Re-Imaging the World through Linear Algebra

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Mathematics Sin Fronteras
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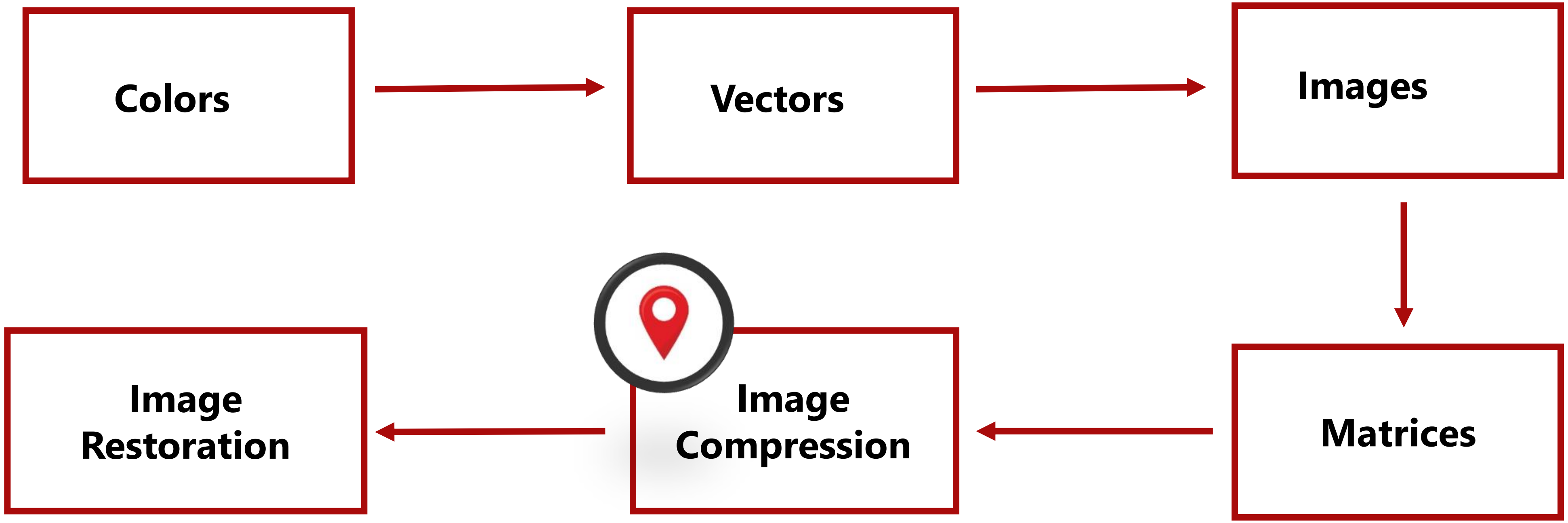
The Plan


Colors
Vectors
Images
Matrices
Image Compression
Image Restoration

Singular Value Decomposition (SVD)

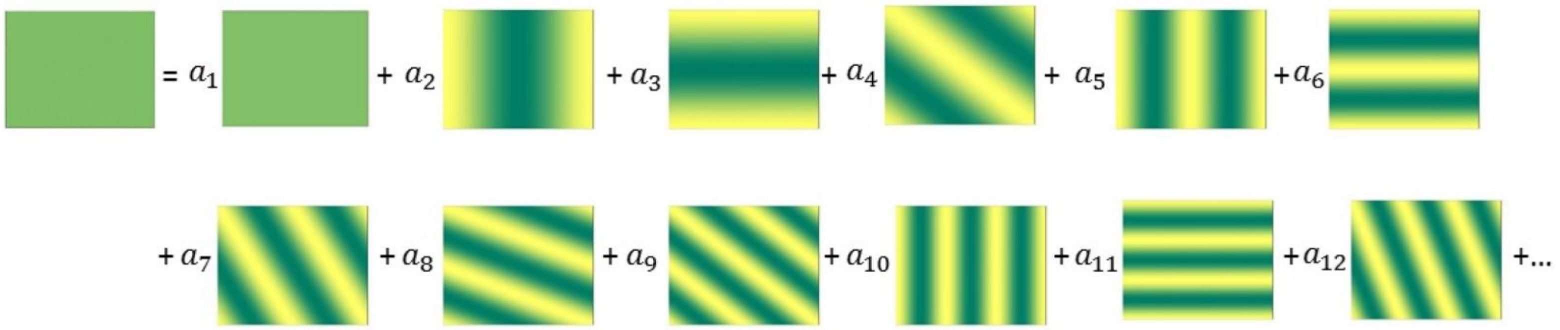
Theorem: Suppose $A \in \mathbb{R}^{n\times n}$. Then, there exist matrices $U \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{n\times n}$ and $\Sigma \in \mathbb{R}^{n\times n}$, with $U^T U = V^T V = I_n$, and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_\mathrm{n})$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, such that $A = U\Sigma V^T$. The columns of U are called the left singular vectors, the columns of V the right singular vectors, and σ_i are called the singular values.

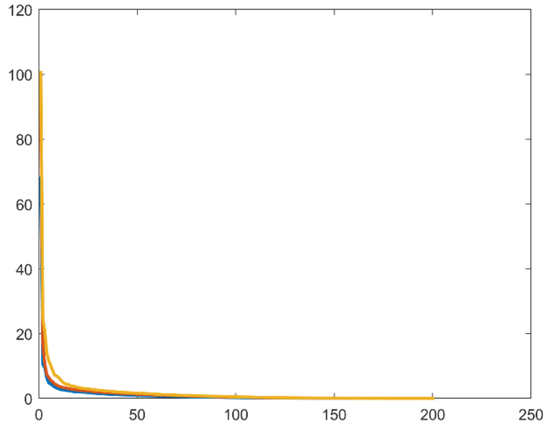
$$A = U\Sigma V^T = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{pmatrix} \begin{pmatrix} — & v_1^T & — \\ & \vdots & \\ — & v_n^T & — \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} — & \sigma_1 v_1^T & — \\ & \vdots & \\ — & \sigma_n v_n^T & — \end{pmatrix} = \Sigma_{i=1}^n \sigma_i u_i v_i^T$$

SVD and TSVD of an Image

$$A = U\Sigma V^T = \Sigma_{i=1}^{n} \sigma_i u_i v_i^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T + \cdots + \sigma_n u_n v_n^T$$



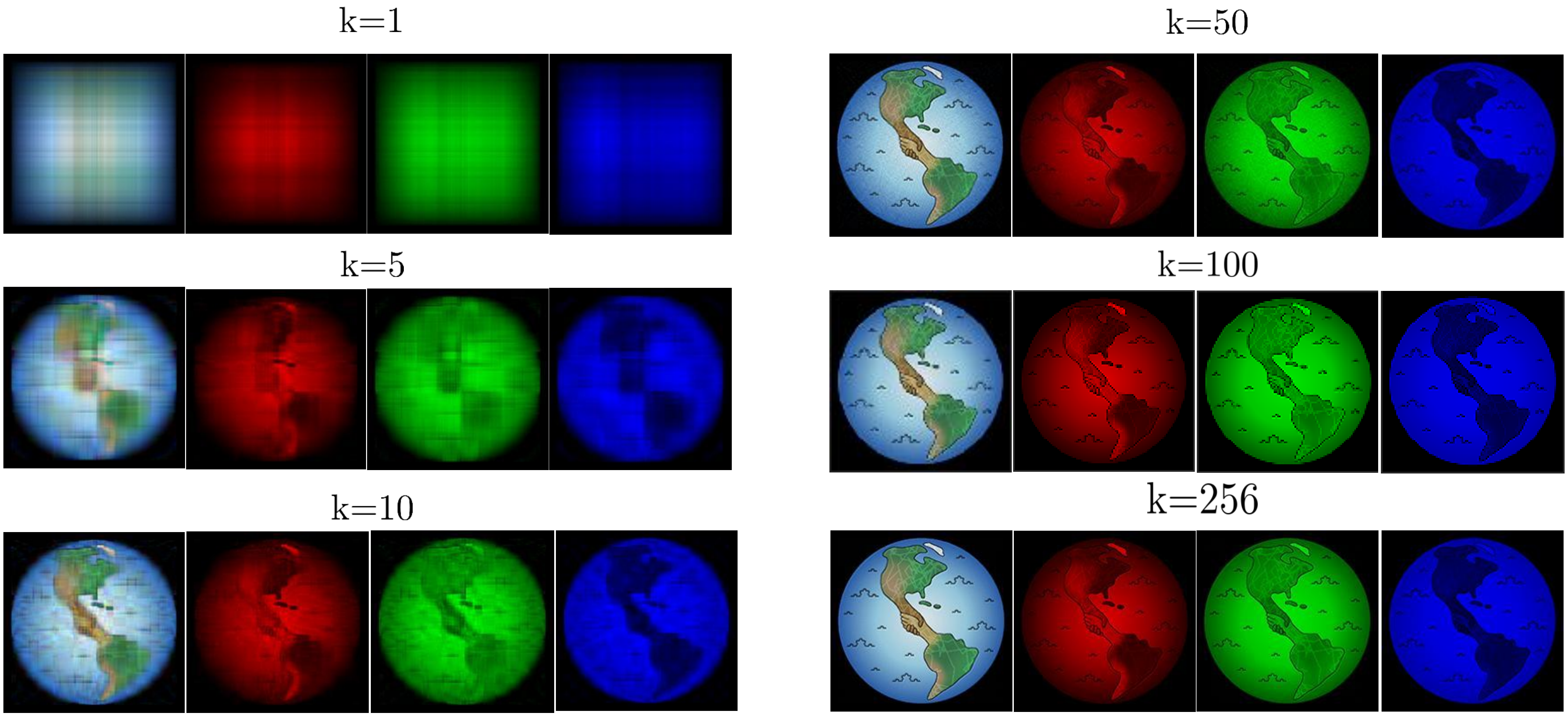


$$A_k = U_k \Sigma_k V_k^T = \Sigma_{i=1}^{k} \sigma_i u_i v_i^T$$

What is the cost of storing these matrices?

What is the cost of computing the matrix A_k?

Image Compression using SVD



Matrix Norms and Distances

Definition: The *Frobenius norm* of a matrix A is defined by

$$\|A\|_F = \sqrt{a_{11}^2 + a_{12}^2 + \cdots + a_{mn}^2}$$

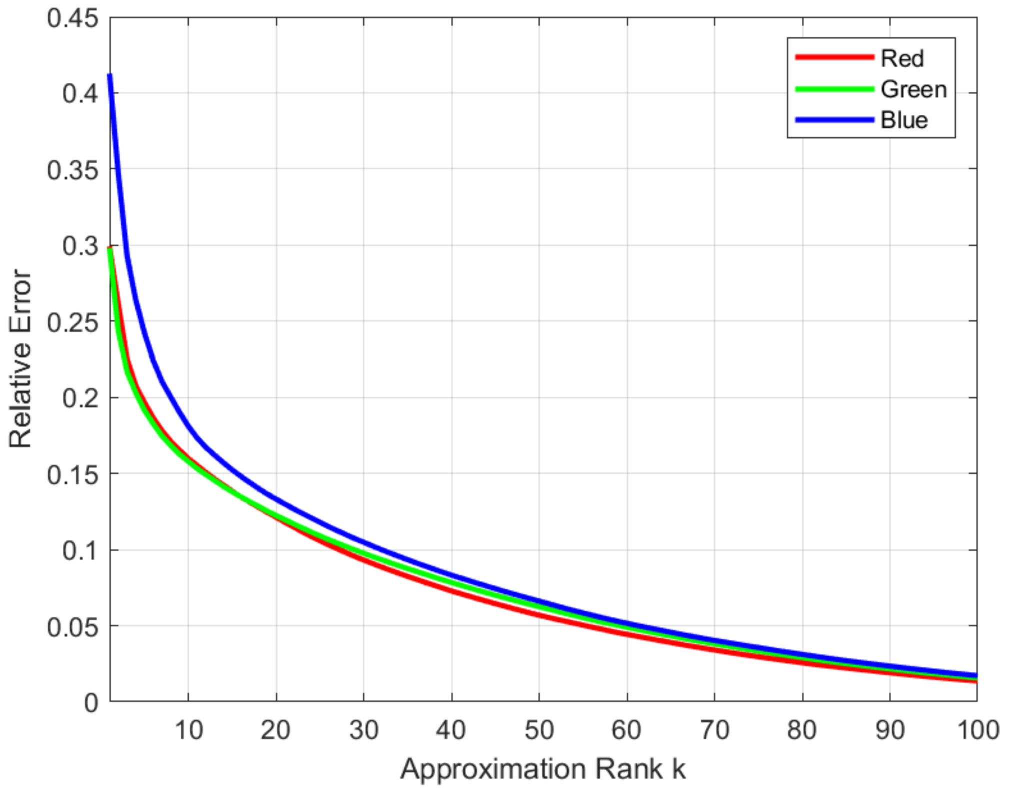
Examples: $\left\| \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\|_F = \sqrt{30}$ y $\left\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\|_F = \sqrt{2}$

Definition: The *Frobenius distance* between two matrices A and B is defined by

$$d(A, B) = \|A - B\|_F = \sqrt{(a_{11} - b_{11})^2 + (a_{12} - b_{12})^2 + \cdots + (a_{mn} - b_{mn})^2}$$

Example: $d\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \left\| \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\|_F = \left\| \begin{pmatrix} 0 & 2 \\ 3 & 3 \end{pmatrix} \right\|_F = \sqrt{22}$

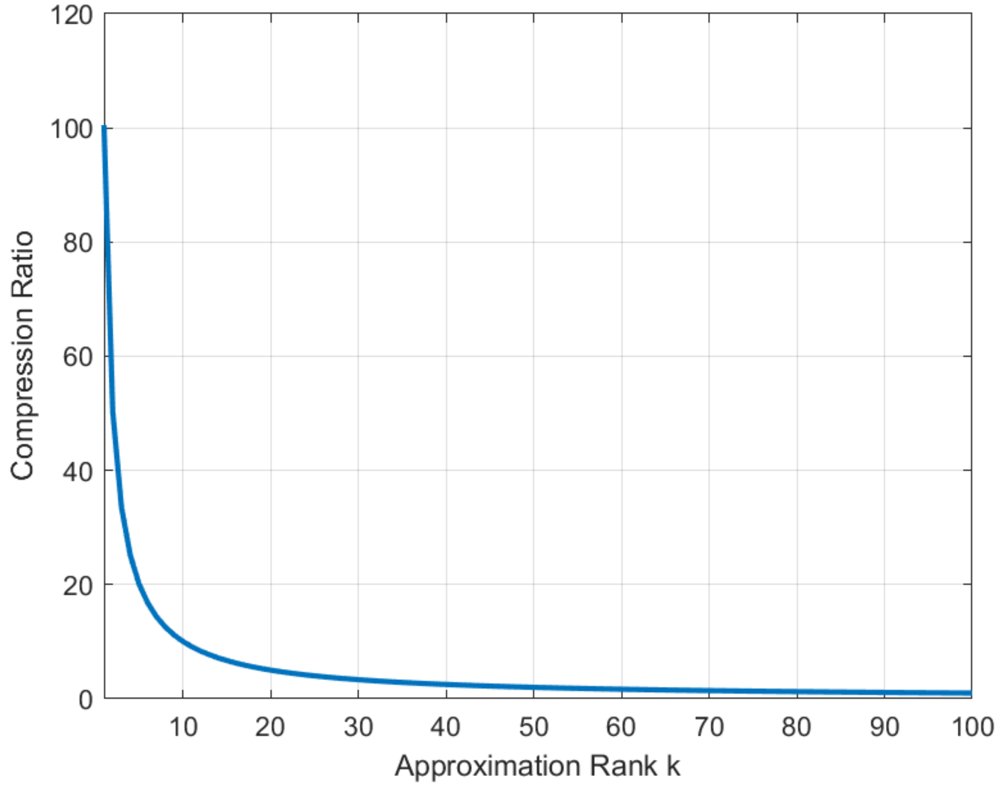
Image Compression: Relative Error



Relative Error:

$$\frac{\|A - A_k\|_F}{\|A\|_F}$$

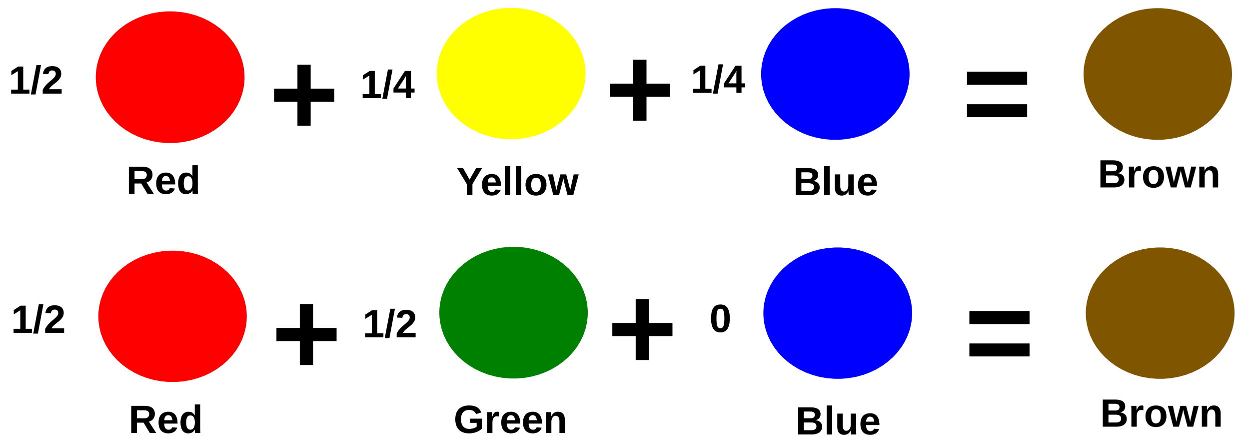
Image Compression: Compression Ratio

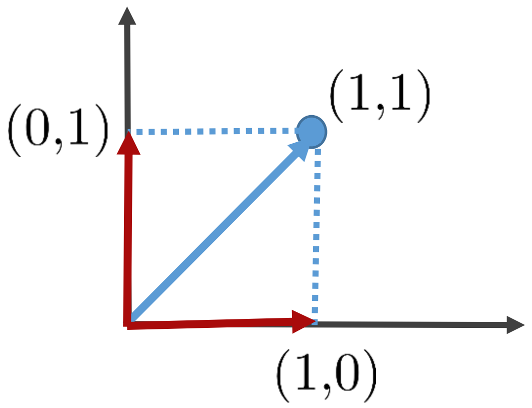


Compression Ration:

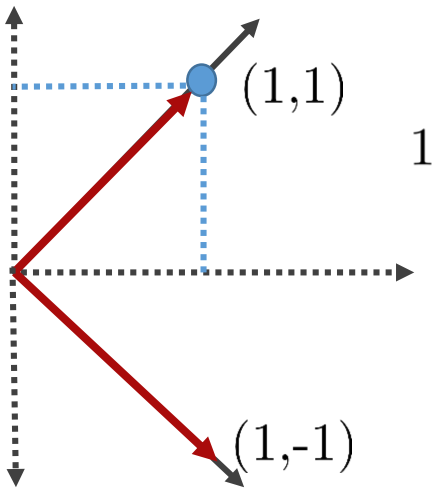
$$\frac{mn}{k(m+n+1)}$$

Coordinate Systems





$$1\ (1,0) + 1\ (0,1) = (1,1)$$



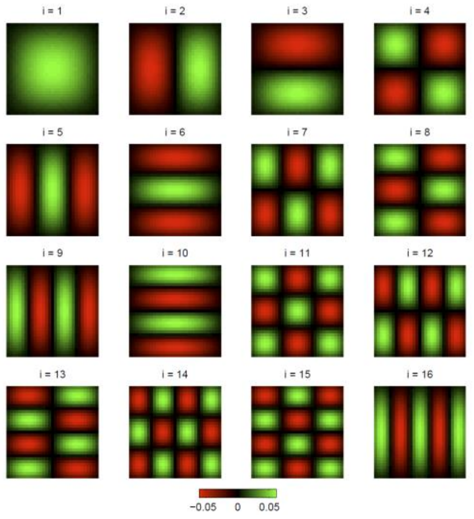
$$1\ (1,1) + 0\ (1,-1) = (1,1)$$

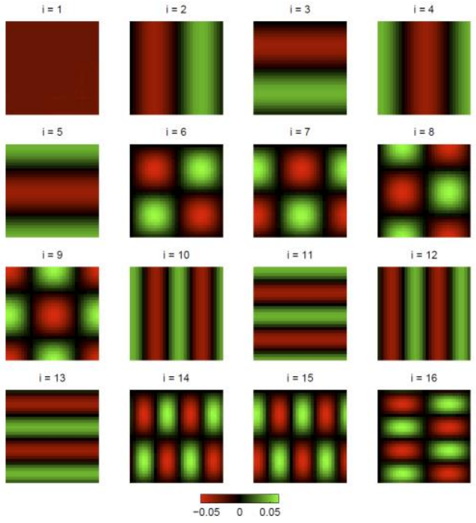
“Coordinate Systems” of Matrices

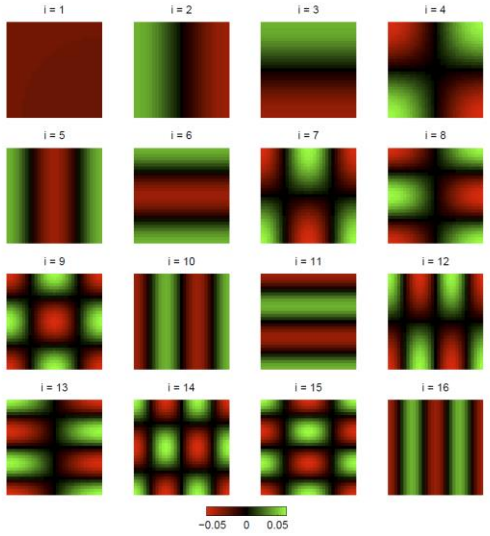
$$\begin{pmatrix} 5 & 5 \\ 3 & 5 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 5\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 5\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

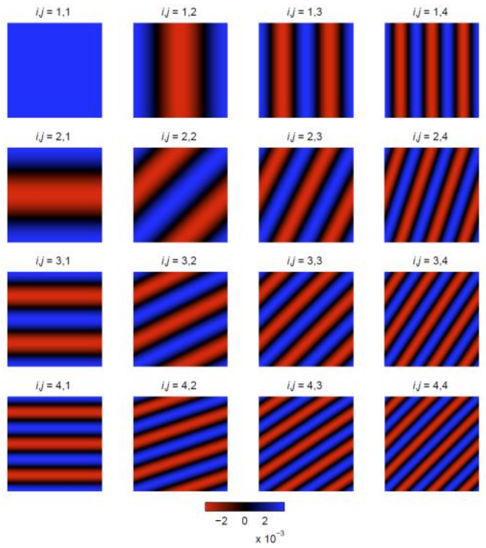
$$\begin{pmatrix} 5 & 5 \\ 3 & 5 \end{pmatrix} = 3\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

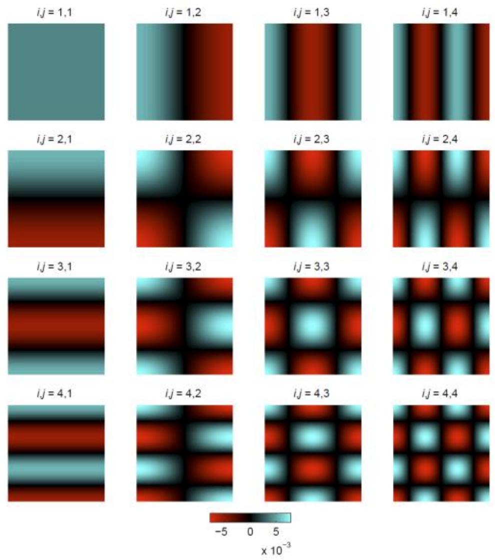
“Coordinate System” for Images

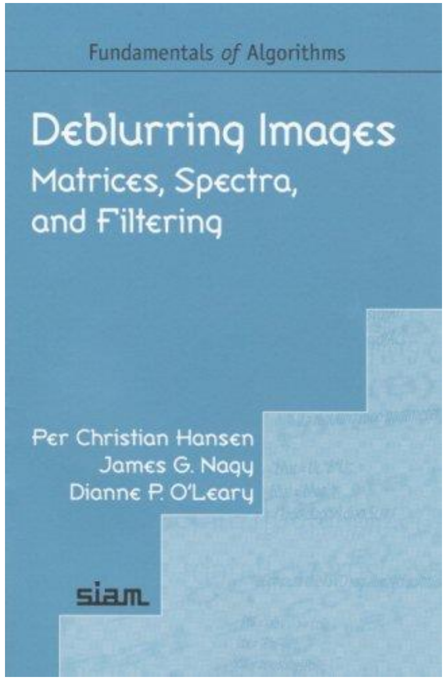










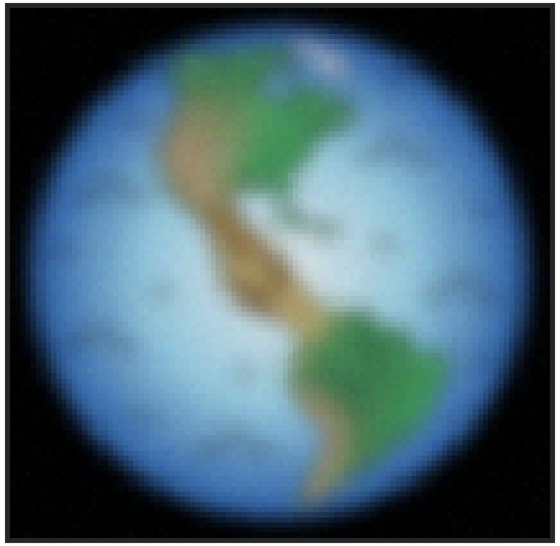


Ingrid Daubechies

She developed very special "coordinate systems" that are called Daubechies Wavelets. They are used to compressed images, videos, and music.

Image Deblurring

True image



Blurred and noisy image

Image deblurring problem: Try to reconstruct the true image from a blurred and noisy one.

System of Equations

If a, c, d, e, b_1 and b_2 are known we have a system of equations

$$ax_1 + cx_2 = b_1$$

$$dx_1 + ex_2 = b_2$$

Example:

$$3x_1 + 6x_2 = 12$$

$$5x_1 + 10x_2 = 24$$

Ill-Conditioned System of Equations

Compare

$$\begin{aligned} x_1 + \quad\quad x_2 &= 1 \\ 2x_1 + 2.0001x_2 &= 2 \end{aligned}$$

$$\text{Exact solution}: x_1 = 1, x_2 = 0,$$

with a slightly different system

$$\begin{aligned} x_1 + \quad\quad x_2 &= .99 \\ 2x_1 + 2.0001x_2 &= 1.89 \end{aligned}$$

$$\text{Exact solution (rounded)}: x_1 = 900, x_2 = -899.$$

Regularization Methods

Original system

$$\begin{aligned} x_1 + x_2 &= .99 \\ 2x_1 + 2.0001x_2 &= 1.89 \end{aligned}$$

Exact solution (rounded): $x_1 = 900, x_2 = -899,$

Desired solution: $x_1 = 1, x_2 = 0,$

Regularized system

$$\begin{aligned} (1 + \boxed{0.05})x_1 + x_2 &= .99 \\ 2x_1 + (2.0001 + \boxed{0.05})x_2 &= 1.89 \end{aligned}$$

Exact solution (rounded): $x_1 = .91, x_2 = .03.$

... the solution of the regularized system is closer to the desired solution!

Systems of Equations in Matrix Form

We can write the system

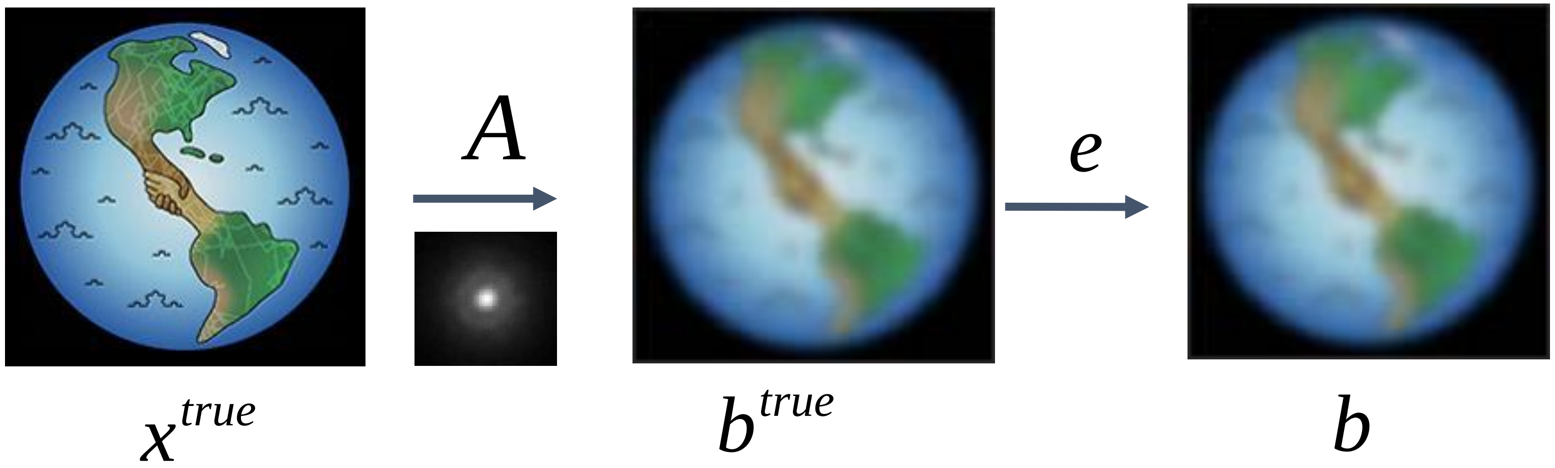
$$\begin{aligned} x_1 + \quad\quad x_2 &= .99 \\ 2x_1 + 2.0001x_2 &= 1.89 \end{aligned}$$

as

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 2.0001 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} .99 \\ 1.89 \end{bmatrix}}_{b}$$

Matrix

Image Deblurring: Math Model



$$Ax = b^{true} + e = b$$

Image deblurring problem: Try to reconstruct the true image from a blurred and noisy one.

The Naïve Solution

A^{-1} ←

x^{true}?

b

...but $x = A^{-1}b =$

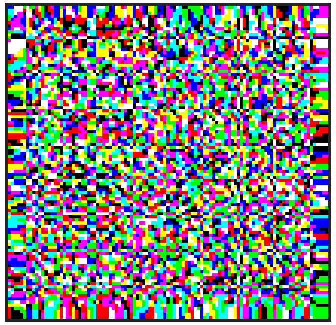
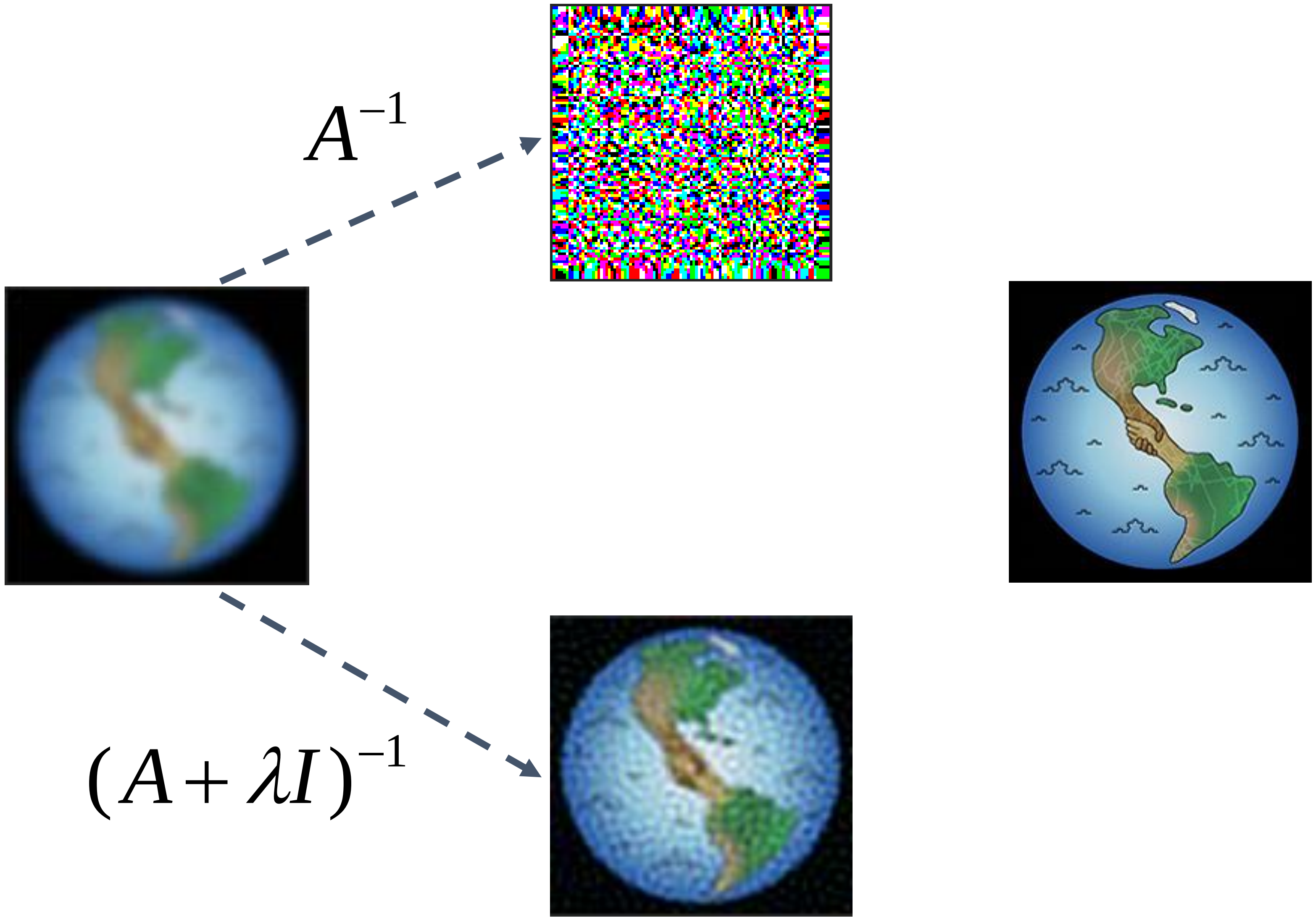


Image deblurring is an <u>ill-posed inverse problem</u>: small perturbations in the data may result in large errors in the solution.

Regularized Solution

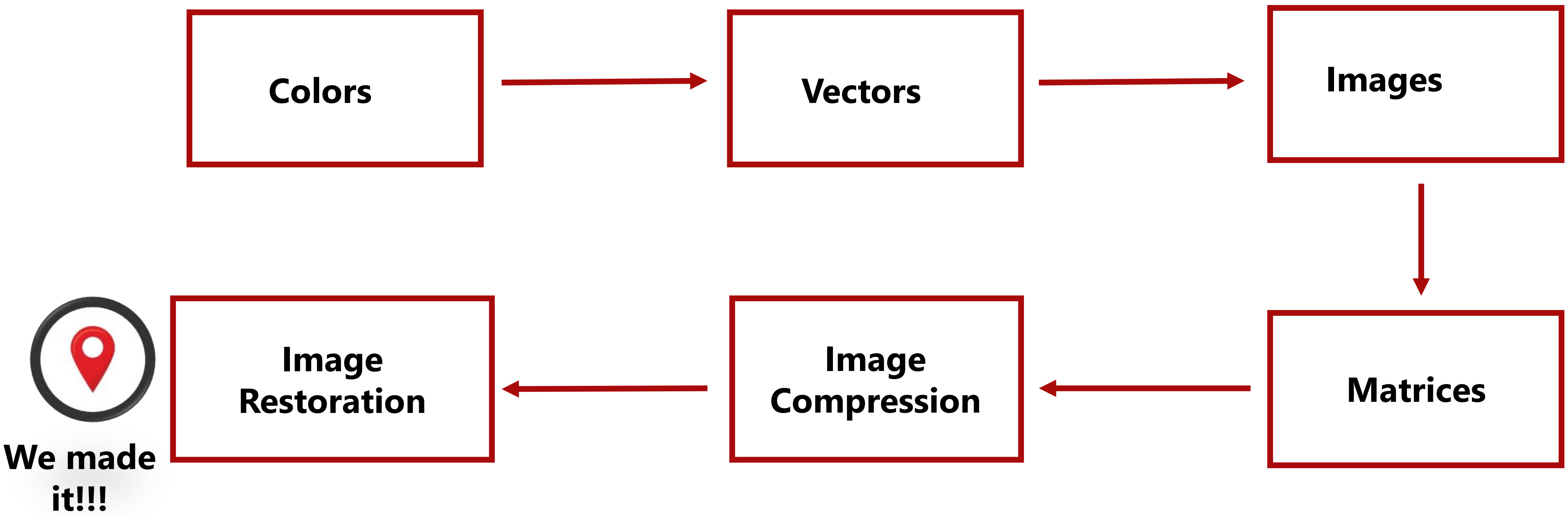


How Important is λ?

Choosing λ is very important!

A whole area of research exists just to develop ways to find optimal values of λ.

The Plan


Colors
Vectors
Images
Matrices
Image Compression
Image Restoration
We made it!!!