Lattice paths, linear algebra and combinatorics
Day 1

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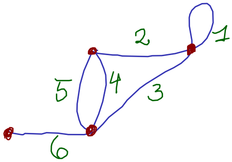
Universidad de los Andes

with Jerónimo Valencia

Mathematics Sin Fronteras
October 28th, 2021

graph theory meets linear algebra



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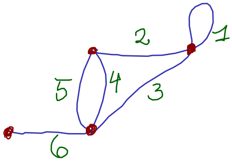

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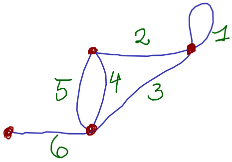

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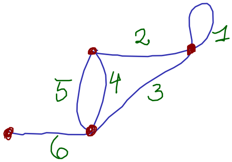

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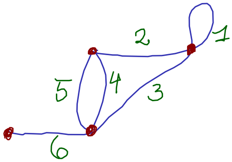

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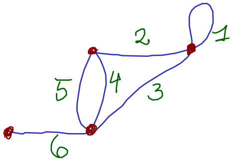

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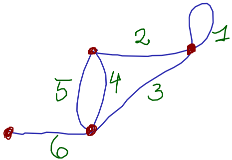

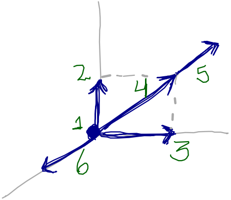

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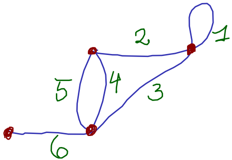

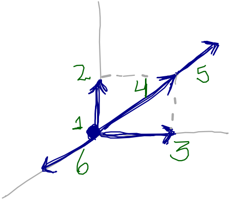

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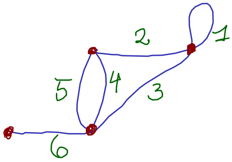

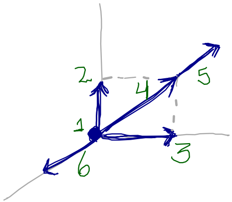

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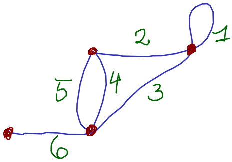

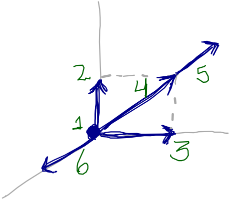

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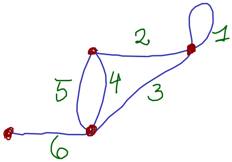

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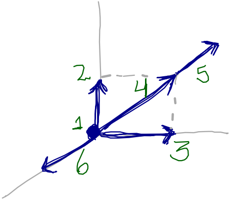

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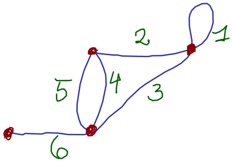

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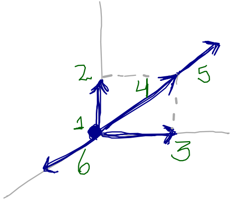

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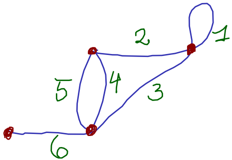

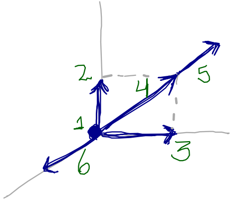

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A matroid $M = (E, \mathcal{B})$ on the ground set E is such that $\mathcal{B}$ consists of subsets of E where:

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Ejemplos:

grafos: $G = (E, \mathcal{B})$ where E are the set of edges and $\mathcal{B}$ is the collection of trees.

k-ev: Let V be a k-dim. vector space in $\mathbb{R}^n$. Then $M_V = ([n], \mathcal{B})$ where $\mathcal{B} = \{J \,:\, |J| = k, p_J \neq 0\}$.

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"Anyone who has worked with matroids has come away with the conviction that matroids are one of the richest and most useful ideas of our day."

-Gian Carlo Rota

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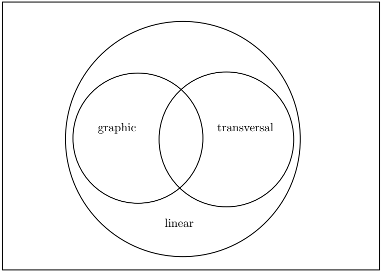

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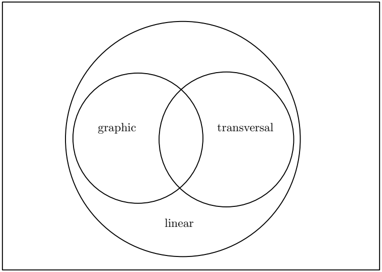

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Let $M = (E, \mathcal{B})$ be a matroid.

- $I \subseteq E$ is *independent* if $I \subset B$ for some basis B.
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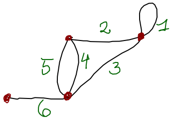

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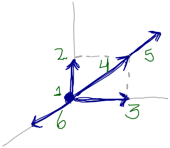

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Lattice path matroids LPMs

Fix $0 \leq k \leq n$ and let $U, L \in \binom{[n]}{k}$.
The **lattice path matroid** $M[U, L]$ is the matroid on $[n]$ whose bases are those $B \in \binom{[n]}{k}$ such that $U \leq B \leq L$.

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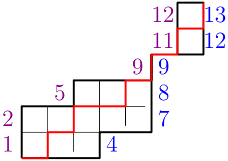
For instance, let $k = 6, n = 13$, $U = \{1, 2, 5, 9, 11, 12\}$, $L = \{4, 7, 8, 9, 12, 13\}$.



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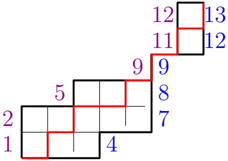

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◦ LPMs are linear.

Lattice path matroids LPMs

What we want: consider matroids M_1, M_2, M_3 as given below:

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{M_1} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}}_{M_2} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}}_{M_3}$$

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◦ How can we understand from a matroidal porint of view this relationship?

• The algebro-geometric importance of this relies on the fact that the sequence $M_1 \subset M_2 \subset M_3$ corresponds to a point in the *full flag variety* $\mathcal{F}\ell_3$.

Coming up next...

- posets

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¡Muchas gracias!