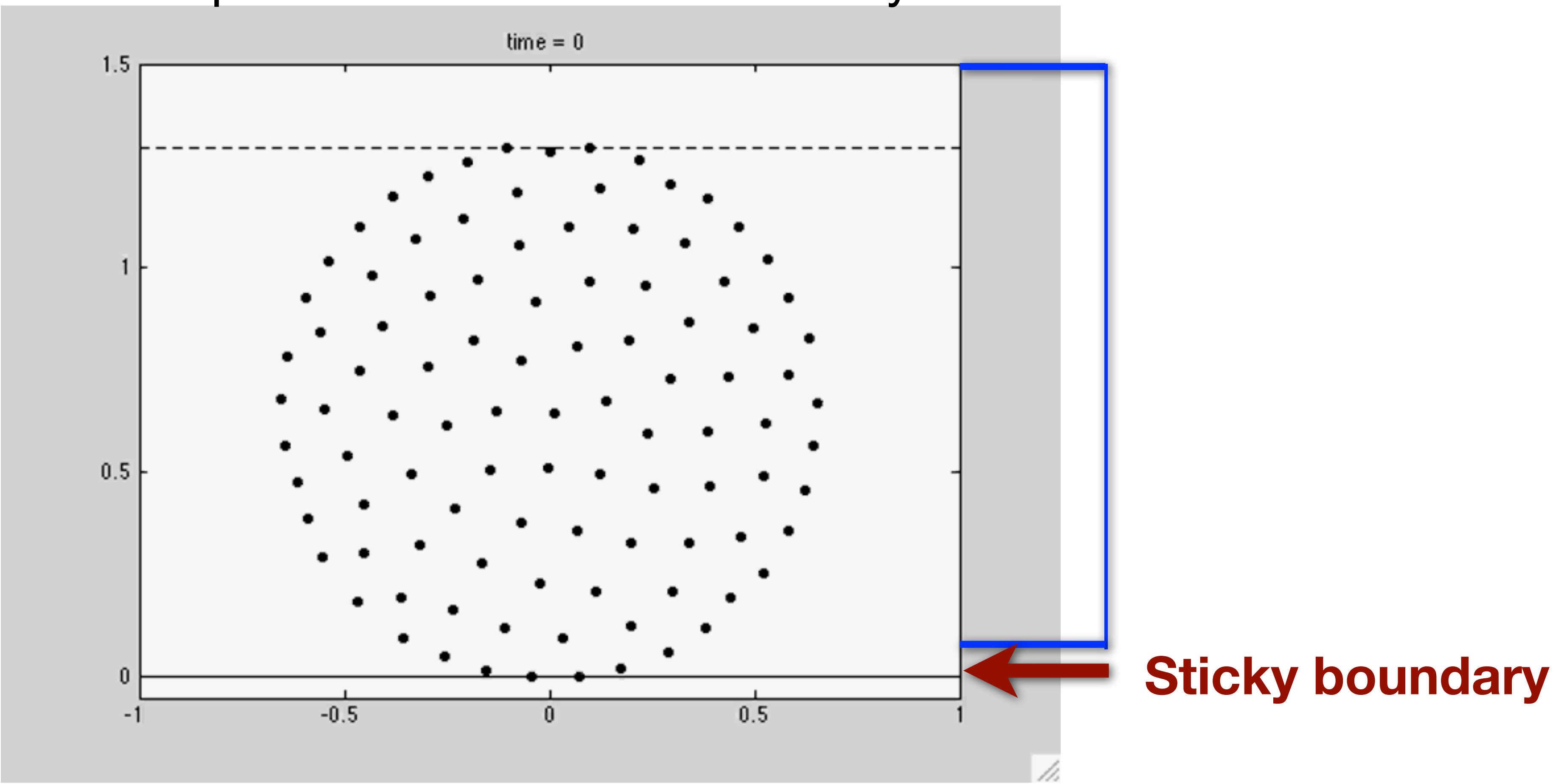
100 km^2

10 - 100 km/day
days or weeks

With gravity, swarm forms a bubble.

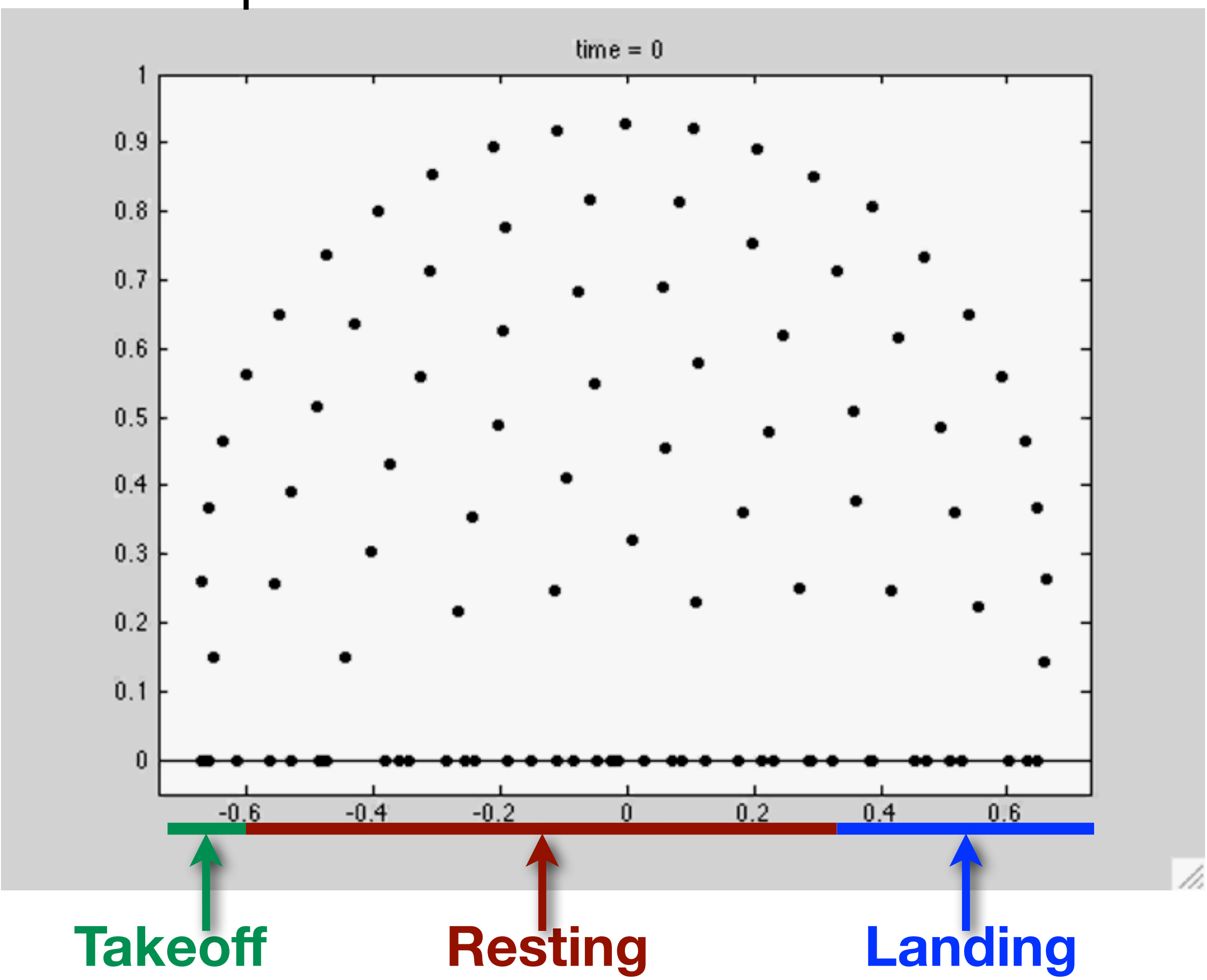
Catastrophic

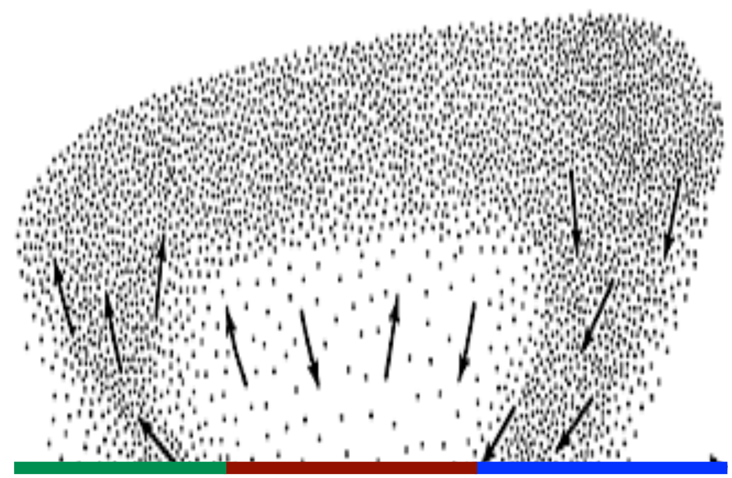
Gravity G = 1



With wind, swarms roll

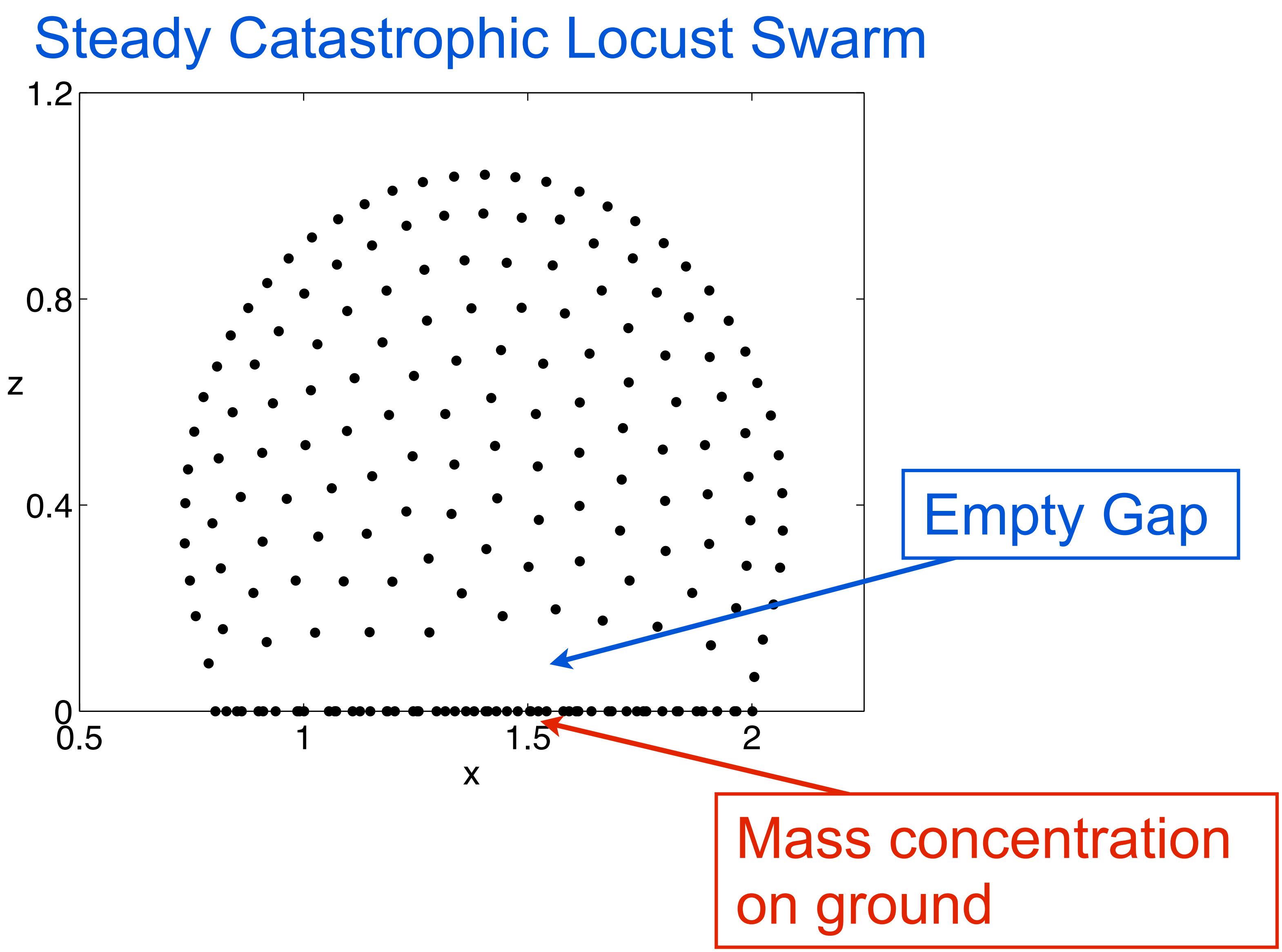
Catastrophic | Wind U = 1





Uvarov, Grasshoppers & Locusts (1977)

Can we explain the swarm morphology?

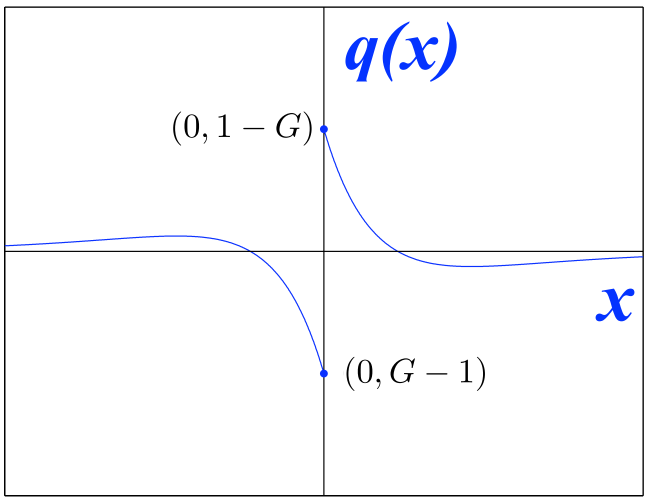


A Continuum Swarming Model

Velocity

$$V(x) = \int_\Omega q(x-y)\rho(y)\,dy + f(x)$$

External Force



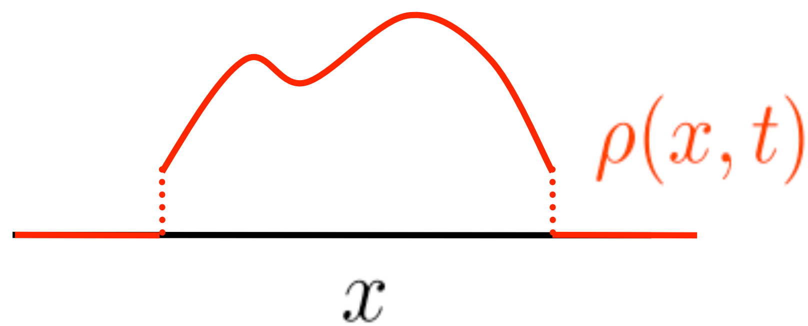
Morse Social Force

$$q(x) = \mathrm{sgn(x)}\left[\mathrm{e^{-x} - Ge^{-x/L}}\right]$$

- Pairwise, additive
- Antisymmetric
- Finite first moment
- Jump discontinuity (x = 0)

Conservation of Mass

$$\rho_t + (\rho V)_x = 0,$$



- Finite Mass
- Compact Support

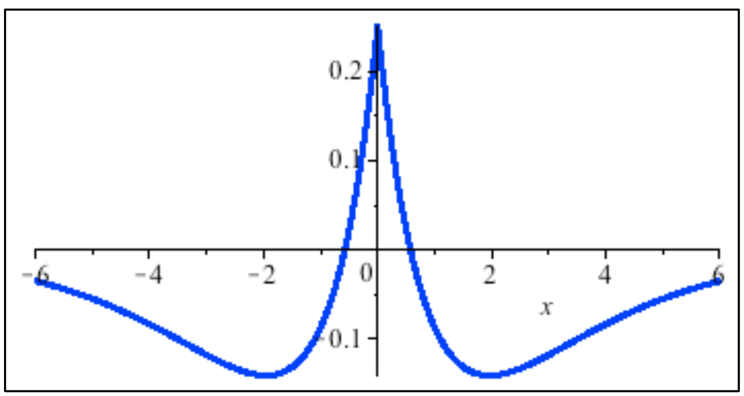
Bodnar & Velasquez (2005,2006)

Energy for the Continuum Model

Energy

$$W[\rho] = \frac{1}{2}\int_{\Omega}\int_{\Omega} \rho(x)\rho(y)Q(x-y)\,dx\,dy + \int_{\Omega} F(x)\rho(x)\,dx$$

External Potential



Morse Potential

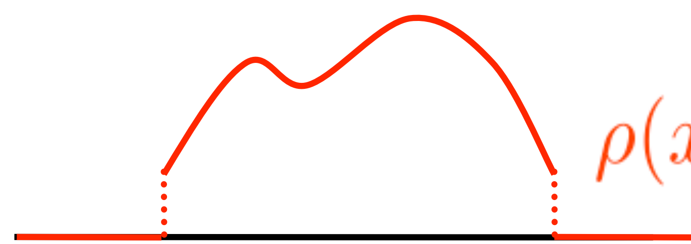
$$Q(x) = e^{-|x|} - GLe^{-|x|/L}$$

- Pairwise, additive [$Q'(x) = -q(x)$, $F'(x) = -f(x)$]
- Repulsive at small x and attractive at large x
- Symmetric
- Pointy at (x = 0)

Energy Dissipation

$$\frac{dW[\rho]}{dt} = -\int_{\Omega} \rho(x)\left\{V(x)\right\}^2\,dx$$

- Energy dissipated unless equilibrium
- Connection to optimal transport



$\rho(x,t)$

x

First Variation and Steady Solutions

Let $\rho(x) = \bar{\rho} + \epsilon\tilde{\rho}(x)$

where $\bar{\rho}$ is a steady solution and $\tilde{\rho}(x)$ is a zero mass perturbation.

Then

$$W[\rho] = W[\bar{\rho}] + \epsilon W_1[\bar{\rho}, \tilde{\rho}] + \epsilon^2 W_2[\tilde{\rho}, \tilde{\rho}],$$

Where the first variation is given by:

$$W_1[\bar{\rho}, \tilde{\rho}] = \int_\Omega \tilde{\rho} \left[\int_\Omega Q(x-y)\bar{\rho}(y)\,dy + F(x) \right] dx$$

For the first variation to vanish, in the support of the solution:

$$\boxed{\int_{\Omega_{\bar{\rho}}} Q(x-y)\bar{\rho}(y)\,dy + F(x) = \lambda \quad \text{for } x \in \Omega_{\bar{\rho}}.}$$

which is a Fredholm Integral Equation.

Constant energy per unit mass

Second Variation and Stability

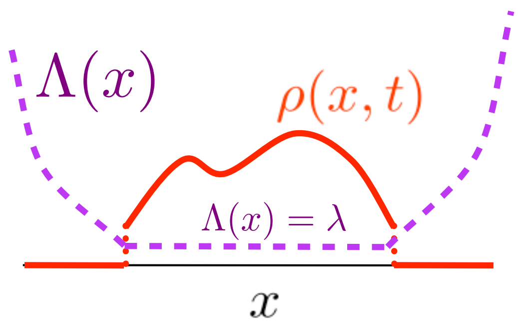
Suppose $\bar{\rho}$ is an equilibrium solution satisfying:

$$\boxed{\int_{\Omega_{\bar{\rho}}} Q(x-y)\bar{\rho}(y)\,dy + F(x) = \lambda \quad \text{for } x \in \Omega_{\bar{\rho}}.}$$

Then for stability two conditions are sufficient:

I) The energy per unit mass λ minimizes the induced potential, $\Lambda(x)$:

$$\int_{\Omega_{\bar{\rho}}} Q(x-y)\bar{\rho}(y)\,dy + F(x) = \Lambda(x) \geq \lambda$$



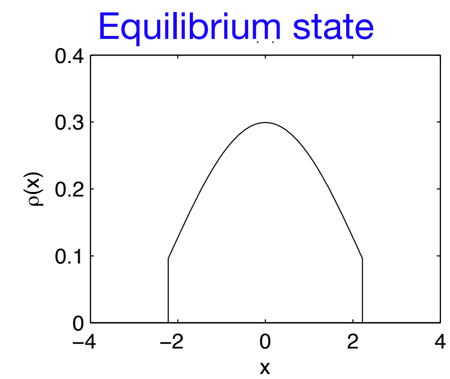
II) Second Variation is positive definite:

$$W_2[\tilde{\rho},\tilde{\rho}] = \int_{\Omega_\rho}\int_{\Omega_\rho} Q(x-y)\tilde{\rho}(x)\tilde{\rho}(y)\,dx\,dy > 0$$

for zero mass perturbations, $\tilde{\rho}(x)$.

Finding Equilibrium States

Equilibrium solutions satisfy an integral equation



Energy

first variation

Fredholm integral equation

Fredholm Integral Equation

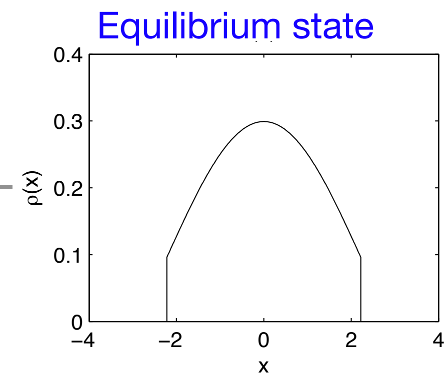
$$\int_{\alpha}^{\beta} Q(x-y)\rho(y)\ dy = \lambda$$

Mass Constraint

$$\int_{\alpha}^{\beta} \rho(y)\ dy = M$$

Where $\alpha < z < \beta$ is the support of the solution

Finding Equilibrium States



Energy $\xrightarrow{\text{first variation}}$ Fredholm integral equation $\xrightarrow{\text{“invert” integral operator}}$ ODE

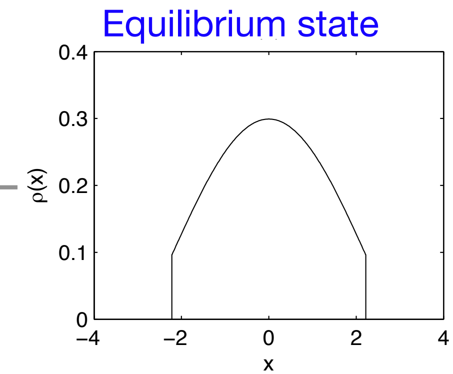
If $\quad Q(x) = e^{-|x|} - GLe^{-|x|/L} \quad$ then

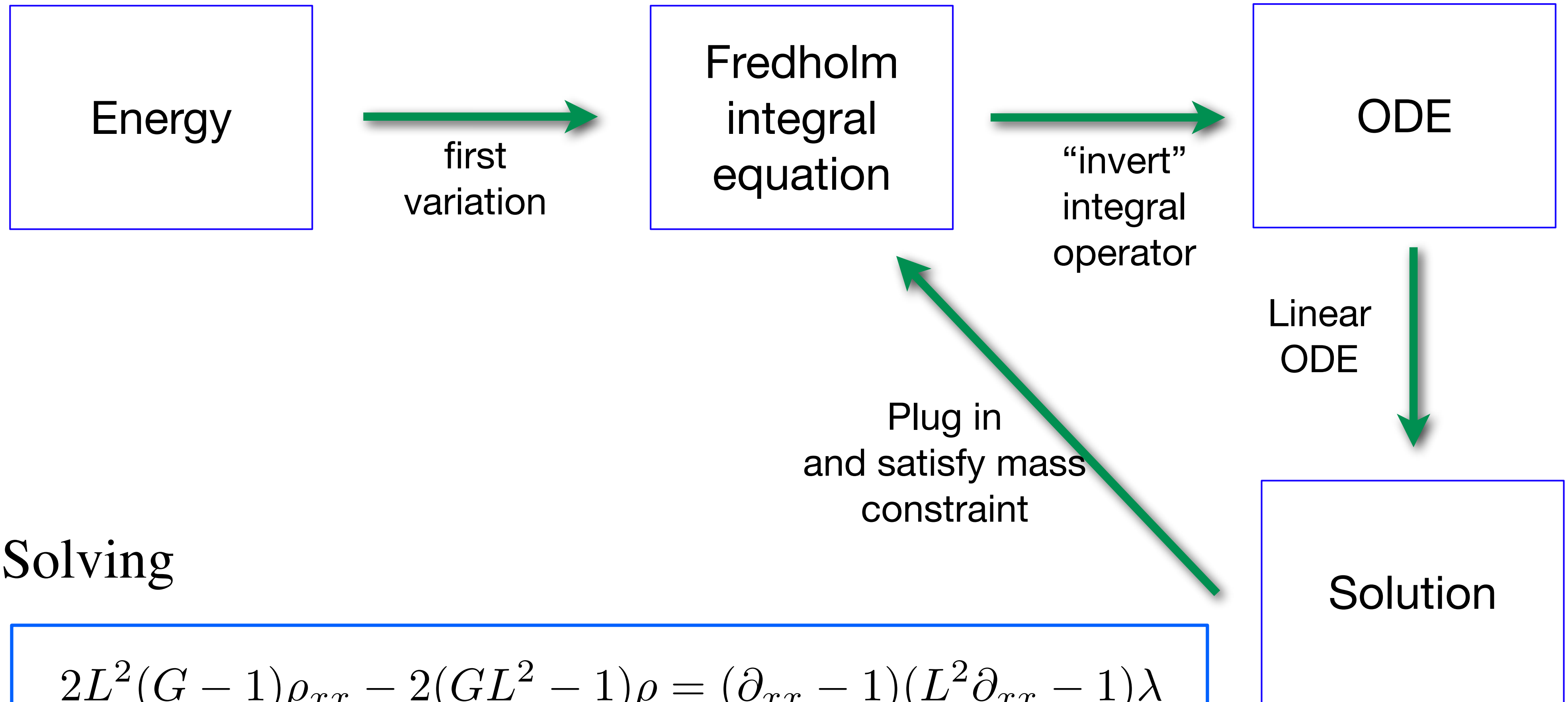
$$(\partial_{xx} - 1)(L^2\partial_{xx} - 1)Q(x) = 2L^2(G-1)\delta_{xx} - 2(GL^2 - 1)\delta$$

So applying $(\partial_{xx} - 1)(L^2\partial_{xx} - 1)$ to $\displaystyle\int_\alpha^\beta Q(x-y)\rho(y)\,dy = \lambda$

yields $\boxed{2L^2(G-1)\rho_{xx} - 2(GL^2-1)\rho = (\partial_{xx} - 1)(L^2\partial_{xx} - 1)\lambda}$

Finding Equilibrium States



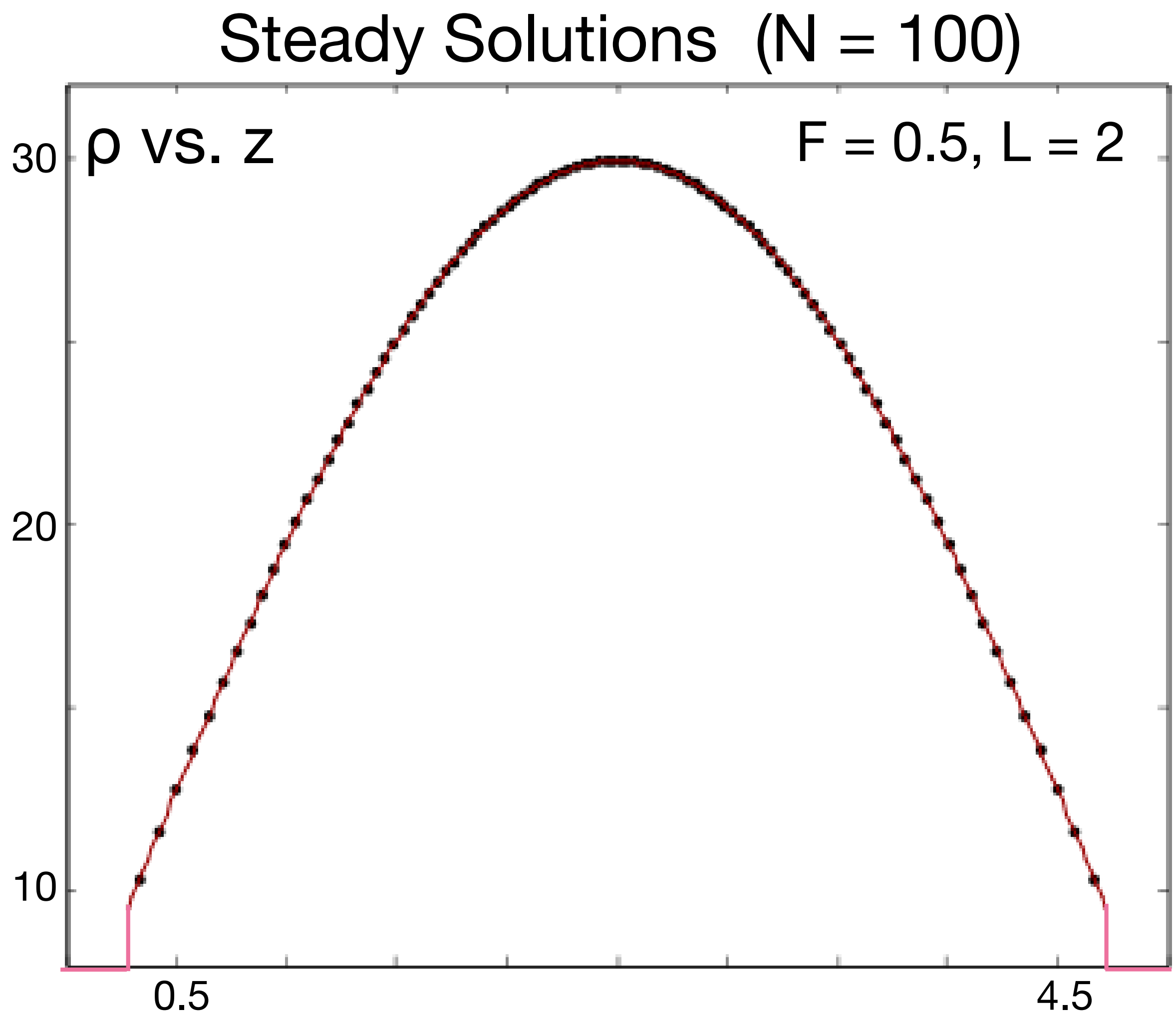


Solving

$$2L^2(G-1)\rho_{xx} - 2(GL^2-1)\rho = (\partial_{xx}-1)(L^2\partial_{xx}-1)\lambda$$

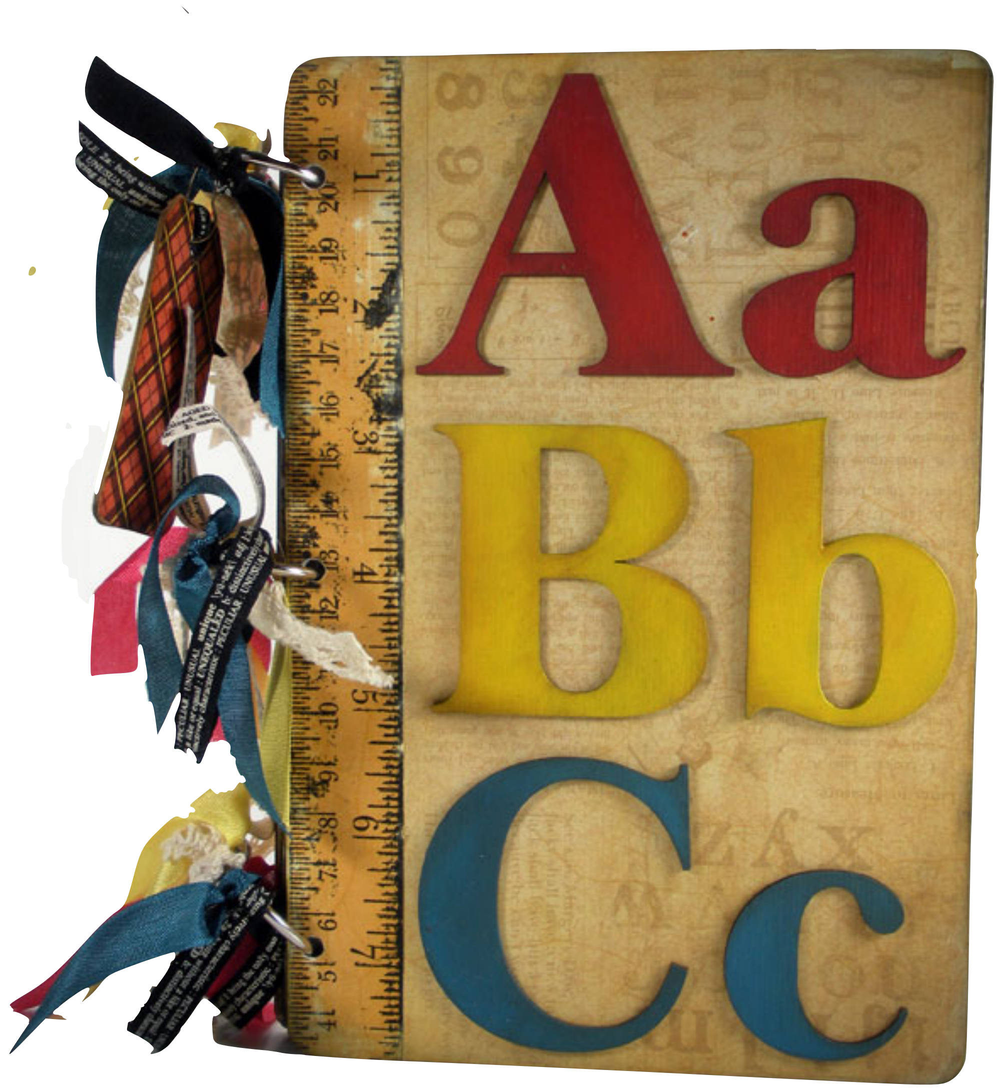
Yields $\boxed{\rho(z) = \mathbf{A} + \mathbf{B}\cos(\mu z)}$ in the support of the solution.

Continuum solutions agree with discrete (numerical) ones.



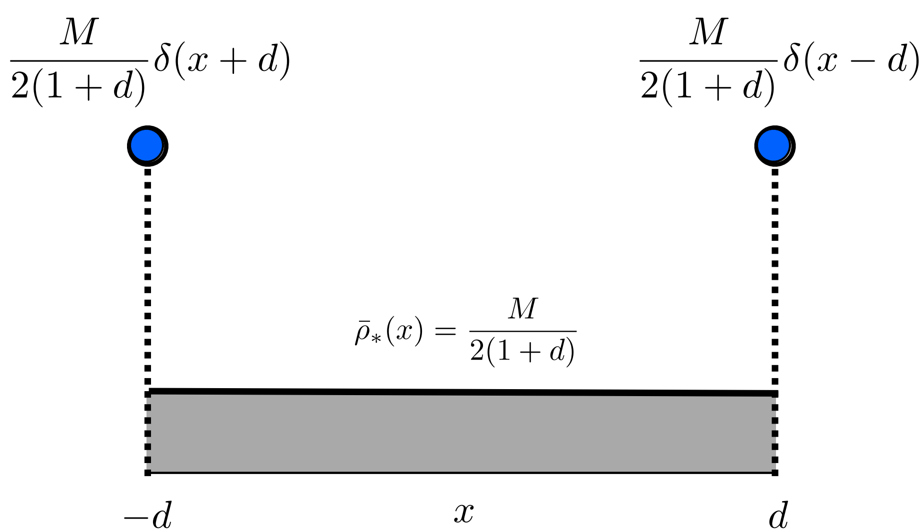
$$\rho(z) = \mathbf{A} + \mathbf{B} \cos(\mu z)$$

A Primer of Swarm Equilibria

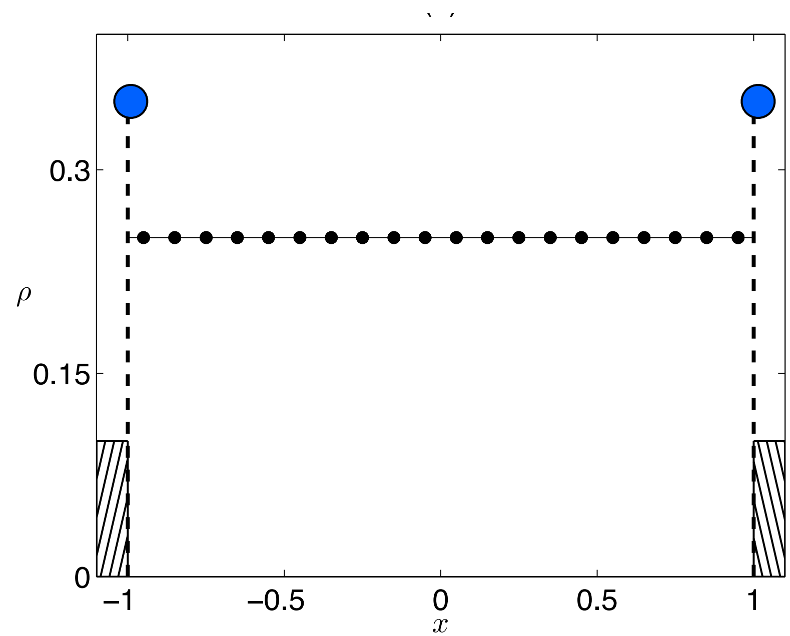


Repulsive Forces on a Finite Interval

- Laplace Repulsive Potential: $Q(x) = e^{-|x|}$
- Finite Interval: -d < x < d
- Equipartition of mass between concentrations at the endpoints and classical solution in interior.
- Mass Concentrations can't be found by classical integral equation methods



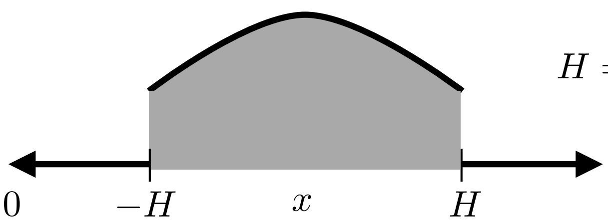
Numerics



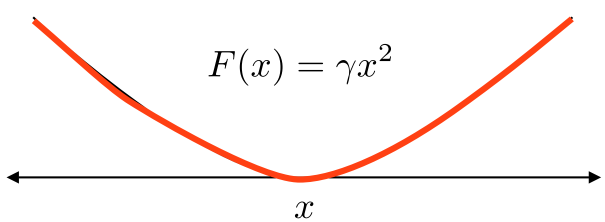
Repulsive Forces in a Quadratic Potential

- Laplace Repulsive Potential: $Q(x) = e^{-|x|}$
- Quadratic external potential: $F(x) = \gamma x^2$
- Solution is compactly supported on $-H < x < H$
- Jump discontinuities at edge of support
- Global energy minimizer.

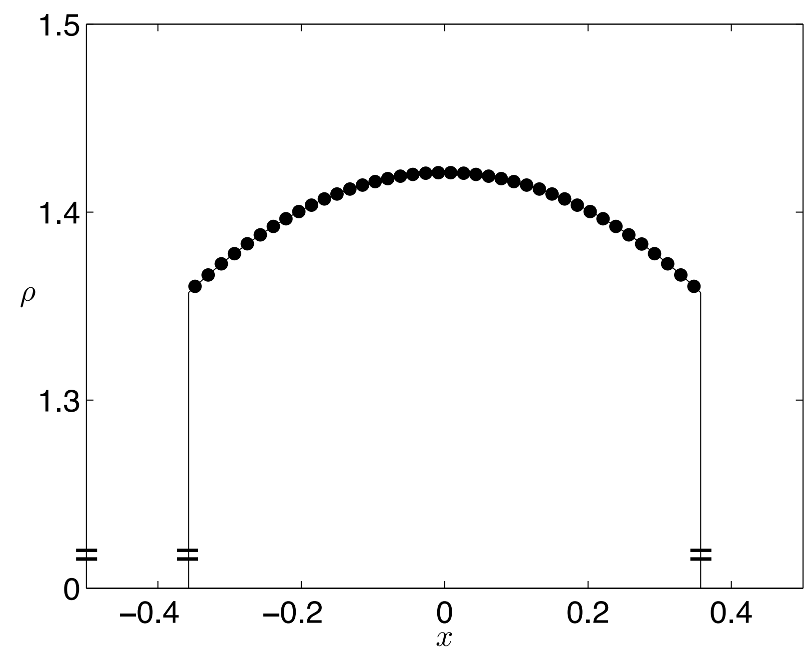
$$\bar{\rho}_*(x) = \frac{\gamma}{2}\left[\left(\frac{3M}{2\gamma}+1\right)^{2/3} + (1-x^2)\right]$$



$$H = \left(\frac{3M}{2\gamma}+1\right)^{1/3} - 1.$$

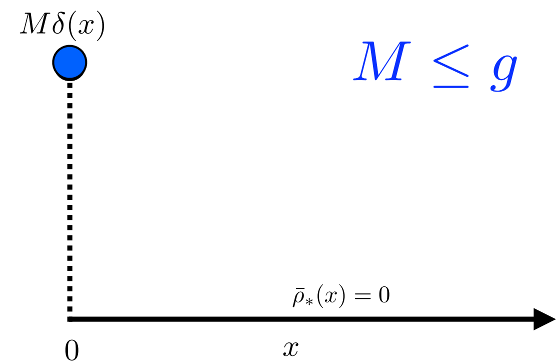


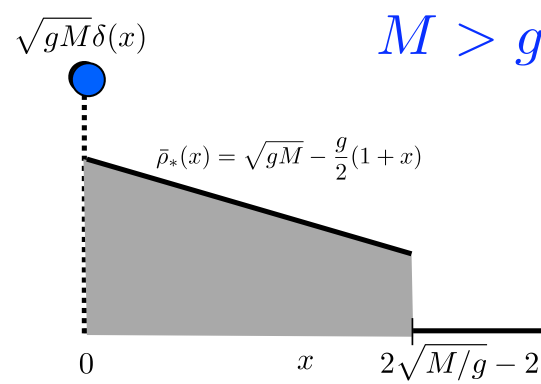
Numerics



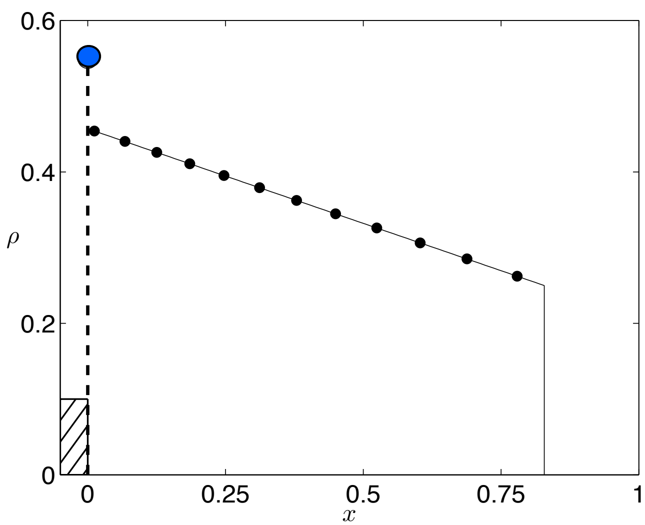
Repulsive forces in a gravitational potential

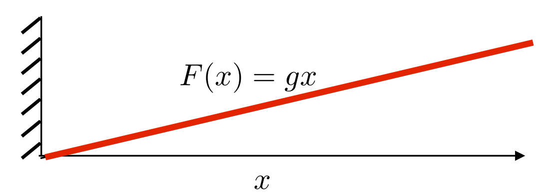
- Laplace Repulsive Potential: $Q(x) = e^{-|x|}$
- Half line: $0 < x < \infty$
- If M < g all mass concentrates at origin.
- If M > g some mass concentrates at origin and the remainder forms a compact swarm.
- Global minimizer and probably global attractor.





Numerics





As a swarm model:

- Concentration on ground (good) but no gap (bad).

Catastrophic Morse Potential in Free Space

- Morse Potential in catastrophic regime $\epsilon < 0$.
- Solution is compactly supported on $-H < x < H$.
- Width of support independent of mass!!
- Jump discontinuities at edge of support.
- Local energy minimizer.
- Probably a global energy minimizer.

Morse Potential

$$Q(x) = e^{-|x|} - GLe^{-|x|/L}$$

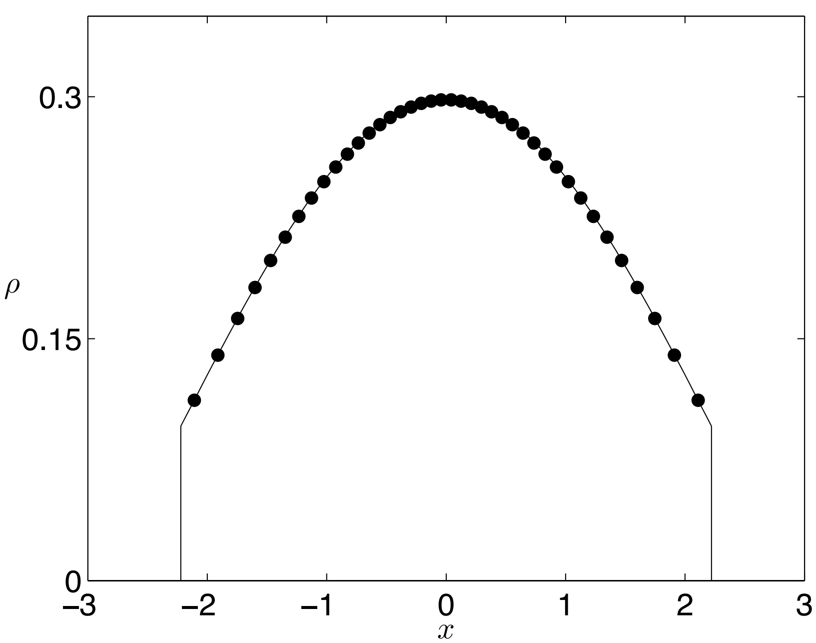
Analytic Solution

$$\bar{\rho}(x) = C\cos(\mu x) - \lambda/\epsilon,$$

$$\epsilon = 2(GL^2 - 1) \qquad \mu = \sqrt{\frac{GL^2-1}{L^2(1-G)}}.$$

$$\begin{aligned} H &= \frac{1}{\mu}\cot^{-1}_{[0,\pi]}\left\{\frac{GL-1}{\sqrt{(1-G)(GL^2-1)}}\right\}, \\ C &= \frac{M}{2(H+L+1)}\frac{\sqrt{G}(L^2-1)}{L(1-G)}, \\ \lambda &= \frac{M(1-GL^2)}{H+L+1}. \end{aligned}$$

Numerics



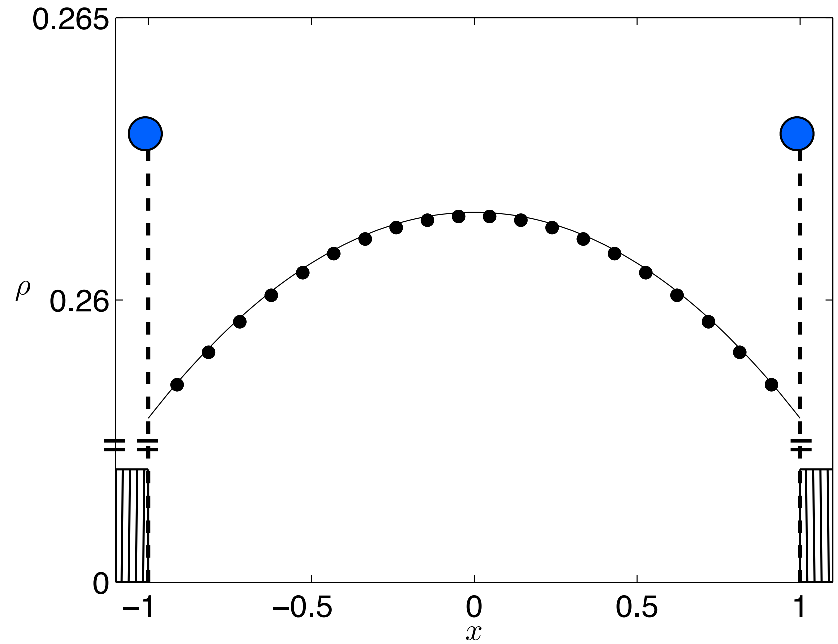
H-Stable Morse Potential on bounded interval

- Morse Potential in H-stable regime.
- No solution in free space (swarm spreads forever).
- Mass concentrations at endpoints of interval.
- Global energy minimizer.
- Probably global attractor.

Morse Potential

$$Q(x) = e^{-|x|} - GLe^{-|x|/L}$$

Numerics



Quasi 2D-Potential

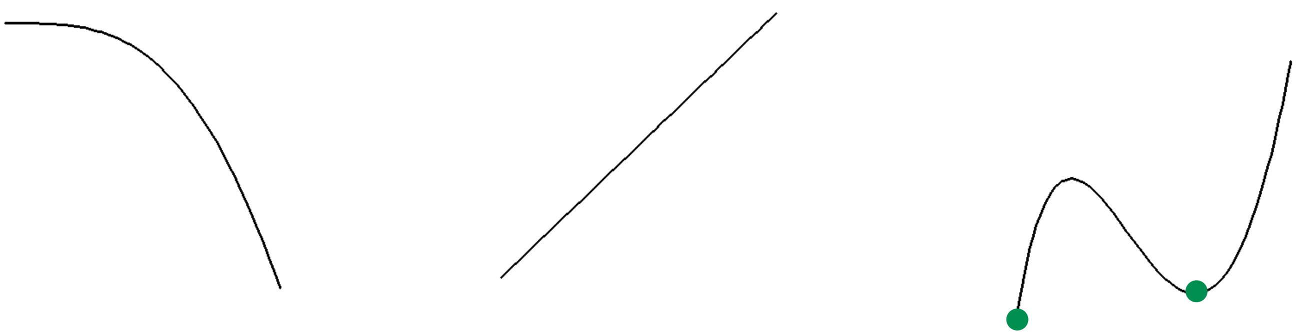
Suppose that we look at a quasi-2D mass distribution

$$\rho(x, z, t) \equiv \rho(z, t)$$

This has an associated potential, $Q_{2D}(z)$:

$$Q_{2D}(z) = \int_{x=-\infty}^{\infty} Q(\sqrt{x^2 + z^2})\ dx = \int_{x=-\infty}^{\infty} e^{-\sqrt{x^2+z^2}}\ dx$$

This allows steady swarms with a gap at zero:



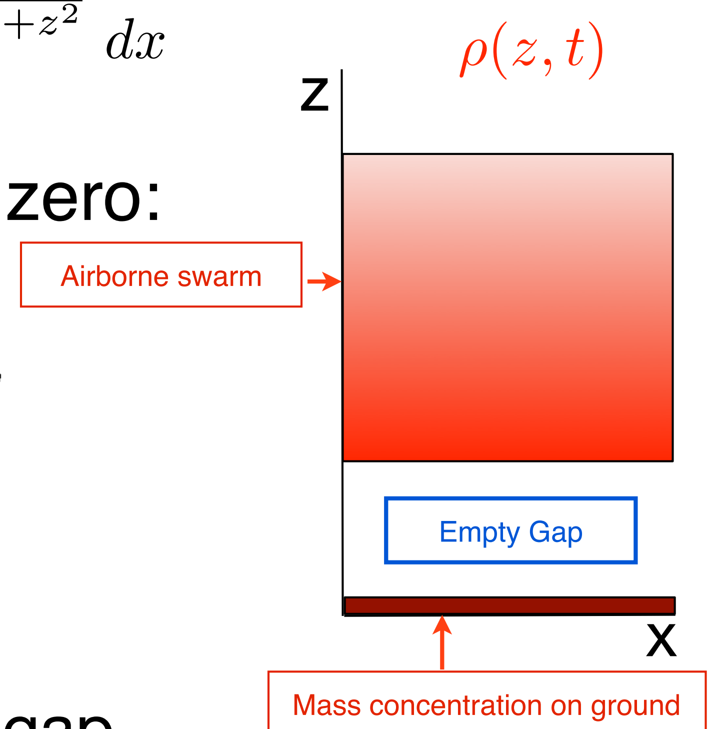
Repulsion
from mass concentrated on ground
$\sim dz^2 \ln(z)$

+

Gravity
gz

=

Stable gap
(for some parameters)



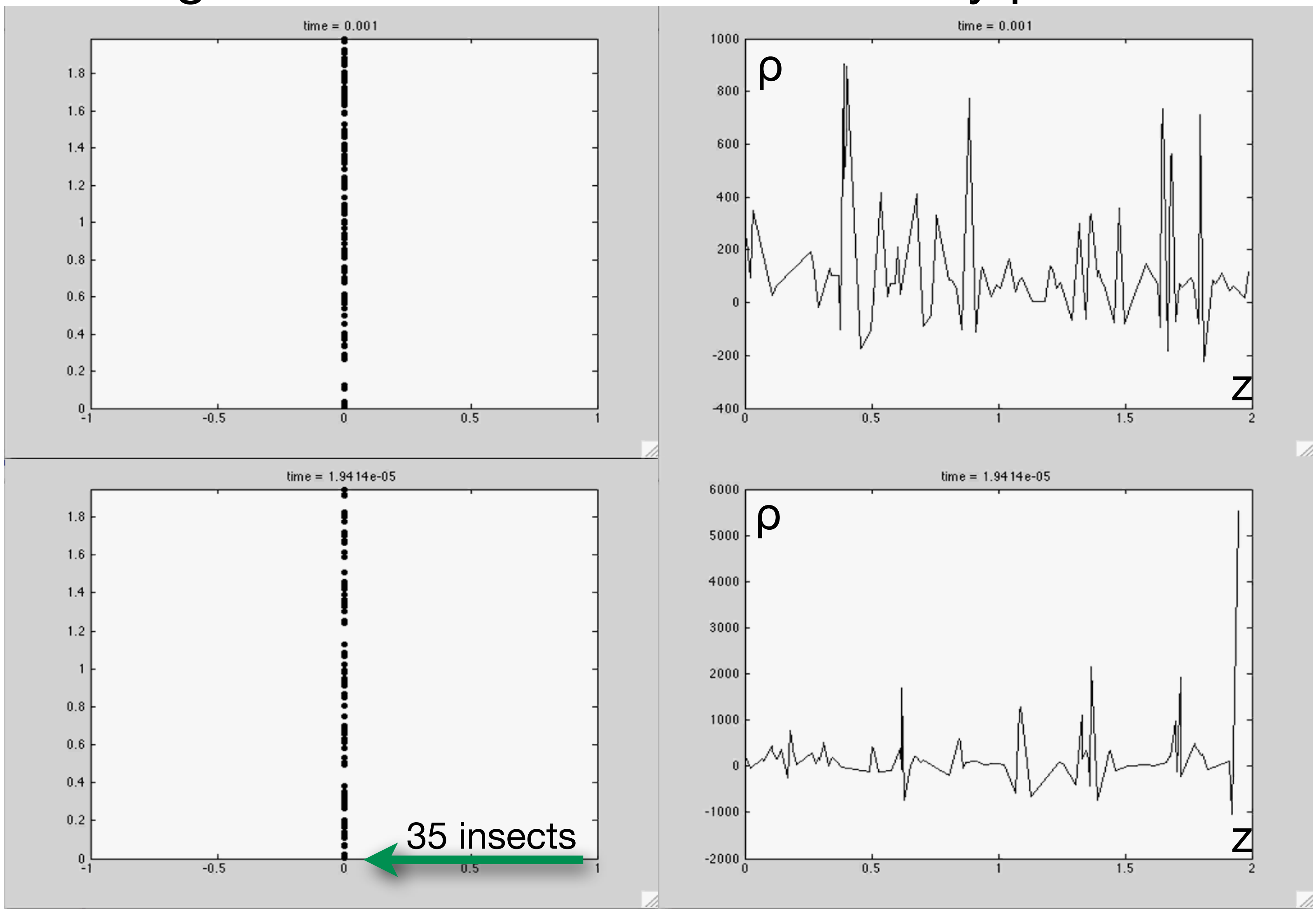
The quasi 2D potential yields dynamically stable bubbles.

Organism column

Density profile

1D

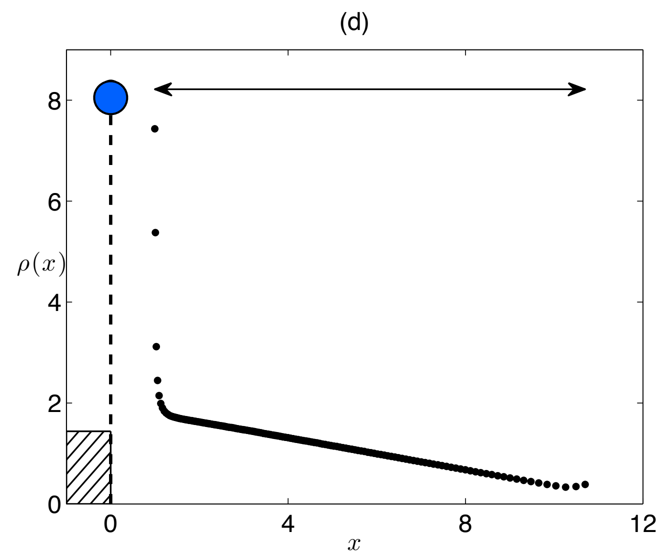
Quasi 2D

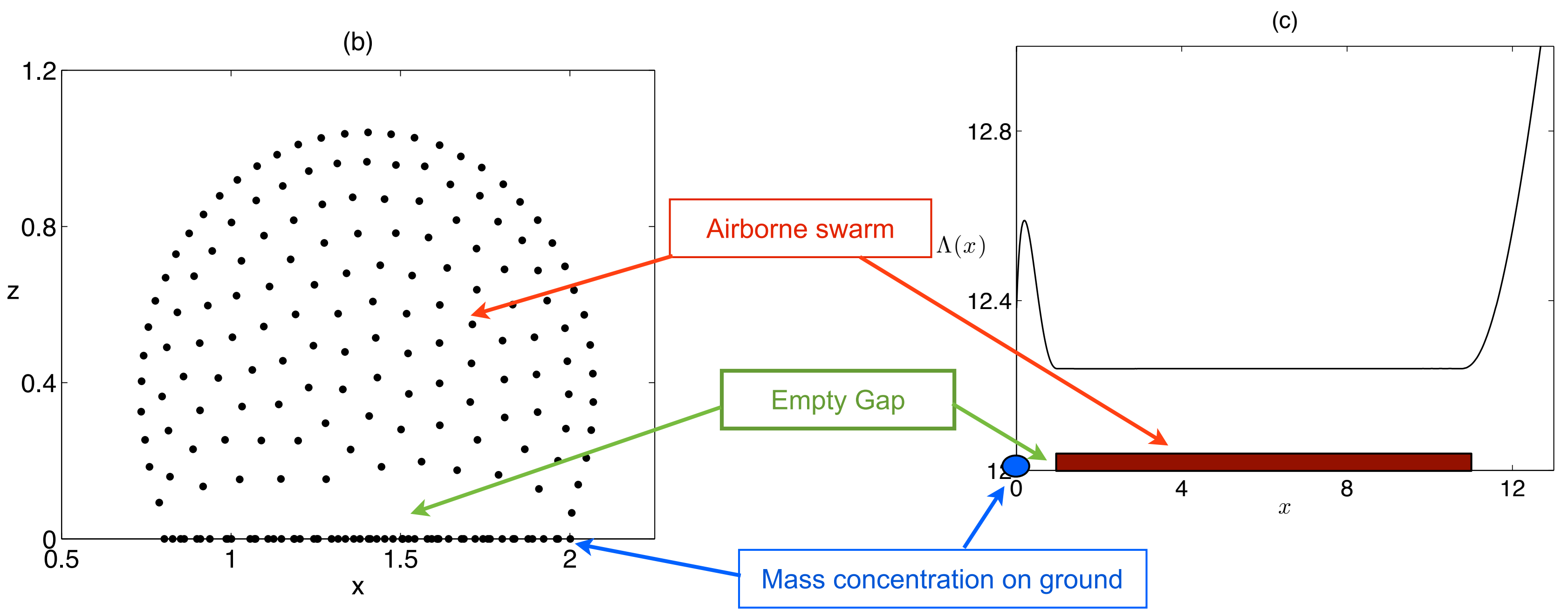


Quasi 2D Locust Swarms

Quasi-2D Numerics

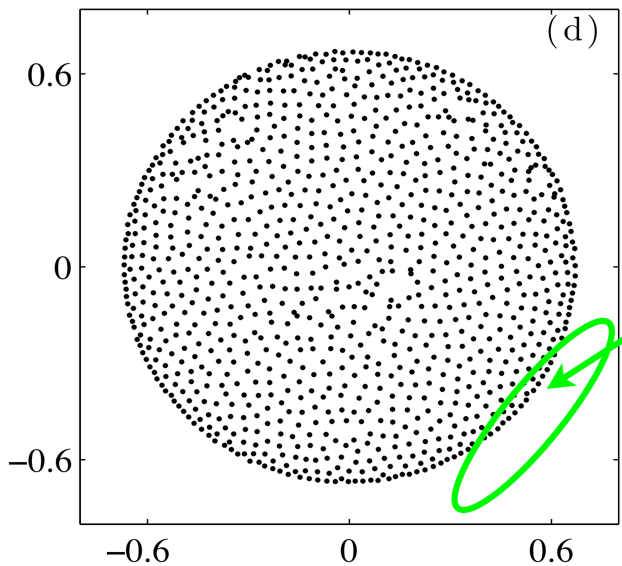
- Quasi 2D Laplace Repulsion Potential.
- Gravity & Ground:
  - Linear potential on a half space.
- Mass concentration on ground
- Gap above ground.
- Continuous family of solutions
- Solution is only a local “swarm” minimizer.





2D Morse Equilibria

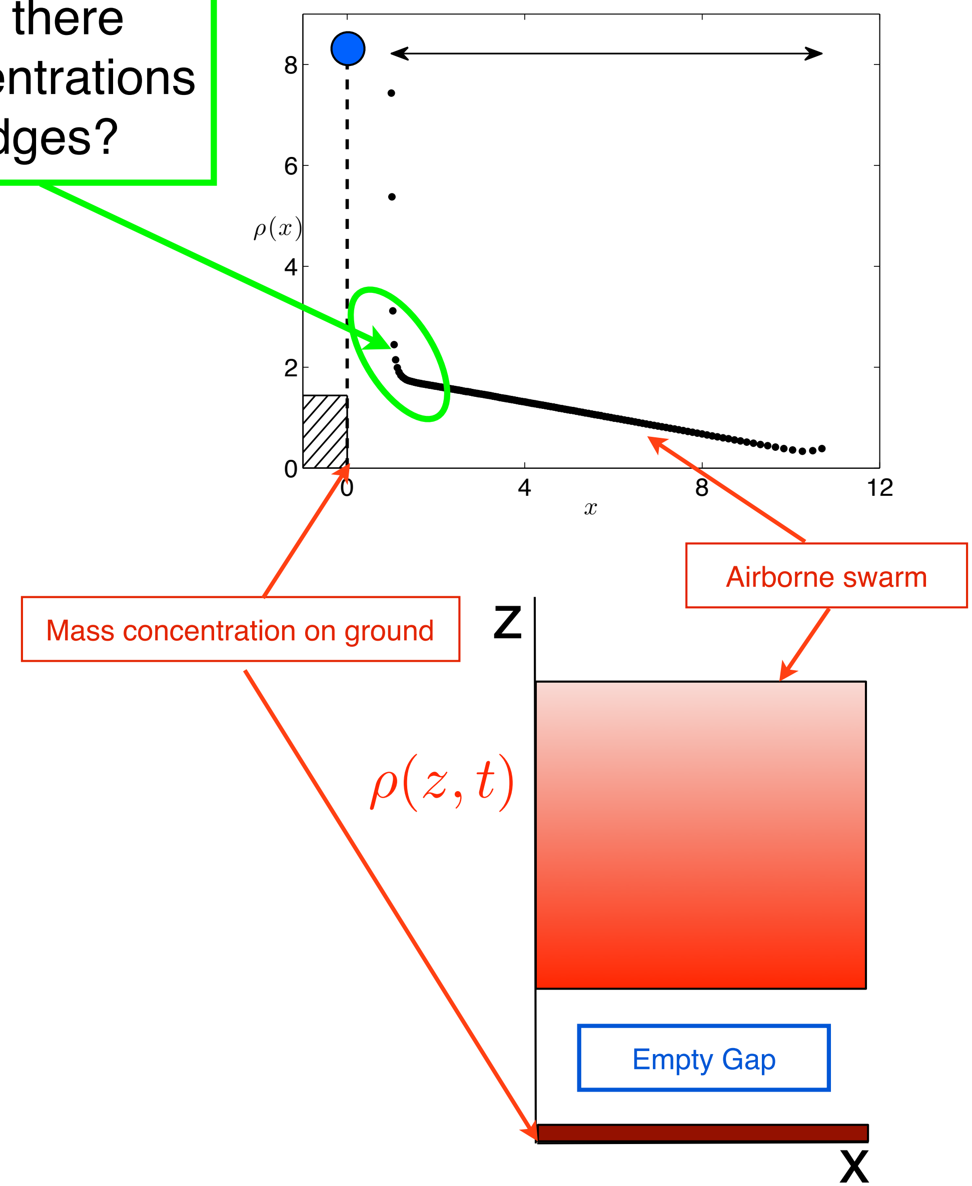
2D Catastrophic Swarm



$$Q(x) = e^{-|x|} - GLe^{-|x|/L}$$

Why are there mass concentrations at the edges?

2D Morse Ribbons



Louis Ryan
HMC 2012

2D Morse - Ribbon

Morse Potential

$$Q(x) = e^{-|x|} - GLe^{-|x|/L}$$

Quasi-2D Morse Potential

$$Q_{2D}(z) = \int_{x=-\infty}^{\infty} Q(\sqrt{x^2+z^2})\ dx$$

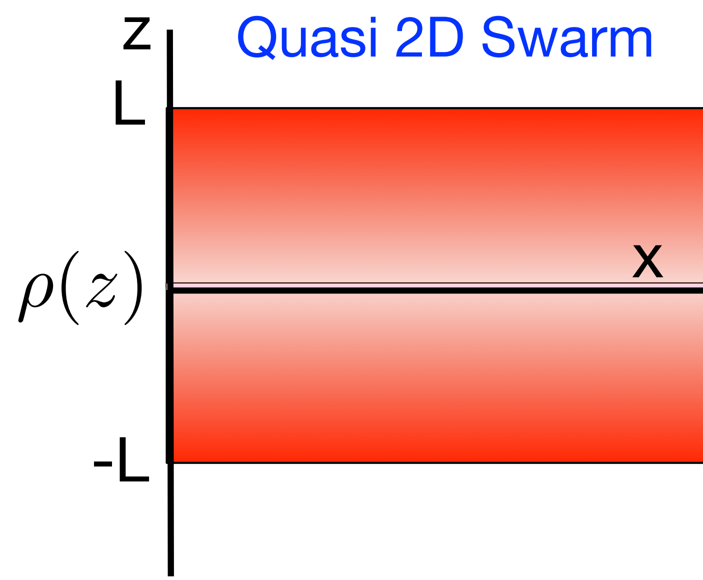
Quasi-2D Morse Potential ($|z| \ll 1$)

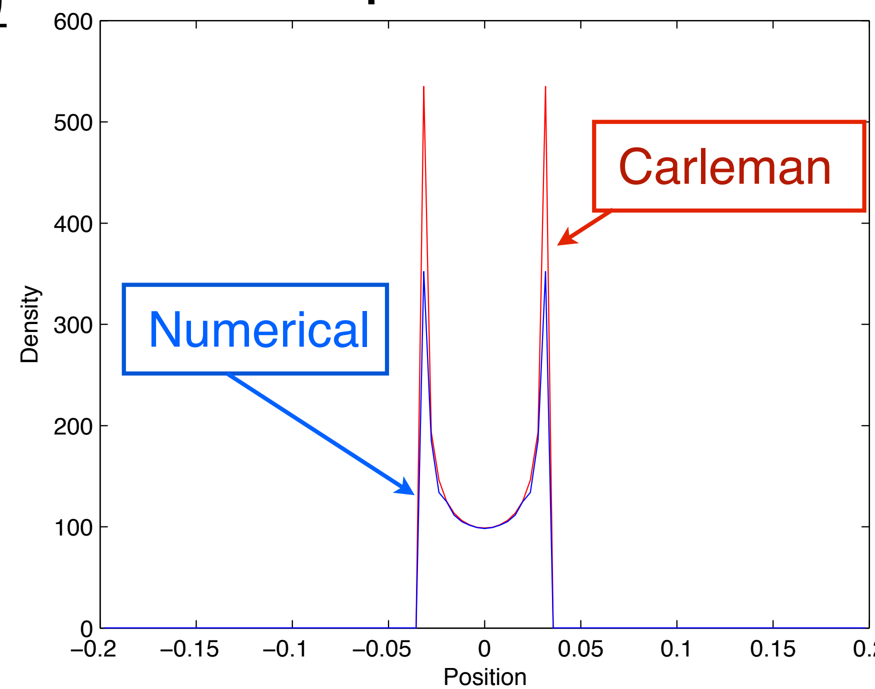
$$Q_{2D}(z) \sim |z|^2 \ln|z|$$

Integral Equation (Carleman)

$$\int_{-L}^{L} |y|^2 \ln|y| \rho(z-y)\ dz = \lambda \qquad |z| < L$$

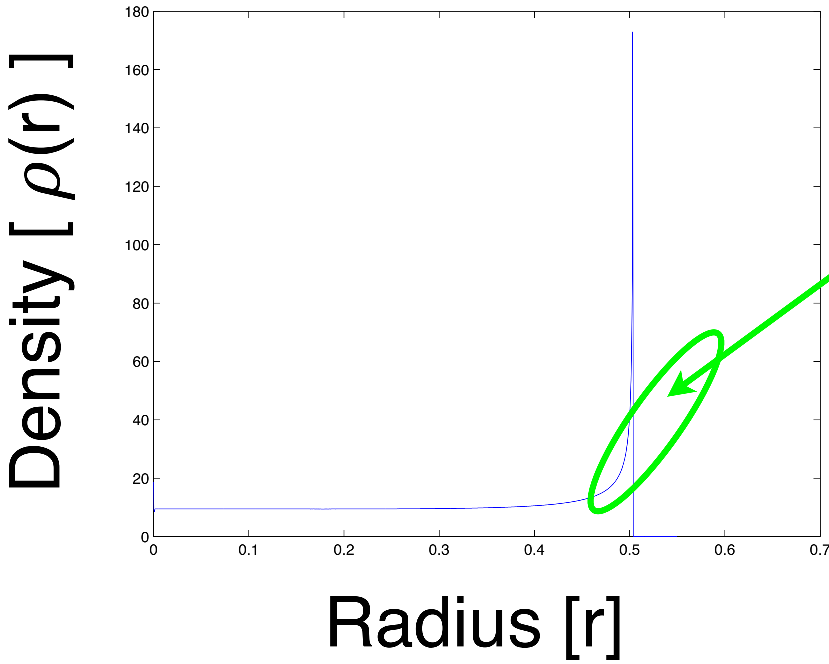
$$\Rightarrow \boxed{\rho(z) = \frac{C}{\sqrt{L^2 - z^2}} \qquad |z| < L}$$





2D Morse - Circular Swarm

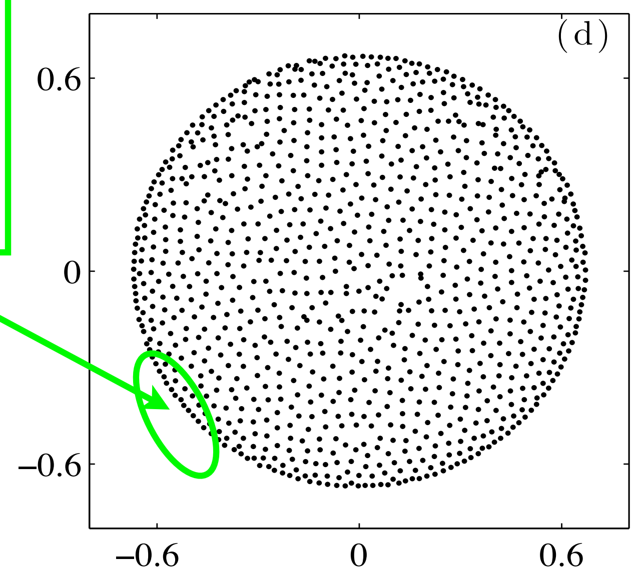
2D Catastrophic Swarm
(Continuum Model)



Singular mass concentration at the edges

$$\rho(r) \sim \frac{C}{\sqrt{R-r}}$$

2D Catastrophic Swarm
(Particle Simulation)



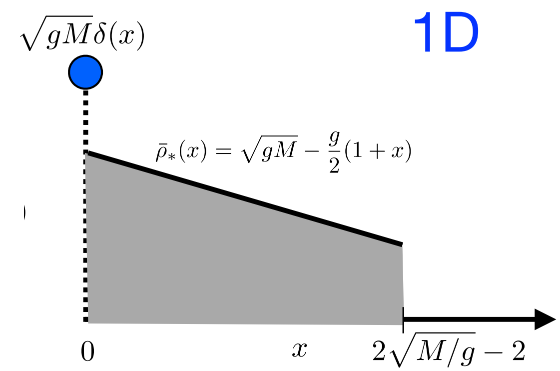
$$Q(x) = e^{-|x|} - GLe^{-|x|/L}$$

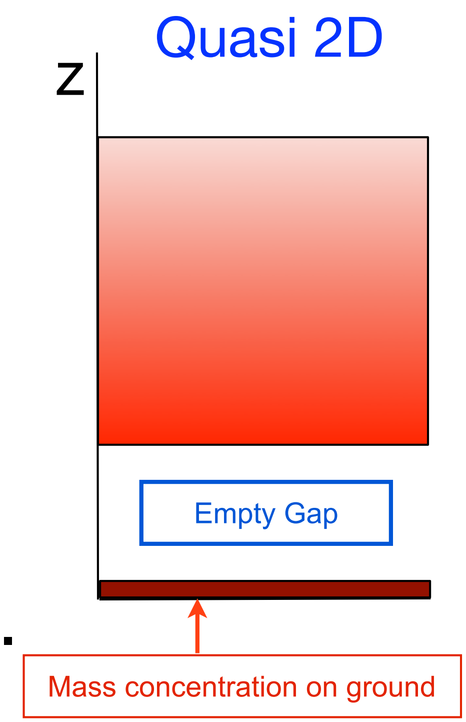
Found as an Energy Minimizer

Dimensionality of local minimizers of the interaction energy
D Balagué, JA Carrillo, T Laurent, G Raoul
Archive for Rational Mechanics and Analysis (2013)

Conclusions

- Energy methods allow us to analyze equlibria and their stability.
- Equilibria satisfy a Fredholm Integral Equation. This yields some analytical solutions.
- Equlibria often contain concentrations on boundaries.
- For locust model, quasi-2D repulsive potential and gravity yield a ground concentration together with a gap.
- 2D Morse yields inverse square-root mass singularities on boundaries.
- Inverse square-root singularities appear generic in 2D for “pointy” repulsive potentials.





Bernoff & Topaz “**Nonlinear Aggregation Equations: A Primer of Swarm Equilibria.**” SIREV (2013).