Mathematics Sin Fronteras
Mini Course 2

Geometric flows: Deforming geometry in time

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Contents

These lecture notes were created to the course of Geometric Flows taught by professor Mariel Sáez at Mathematics Sin Fronteras meeting. If you find any mistakes, please send us an email.

1 Lecture I

In the first lecture we define what is a curve, its length and other geometric quantities such as the tangent vector and the curvature.

1.1 Basic definitions and examples

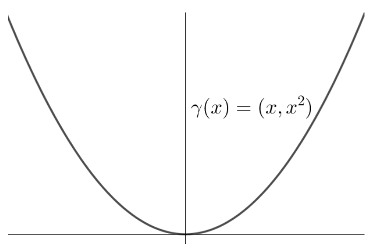
Definition 1.1. A *curve* on M is a function $\gamma : I \to M$, with $I \subseteq \mathbb{R}$.

Remark.

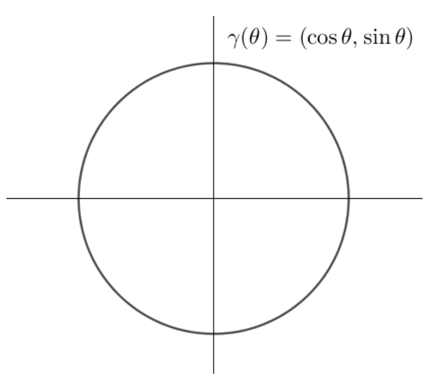
1. In general, I will be an interval.
2. In this course M will always be some Euclidean space $\mathbb{R}^n$. Nevertheless, the previous definition can be generalized when M is a manifold.
3. We will always think that γ is smooth enough, i.e., that $\gamma_i(t)$ are smooth enough for $i = 1, \ldots, n$, where $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$ (C^3 is more than necessary for our purposes).

Example 1.2.

1.
$$\begin{aligned}\gamma : \mathbb{R} &\to \mathbb{R}^2,\\ x &\mapsto (x, x^2).\end{aligned}$$

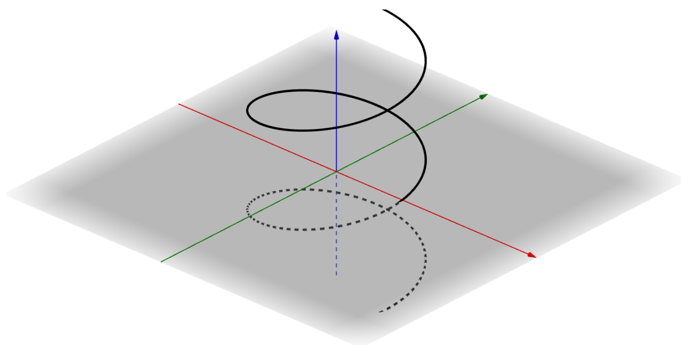


2.
$$\begin{aligned}\gamma : [0, 2\pi) &\to \mathbb{R}^2,\\ \theta &\mapsto (\cos\theta, \sin\theta).\end{aligned}$$



3.

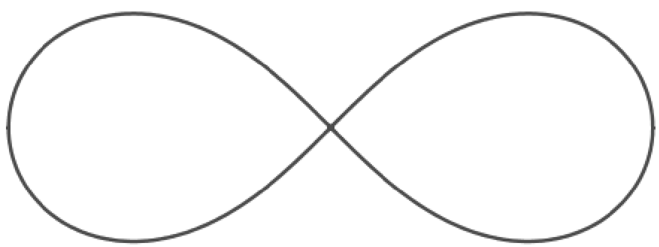
$$\begin{aligned}\gamma : \mathbb{R} &\to \mathbb{R}^3, \\ t &\mapsto (\cos t, \sin t, t).\end{aligned}$$



Definition 1.3. A curve $\gamma : [a, b] \to M$ is *closed* if $\gamma(a) = \gamma(b)$.

Definition 1.4. A curve is *embedded* if it does not self intersect. That is, the function is injective. Otherwise, we say that the curve is *immersed*.

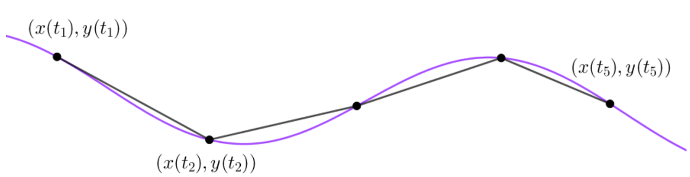
Example 1.5. The *lemniscate* is an example of a curve which is immersed but not embedded.



1.2 How do we compute the length?

Consider a curve $\gamma : I \to \mathbb{R}^2$. If we want to obtain the length of a portion of the curve, we first approximate this portion by a polygonal path and then we calculate the length of the polygonal path, in order to obtain an approximation of the wanted length.

Example 1.6. Consider the following curve with the following polygonal path



In this case, if we write $\Delta x_i = x(t_{i+1}) - x(t_i)$ and $\Delta y_i = y(t_{i+1}) - y(t_i)$, then we have the approximation

$$L \sim \sum_{i=1}^{4} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

This motivates the following definition

Definition 1.7. Consider $\gamma : [a, b] \to \mathbb{R}^2$. The *length* of γ is defined by

$$L(\gamma) := \int_a^b |\gamma'(t)| dt.$$

Example 1.8.

1. Consider the curve

$$\begin{aligned} \gamma : [0, 2\pi] &\to \mathbb{R}^2, \\ \theta &\mapsto 2(\cos\theta, \sin\theta). \end{aligned}$$

   We have that $\gamma'(\theta) = 2(-\sin\theta, \cos\theta)$. Then,

$$L(\gamma) = \int_0^{2\pi} \sqrt{4(\sin^2\theta + \cos^2\theta)} d\theta = 4\pi.$$

2. Now consider

$$\begin{aligned} \gamma : [-1, 1] &\to \mathbb{R}^2, \\ t &\mapsto (t, 2t + 1). \end{aligned}$$

   Notice that $\gamma'(t) = (1, 2)$. Then,

$$L(\gamma) = \int_{-1}^{1} \sqrt{1 + 4} dt = 2\sqrt{5}.$$

1.3 Some geometric quantities

1.3.1 The tangent vector

Definition 1.9. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a curve. The *unit tangent vector at* $t_0 \in (a, b)$ is defined by

$$\tau(t_0) = \frac{\gamma'(t_0)}{|\gamma'(t_0)|}.$$

Remark. This is only well defined if $\gamma'(t_0) \neq 0$. Points at which this holds are called *regular*.

Example 1.10.

1. For the curve

$$\begin{aligned} \gamma : [0, 2\pi] &\to \mathbb{R}^2, \\ \theta &\mapsto 2(\cos\theta, \sin\theta), \end{aligned}$$

   we have that $\gamma'(\theta) = 2(-\sin\theta, \cos\theta)$. Then, $|\gamma'(\theta)| = \sqrt{4\sin^2\theta + 4\cos^2\theta} = 2$. Therefore,

$$\tau(\theta_0) = \frac{\gamma'(\theta_0)}{|\gamma'(\theta_0)|} = (-\sin\theta_0, \cos\theta_0).$$

2. If we consider

$$\begin{aligned} \gamma : [-1, 1] &\to \mathbb{R}^2, \\ t &\mapsto (t, 2t+1), \end{aligned}$$

   then, $\gamma'(t) = (1, 2)$. From this, $|\gamma'(t)| = \sqrt{1+4} = \sqrt{5}$. Thus

$$\tau(t_0) = \frac{\gamma'(t_0)}{|\gamma'(t_0)|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right).$$

1.3.2 The arc-length parameter

The same geometric object can be described in different ways, to see this, consider the following.

Example 1.11. The following maps define the same geometric object, the unitary semicircle centered at the origin:

$$\begin{aligned} \gamma_1 : [0, \pi] &\to \mathbb{R}^2, \\ \theta &\mapsto (\cos\theta, \sin\theta), \end{aligned}$$

$$\begin{aligned} \gamma_2 : [-1, 1] &\to \mathbb{R}^2, \\ x &\mapsto (x, \sqrt{1-x^2}), \end{aligned}$$

A choice of description is called a *parametrization*. The curve will be from now on the *geometric object*, that is, the set

$$\{\gamma(t); t \in [a, b]\}.$$

Let $\gamma : [a,b] \to \mathbb{R}^2$. We define the function

$$s(x) = \int_a^x |\gamma'(\lambda)| d\lambda.$$

Notice that s is a function from the interval $[a,b]$ to $[0, L(\gamma)]$ and is increasing. This implies that s is invertible and we can use to it to parametrize γ:

$$\begin{aligned} \gamma : [0, L(\gamma)] &\to \mathbb{R}^2, \\ s &\mapsto \gamma(x(s)). \end{aligned}$$

Here, x is the inverse of s.

Definition 1.12. The previous parametrization is called the *arc-length parametrization.*

Remark. Using the chain rule we obtain

$$\frac{d\gamma}{ds} = \frac{d\gamma}{dx} \cdot \frac{dx}{ds} = \frac{\gamma'(x)}{|\gamma'(x)|} = \tau(s).$$

From this calculation, we see that why is useful consider a curve with its arc-length parametrization.

Example 1.13.

1. As before, consider the curve

$$\begin{aligned} \gamma : [0, 2\pi] &\to \mathbb{R}^2, \\ \theta &\mapsto 2(\cos\theta, \sin\theta). \end{aligned}$$

We computed that $\gamma'(\theta) = 2(-\sin\theta, \cos\theta)$. Then

$$s = \int_0^x \sqrt{4(\sin^2\theta + \cos^2\theta)} d\theta = 2x.$$

Using this, we find that

$$\begin{aligned} \gamma : [0, 4\pi] &\to \mathbb{R}^2, \\ s &\mapsto 2\left(\cos\frac{s}{2}, \sin\frac{s}{2}\right), \end{aligned}$$

is the arc- length parametrization of γ. Also,

$$\tau(s) = \gamma'(s) = \left(-\sin\frac{s}{2}, \cos\frac{s}{2}\right).$$

2. Similarly, consider

$$\gamma : [-1, 1] \to \mathbb{R}^2,$$
$$t \mapsto (t, 2t + 1).$$

As before, $\gamma'(t) = (1, 2)$. Then,

$$s = \int_{-1}^{x} \sqrt{1 + 4} dt = \sqrt{5}(x + 1).$$

Finally,

$$\gamma : [0, 2\sqrt{5}] \to \mathbb{R}^2,$$
$$s \mapsto \left(\frac{s}{\sqrt{5}} - 1, 2\left(\frac{s}{\sqrt{5}} - 1 \right) + 1 \right),$$

is the arc-length parametrization of γ. From this, we can calculate the tangent vector of the curve:

$$\tau(s) = \gamma'(s) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right).$$

1.3.3 What is curvature?

Definition 1.14. Consider $\gamma : [0, L(s)] \to \mathbb{R}^2$ parametrized in arc-length parameter. We define the *curvature* of γ by

$$\kappa(s) := |\gamma''(s)|.$$

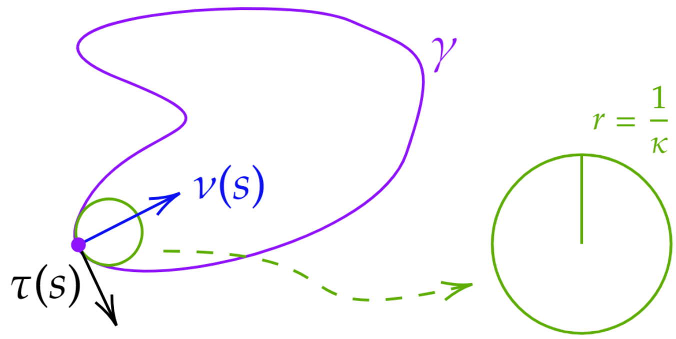
When $\kappa(s) \neq 0$, we can define the normal vector of a curve:

Definition 1.15. Let $\gamma : [0, L(s)] \to \mathbb{R}^2$ be a curve parametrized in arc-length. arco-parametrizada. We define the *normal vector* of γ (at s_0) ν as the vector such that $\{\tau, \nu\}$ is a positive basis. Also, we define the *signed curvature* by

$$\gamma''(s) = \kappa\nu.$$

Remark. Curves that satisfy $\gamma''(s) \neq 0$ for all s are usually called *biregular*.

Let us give the geometric intuition that is behind the definition of curvature. In order to do this, let $\gamma(s)$ be a biregular curve. At a point s_0, consider a circle that passes through $\gamma(s_0)$ and that have common tangent line to γ at s_0. It can be shown that the curvature of γ at s_0 is exactly the inverse of the radius of this circle. This is the "geometric equivalent" of a Taylor approximation of order 2.



Example 1.16.

1. Consider the curve

$$\begin{aligned} \gamma : [0, 2\pi] &\to \mathbb{R}^2, \\ \theta &\mapsto 2(\cos\theta, \sin\theta). \end{aligned}$$

We already know that its arc-length parametrization is

$$\begin{aligned} \gamma : [0, 4\pi] &\to \mathbb{R}^2, \\ s &\mapsto 2\left(\cos\frac{s}{2}, \sin\frac{s}{2}\right), \end{aligned}$$

and $\tau(s) = \left(\cos\frac{s}{2}, \sin\frac{s}{2}\right)$. From this, $\gamma''(s) = \frac{1}{2}\left(-\sin\frac{s}{2}, \cos\frac{s}{2}\right)$, and we obtain $\kappa(s) = |\gamma''(s)| = \frac{1}{2}$. Also, $\nu(s) = \left(-\sin\frac{s}{2}, \cos\frac{s}{2}\right)$.

2. Consider the curve

$$\begin{aligned} \gamma : [-1, 1] &\to \mathbb{R}^2, \\ t &\mapsto (t, 2t+1). \end{aligned}$$

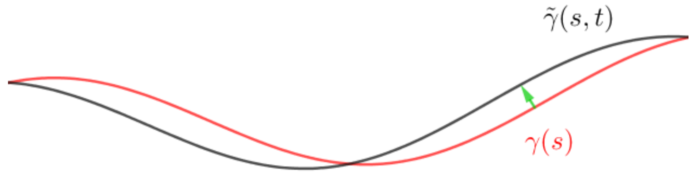
Recall that

$$\begin{aligned} \gamma : [0, 2\sqrt{5}] &\to \mathbb{R}^2, \\ s &\mapsto \left(\frac{s}{\sqrt{5}} - 1, 2\left(\frac{s}{\sqrt{5}} - 1\right) + 1\right), \end{aligned}$$

is the arc-length parametrization of γ and that $\tau(s) = \frac{1}{\sqrt{5}}(1, 2)$. This implies that $\kappa(s) = |\gamma''(s)| = 0$.

So far we have defined the curvature and we gave an geometric interpretation, but why is the curvature a natural quantity to study? To answer this question, given a arc-length parametrized curve $\gamma(s)$, consider a normal variation of it, that is

$$\tilde{\gamma}(s,t) = \gamma(s) + t\varphi(s)\nu(s)$$



Clearly

$$\left|\frac{d\tilde{\gamma}}{ds}\right| = |\gamma'(s) + t\varphi\nu + t\varphi\nu'|,$$

where the prime denotes the derivative with respect the arc-length parameter. Then,

$$L(\tilde{\gamma})(t) = \int_0^{L(\gamma)} |\tau(s) + t\varphi'\nu + t\varphi\nu'| ds.$$

We know all the terms in this expression, except for ν'. To find it, first recall that at each s, $\{\tau(s), \nu(s)\}$ is an orthonormal basis of $\mathbb{R}^2$. This allows us to write

$$\nu'(s) = a(s)\tau(s) + b(s)\nu(s).$$

Since $\langle \nu, \nu \rangle = 1$, we obtain $\langle \nu', \nu \rangle = 0$, and from $\langle \nu, \tau \rangle = 0$, we conclude $\langle \nu', \tau \rangle = -\langle \nu, \tau' \rangle$. Therefore,

$$\nu'(s) = -\kappa\tau.$$

In resume, we have that

$$L(\tilde{\gamma})(t) = \int_0^{L(\gamma)} |(1 - t\kappa\varphi)\tau(s) + t\varphi'\nu| ds.$$

To calculate the derivative of this expression, first notice that

$$\frac{d}{dt}|(1 - t\kappa\varphi)\tau(s) + t\varphi'\nu|^2 = \frac{d}{dt}[(1 - t\kappa\varphi)^2 + t^2(\varphi')^2] = -2\kappa\varphi(1 - t\kappa\varphi) + 2t(\varphi')^2.$$

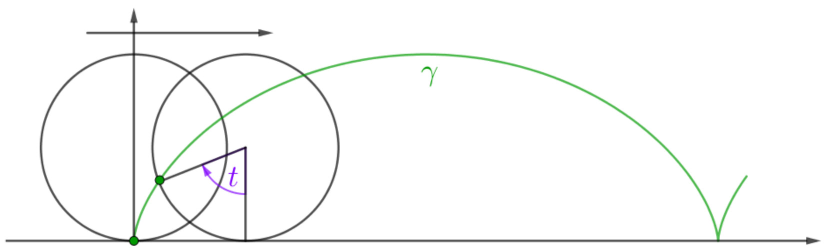
Therefore,

$$\left.\frac{d}{dt}L(\tilde{\gamma})(t)\right|_{t=0} = -\int_0^{L(\gamma)} \kappa\varphi ds.$$

1.4 Exercises

Exercise 1.1. Let γ be a curve such that $\gamma'' \equiv 0$. ¿What can you say about γ?

Exercise 1.2. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a cycloid



1. Obtain a parametrization of the cycloid, and determine its singular points (i.e., the points where the parametrization is not regular).
2. Compute the length of the cycloid corresponding to a complete rotation of the disk.

Exercise 1.3. Compute the tangent and the normal vector, in addition to the curvature of the curve $(x, -\ln(\cos x))$.

Exercise 1.4. Compute the same quantities of the previous exercises but now for the curve $(x, u(x))$, where u is smooth enough.

Exercise 1.5. Compute the same quantities of the previous exercises but now for the curve $r(\theta)(\cos\theta, \sin\theta)$

Exercise 1.6. Consider $\gamma : [a, b] \to \mathbb{R}^2$ a parametrized curve with regular parametrization and let φ to be a (strictly) monotone function. Prove that $\gamma \circ \varphi$ represent the same curve. What happens with the geometric quantities if φ is decreasing?. Give and example of this.

2 Lecture II

In this class we introduce the curve shortening flow and some examples. Then, we study the maximum principle and how to apply it in order to obtain information of the flow.

2.1 What is curvature? (Continuation)

Recall that

$$\frac{d}{dt}L(\tilde{\gamma})(t)\bigg|_{t=0} = -\int_0^{L(\gamma)} \kappa\varphi ds.$$

If we set $\varphi = \kappa$, then this is decreasing (and it is the fastest decreasing direction).

Definition 2.1. The *curve shortening flow* (CSF) is the deformation of a curve in the normal direction with a speed equal to its curvature. More precisely, we consider $\gamma : I \times [0, \infty) \to \mathbb{R}^2$ satisfying

$$\frac{d\gamma}{dt} = \kappa\nu = \gamma_{ss},$$

where κ is the signed curvature.

Remark. For each t, $\gamma(\cdot, t)$ is a curve.

Example 2.2. Assume that $r(t)(\cos\theta, \sin\theta) = \gamma(\theta, t)$ is a solution to the curve shortening flow. In this example, $\nu = -(\cos\theta, \sin\theta)$ and $\kappa = \frac{1}{r}$. Now, notice that

$$\frac{d\gamma}{dt} = \frac{dr}{dt}(\cos\theta, \sin\theta).$$

Then,

$$\frac{d\gamma}{dt} \cdot \nu = -\frac{dr}{dt},$$

but $\frac{d\gamma}{dt} \cdot \nu = \kappa = \frac{1}{r}$. Thus,

$$r(t)\frac{dr}{dt} = -1,$$

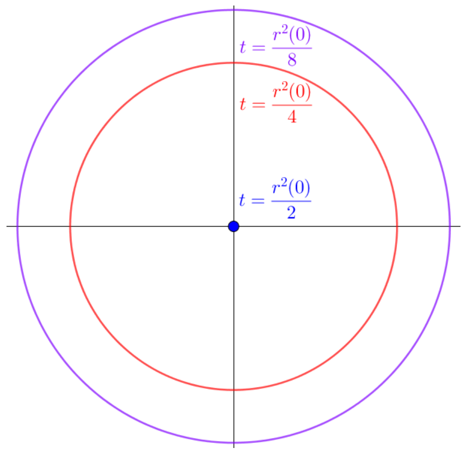
or, equivalently,

$$\frac{1}{2}\frac{dr^2}{dt} = -1.$$

Integrating we obtain

$$r(t) = \sqrt{r^2(0) - 2t}.$$

Geometrically, we see that the curves, which are concentric circles, shrink to a point as $t \to \frac{r^2(0)}{2}$.



Example 2.3. We look for a solution of the CSF of the form $\gamma(x,t) = (x, u(x)+t)$. In this case,

$$\frac{d\gamma}{dt} = (0,1), \quad \text{and} \quad \nu = \frac{\left(\frac{du}{dx}, -1\right)}{\sqrt{1+\left(\frac{du}{dx}\right)^2}},$$

which implies

$$\frac{d\gamma}{dt}\cdot\nu = \frac{-1}{\sqrt{1+u_x^2}}.$$

Now we want to compute γ_{ss}. For this, notice that from

$$s = \int_0^x \sqrt{1+u_x^2}\,d\lambda,$$

it follows that

$$\frac{dx}{ds} = \frac{1}{\sqrt{1+u_x^2}}.$$

Using this last calculation, we obtain

$$\frac{d\gamma}{ds} = (1, u_x)\frac{dx}{ds} = \frac{(1,u_x)}{\sqrt{1+u_x^2}},$$

and from this

$$\frac{d^2\gamma}{ds^2} = \frac{1}{\sqrt{1+u_x^2}}\left(\frac{-u_x u_{xx}}{(1+u_x)^{\frac{3}{2}}}, \frac{u_{xx}}{\sqrt{1+u_x^2}} - \frac{u_x^2 u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right).$$

Then

$$\frac{d^2\gamma}{ds^2}\cdot\nu = \frac{-u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}.$$

Since γ is a solution of the CSF, we obtain

$$\frac{-1}{\sqrt{1+u_x^2}} = \frac{d\gamma}{dt} \cdot \nu = \frac{d^2\gamma}{ds^2} \cdot \nu = \frac{-u_{xx}}{(1+u_x^2)^{\frac{3}{2}}},$$

or equivalently

$$1 = \frac{u_{xx}}{1+u_x^2}.$$

Integrating this expression we obtain

$$x = \arctan u_x.$$

and then

$$\frac{du}{dx} = \frac{\sin x}{\cos x} = -\frac{d}{dx}\ln(\cos x).$$

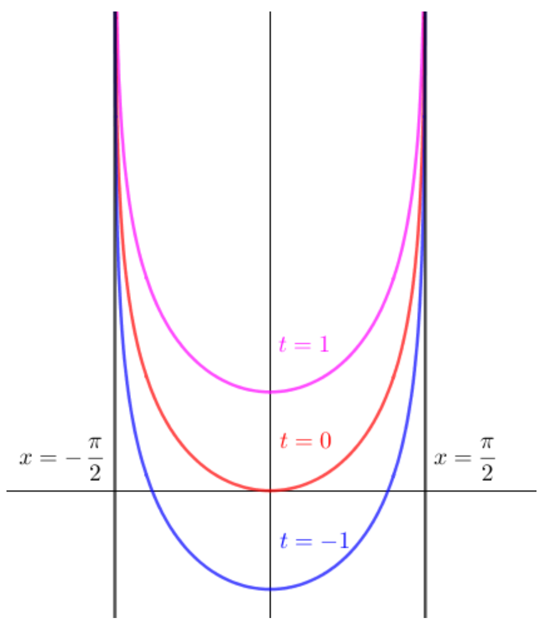
From this, we conclude that

$$u(x) = -\ln\cos(x) + u(0).$$

If we choose $u(0) = 0$ we find another example of solution to the CSF:

$$(x, t - \ln(\cos x)), \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

This solution is called the *grim reaper*. Below we give the curves for $t = -1, 0$ and 1.



Notice that $\gamma(x, t)$ is the translation of the curve $x \mapsto (x, -\ln(\cos x))$ along the $y-$axis.

2.2 The maximum principle

Now we focus our attention in a very useful tool of differential equations, which we will use later.

Assume that we have a solution to an ODE of the form

$$-af'' + bf' + c(x) = 0, \quad \text{for } x \in (A, B),$$

with $a > 0$ and $c > 0$.

Theorem 2.4. *If f satisfies the previous ODE, then does not have a maximum in the interior.*

Proof. Assume by contradiction that there exists $x \in (A, B)$ in which f attains its maximum. Then, $f''(x) \leq 0$ and $f'(x) = 0$. These conditions and the hypothesis that f solve the ODE implies that

$$c(x) \leq -af'' + bf' + c(x) = 0,$$

which is a contradiction since $c > 0$. □

Remark. The same statement is true for the minimum if $c < 0$.

Theorem 2.5. *If f satisfies the same as before but now with $c \geq 0$, the same statements hold.*

Proof. Take $f_\varepsilon = f + \varepsilon e^{Lx}$ and compute as follows:

$$\begin{aligned} -af_\varepsilon'' + bf_\varepsilon' &= -a(f'' + \varepsilon L^2 e^{Lx}) + b(f' + \varepsilon L e^{Lx}) \\ &= -af'' + bf' + c(x) - c(x) + \varepsilon e^{Lx}(-La^2 + Lb) \\ &= -c(x) + \varepsilon e^{Lx}(-L^2 a + Lb). \end{aligned}$$

For L large enough, $-L^2 a + Lb < 0$. Therefore, f_ε satisfies the condition of the previous theorem and we can conclude that

$$f_\varepsilon \leq \max\{f_\varepsilon(A), f_\varepsilon(B)\}.$$

If we let $\varepsilon \to 0$, we conclude what we wanted. □

Exercise. Prove that the minimum is attained at the boundary if $c \leq 0$.

2.3 An application of the maximum principle

Proposition 2.6. *Assume that f_1 and f_2 are defined on $[0,1]$ and satisfy*

$$-af_i'' + bf_i' + c = 0,$$

and $f_1(0) = f_2(0)$, $f_1(1) = f_2(1)$. Then, $f_1 \equiv f_2$.

Proof. Define $g := f_1 - f_2$. Notice that

$$\begin{aligned} -ag'' + bg' &= -a(f_1 - f_2) + b(f_1 - f_2) \\ &= -af_1 + bf_1 + c - af_2 + bf_2 - c = 0. \end{aligned}$$

Also, $g(0) = g(1) = 0$. From the maximum principle, we have that $g \le 0$ and $-g \le 0$. Therefore, $g \equiv 0$, i.e., $f_1 \equiv f_2$. □

We will be interested in equations of the form

$$-af_{xx} + bf_x + c = f_t.$$

Following the preceding ideas, we can define

$$\tilde{f}(t) = \max_{x\in[0,1]} f(x,t).$$

If $\tilde{f}$ is regular enough, we would have

$$c \le (\tilde{f})_t \quad (\text{or } \tilde{f} = f(1,t) \text{ or } \tilde{f} = f(0,t)\ ).$$

In particular, if $c \ge 0$, $\tilde{f}$ is increasing.
Similarly, we can define

$$\underset{\sim}{f}(t) = \min_{x\in[0,1]} f(x,t)$$

and

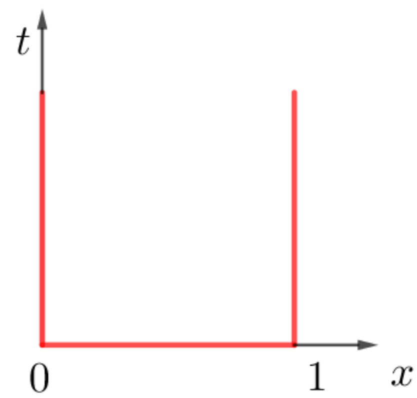
$$(\underset{\sim}{f})_t \le c \quad (\text{or } \underset{\sim}{f} = f(1,t) \text{ or } \underset{\sim}{f} = f(0,t)\ ).$$

If $c \le 0$, $\underset{\sim}{f}$ is decreasing.

Therefore

$$\max_{x,t} f(x,t) = \max\left\{ \max_{x\in[0,1]} f(x,0), \max_t\{f(0,t), f(1,t)\} \right\}.$$

The set of points where f attains its maximum is called *parabolic boundary*



2.4 A return to CSF

Theorem 2.7. *Assume that $(x, u_1(x,t))$ and $(x, u_2(x,t))$ are solutions of the CSF. Assume in addition that*

1. $u_1(x,0) > u_2(x,0)$ *for* $x \in [-1,1]$,
2. $u_1(\pm 1, t) > u_2(\pm 1, t)$.

Then $u_1(x,t) > u_2(x,t)$ *for every* $x \in [-1,1]$ *and* $t \in [0,T)$.

Exercise. Show that

$$(u_i)_t = \frac{(u_i)_{xx}}{1+(u_i)_x^2}.$$

Proof. Let $v_\varepsilon = u_1 - u_2 + \varepsilon t$. The conditions 1 and 2 of the hypothesis implies that

1. $v_\varepsilon(x,0) > 0$
2. $v_\varepsilon(\pm 1, t) > 0$.

Assume that there is an $x_0 \in (-1,1)$ and a first $t_0 \in (0,T)$ such that $v_\varepsilon(x_0,t_0) = 0$ (that is, $u_2(x_0,t_0) = u_1(x_0,t_0) + \varepsilon t_0 > u_1(x_0,t_0)$. Since for $t < t_0$ we have $v_\varepsilon(x_0,t) > 0$, then we obtain

$$\frac{dv_\varepsilon}{dt}(x_0,t_0) \leq 0. \tag{*}$$

Since this is a minimum we have

$$\frac{dv_\varepsilon}{dx} = 0 \quad \text{and} \quad \frac{d^2 v_\varepsilon}{dx^2} \geq 0. \tag{**}$$

Also, by the previous exercise

$$\frac{dv_\varepsilon}{dt} = \frac{du_1}{dt} - \frac{du_2}{dt} + \varepsilon = \frac{(u_1)_{xx}}{1+(u_1)_x^2} - \frac{(u_2)_{xx}}{1+(u_2)_x^2} + \varepsilon.$$

If we use (*) and (**), at (x_0, t_0) we obtain

$$0 \geq \frac{dv_\varepsilon}{dt} = \frac{(v_\varepsilon)_{xx}}{1+(u_1)_x^2} + \varepsilon > 0,$$

which is a contradiction. Therefore, $v_\varepsilon > 0$, i.e., for every $\varepsilon > 0$

$$u_1(x,t) \geq u_2(x,t) - \varepsilon t.$$

Finally, we let $\varepsilon \to 0$ to conclude what we wanted. □

Theorem 2.8. *If $\gamma_1(x,t)$ and $\gamma_2(x,t)$ are two bounded closed curves such that $\gamma_1(x,0) \cap \gamma_2(x,0) = \emptyset$, then $\gamma_1(x,t) \cap \gamma_2(x,t) = \emptyset$ while solutions are defined.*

Proof. If they touch at some point, we can locally (after rotation and translation) write them as graphs and then we can apply the previous result. □

Corollary 2.9. *If $\gamma(x,t)$ is a compact (bounded) curve, the solution can exists at most for a finite time.*

2.5 Exercises

Exercise 2.1. Assume that γ satisfies the CSF and $\phi(x,t)$ is an increasing function en x for every t. Compute the equation that satisfies $\gamma(\phi(x,t),t)$.

Exercise 2.2. Prove that if

$$-af'' + bf' + cf = 0 \quad \text{in } (0,1),$$

and $a > 0$, $c > 0$, $f(0) > 0$, $f(1) > 0$, then $f \geq 0$ in $(0,1)$.

3 Lecture III

Theorem 3.1 (Main Theorem)**.** *Every compact, embedded, C^2 curve converges to a point in finite time.*

This foundational theorem was first proved in the mid 80's by a combination of results due to Gage & Hamilton and Grayson, see [GH86] and [Gra87]. A few alternative proofs have been published later. Here we will analyze one provided by G. Huisken in [Hui98].

In order to reach our objective, we first check the following simpler result:

Theorem 3.2. *Let $\gamma : I \times [0,T] \to \mathbb{R}^2$ be an open curve that evolves under curve shortening flow (in its interior). Let*

$$\delta(x,y,t) = |\gamma(x,t) - \gamma(y,t)| \quad \text{and} \quad \ell(x,y,t) = \int_x^y |\gamma'(\lambda,t)| d\lambda.$$

Assume that $\frac{d}{\ell}$ attains an infimum in the interior at time t_0. Then

$$\frac{d}{dt}\left(\frac{\delta}{\ell}\right)(x,y,t_0) \geq 0,$$

whith equality when γ is a straight line.

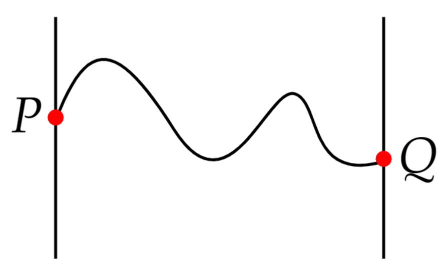
Remark. In general $\frac{\delta}{\ell} \leq 1$, since ℓ is the shortest distance.

The "isoperimetric quantity" $\frac{\delta}{\ell}$ gives a quantitative measure of how γ differs from a straight line.

Remark. For an open curve we need to specify the behavior at the boundary. Two standard choices to have a well defined problem are the following:

1. To prescribe the points.

2. To prescribe an angle with fixed lines.

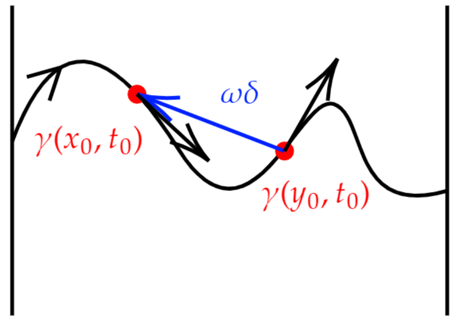


Note that if we fix the end points, then

$$\frac{d}{dt}\left(\frac{\delta}{\ell}(P,Q,t)\right) = -\frac{\delta}{\ell^2}\frac{d\ell}{dt} = \frac{\delta}{\ell^2}\int_P^Q k^2 ds \geq 0,$$

with equality only for a straight line.

Sketch of the proof. Assume that at t_0 the infimum is attained at x_0, y_0.



Without loss of generality, we can assume that $\gamma(x, t_0)$ is parametrized in arc–length parameter, that is, $x \in [0, L(t_0)]$. To show that

$$\frac{d}{dt}\left(\frac{\delta}{\ell}\right) = \frac{\delta'\ell - \ell'\delta}{\ell^2} \geq 0,$$

we first consider the following one variable function:

$$f_1(\lambda) = \frac{\delta}{\ell}(x_0 + \lambda, y_0, t_0).$$

Since f_1 has a minimum at $\lambda = 0$, we obtain

$$\frac{df_1}{d\lambda}(x_0, y_0, t_0) = 0.$$

In order to understand what this means, we will compute this derivative. First, we will compute their ingredients: since $\gamma(\cdot, t_0)$ is parametrized in arc-length parameter,

$$\ell(x_0 + \lambda, y_0, t_0) = \int_{x_0+\lambda}^{y_0} 1 d\alpha = y_0 - x_0 - \lambda.$$

This implies that $\frac{d\ell}{d\lambda} = -1$. Also, since

$$\delta^2(x_0 + \lambda, y_0 + \lambda) = |\gamma(x_0 + \lambda, t_0) - \gamma(y_0, t_0)|^2,$$

then

$$\left.\frac{d\delta^2}{d\lambda}\right|_{\lambda=0} = 2\delta \left.\frac{d\delta}{d\lambda}\right|_{\lambda=0} = 2\langle \gamma(x_0, t_0) - \gamma(y_0, t_0), \tau(x_0, t_0)\rangle.$$

Thus,

$$\begin{aligned} 0 = \frac{df_1}{d\lambda}(0) &= \frac{\ell \frac{d\delta}{d\lambda} - \delta \frac{d\ell}{d\lambda}}{\ell^2} \\ &= \frac{\frac{d\delta}{d\lambda}}{\ell} + \frac{\delta}{\ell^2} \\ &= \frac{\langle \omega, \tau(x_0, t_0)\rangle}{\ell} + \frac{\delta}{\ell^2}, \end{aligned}$$

where $\omega := \frac{\gamma(x_0,t_0)-\gamma(y_0,t_0)}{|\gamma(x_0,t_0)-\gamma(y_0,t_0)|}$. In a similar way, if we consider $f_2(\lambda) = \frac{\delta}{\ell}(x_0, y_0+\lambda, t_0)$, we notice that

$$\frac{df_2}{d\lambda}(0) = 0,$$

and

$$0 = \frac{df_2}{d\lambda}(0) = -\frac{\langle \omega, \tau(y_0,t_0)\rangle}{\ell} - \frac{\delta}{\ell^2}.$$

Therefore,

$$\langle \omega, \tau(x_0,t_0)\rangle = \langle \omega, \tau(y_0,t_0)\rangle.$$

From here, we separate the proof into two cases.

- **Case 1:** $\tau(x_0,t_0) = \tau(y_0,t_0)$. Here we consider the function $\tilde{f}_1(\lambda) = \frac{\delta}{\ell}(x_0+\lambda, y_0+\lambda, t_0)$. Note that

$$\frac{d\tilde{f}_1}{d\lambda}(0) = 0 \quad \text{and} \quad \frac{d^2\tilde{f}_1}{d\lambda^2}(0) \geq 0.$$

Similar to the previous computations, using the second order quotient rule we obtain

$$\begin{aligned}
0 \leq \frac{d^2\tilde{f}_1}{d\lambda^2}(0) &= \frac{\frac{d^2\delta}{d\lambda^2} - 2\frac{d\tilde{f}_1}{d\lambda}\frac{d\ell}{d\lambda} - \tilde{f}_1\frac{d^2\ell}{d\lambda}}{\ell} \\
&= \frac{1}{\ell}\frac{d^2\delta}{\lambda^2} \\
&= \frac{\langle \omega, \kappa(x_0,t_0)\nu(x_0,t_0) - \kappa(y_0,t_0)\nu(y_0,t_0)\rangle}{\ell} + \frac{|\tau(x_0,t_0)-\tau(y_0,t_0)|^2}{\delta\ell} \\
&- \frac{\langle \omega, \tau(x_0,t_0) - \tau(y_0,t_0)\rangle}{\delta^2\ell}\frac{d\delta}{d\lambda} \\
&= \frac{\langle \omega, \kappa(x_0,t_0)\nu(x_0,t_0) - \kappa(y_0,t_0)\nu(y_0,t_0)\rangle}{\ell}.
\end{aligned}$$

Now notice that since γ satisfies the CSF, then

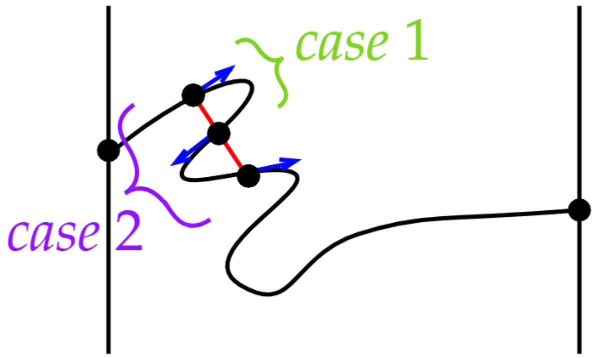
$$\frac{d\delta}{dt} = \frac{\langle \omega, \kappa(x_0,t_0)\nu(x_0,t_0) - \kappa(y_0,t_0)\nu(y_0,t_0)\rangle}{\ell} \geq 0,$$

and (from the computations of the previous lectures)

$$\frac{d\ell}{dt} = -\int_{x_0}^{y_0} \kappa^2 ds.$$

Therefore,

$$\frac{d}{dt}\left(\frac{\delta}{\ell}\right) = \frac{\delta'\ell - \ell'\delta}{\ell^2} = \frac{\langle \omega, \kappa(x_0,t_0)\nu(x_0,t_0) - \kappa(y_0,t_0)\nu(y_0,t_0)\rangle}{\ell} + \frac{\delta}{\ell}\int_{x_0}^{y_0}\kappa^2 ds \geq 0.$$



- **Case 2:** $\tau(x_0,t_0) \neq \tau(y_0,t_0)$. In this case, we consider $\tilde{f}_2(\lambda) = \frac{\delta}{\ell}(x_0+\lambda, y_0-\lambda, t_0)$. As before,

$$\frac{d\tilde{f}_2}{d\lambda}(0) = 0 \quad \text{and} \quad \frac{d^2\tilde{f}_2}{d\lambda^2}(0) \geq 0.$$

Also,

$$0 = \frac{d\tilde{f}_2}{d\lambda} = \frac{\langle \omega, \tau(x_0,t_0) + \tau(y_0,t_0)\rangle}{\ell} + \frac{2\delta}{\ell^2},$$

and

$$\frac{d^2\tilde{f}_2}{d\lambda^2} = \frac{\langle \omega, \kappa(x_0,t_0)\nu(x_0,t_0) - \kappa(y_0,t_0)\nu(y_0,t_0)\rangle}{\ell} + \frac{|\tau(x_0,t_0)+\tau(y_0,t_0)|^2}{d\ell} - \frac{\langle \omega, \tau(x_0,t_0)+\tau(y_0,t_0)\rangle^2}{d\ell}.$$

Let us write $\tau_1 = \tau(x_0,t_0)$ and $\tau_2 = \tau(y_0,t_0)$. Then

$$\langle \tau_1 - \tau_2, \tau_1 + \tau_2\rangle = \langle \tau_1, \tau_1\rangle + \langle \tau_1, \tau_2\rangle - 2\langle \tau_2, \tau_1\rangle + \langle \tau_2, \tau_2\rangle = 0,$$

that is, $\tau_1 - \tau_2 \perp \tau_1 + \tau_2$. Since we also have $\tau_1 - \tau_2 \perp \omega$, we obtain $\tau_1 + \tau_2 /\!/ \omega$ and (since now $\omega = \frac{\tau_1+\tau_2}{|\tau_1+\tau_2|}$) $\langle \omega, \tau_1+\tau_2\rangle^2 = |\tau_1+\tau_2|^2$. Therefore,

$$0 \leq \frac{d^2\tilde{f}_2}{d\lambda^2} = \frac{\langle \omega, \kappa(x_0,t_0)\nu(x_0,t_0) - \kappa(y_0,t_0)\nu(y_0,t_0)\rangle}{\ell},$$

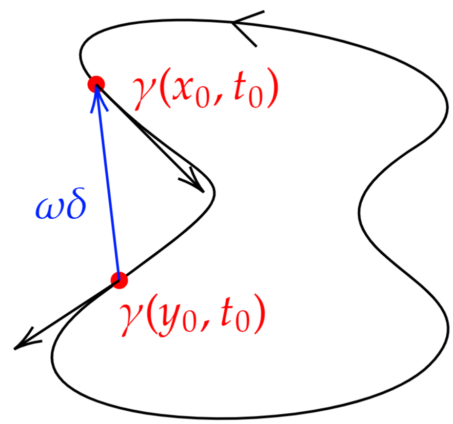
and we can conclude as in the previous case.

Finally, note that for the equality necessarily

$$\int_{x_0}^{y_0} \kappa^2 ds = 0,$$

which implies that $\kappa \equiv 0$, that is, the curve is a straight line. $\square$

Remark. The situation is similar for closed curves, and the picture at the critical values looks as follows:



For our Main Theorem a similar idea can be used, but we need to consider a different "isoperimetric profile". Instead of $\frac{\delta}{\ell}$ we use $\frac{\delta}{\psi(\ell)}$, were

$$\psi(\ell) = \frac{L(t)}{\pi} \sin\left(\frac{\ell\pi}{L(t)}\right),$$

where $L(t)$ is the total length at time t.The computation in this case is similar and we leave it to the interested reader.

For further study about the CSF and some of its generalizations, we highly recommend the book [CZ01] and [Zhu02].

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