Re-Imaging the World through Linear Algebra

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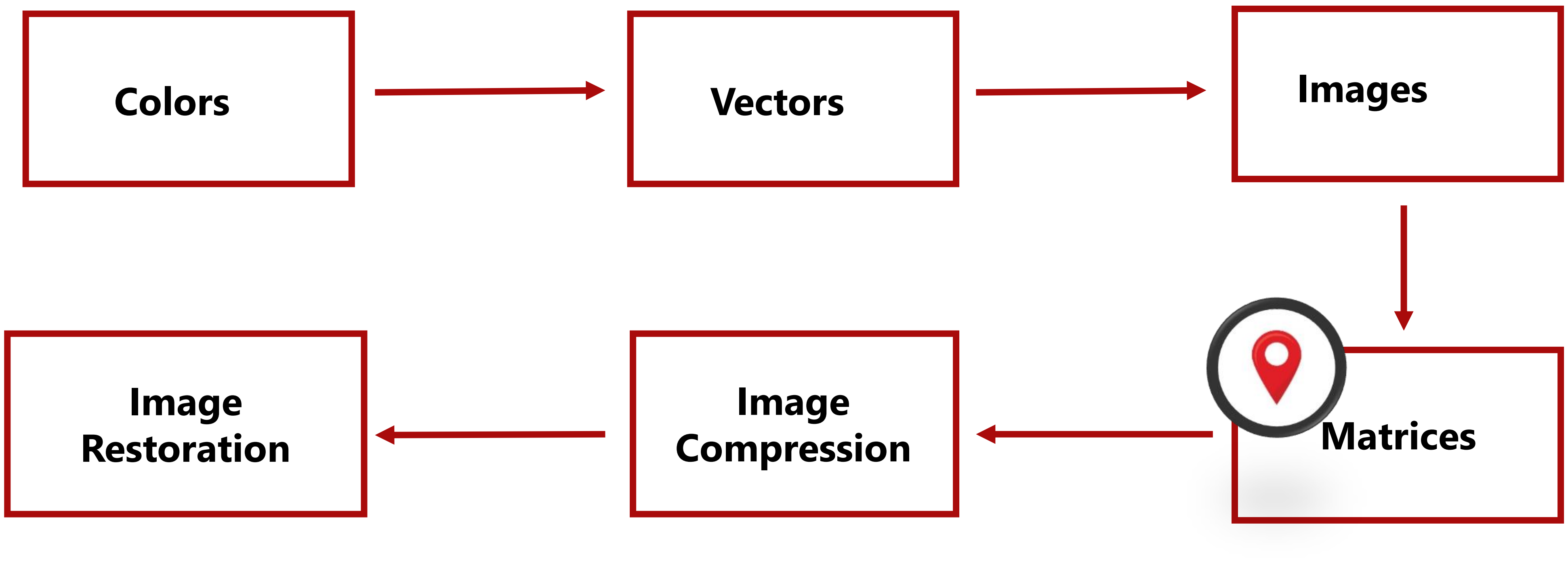
Mathematics Sin Fronteras
March10, 17, 24, 2021

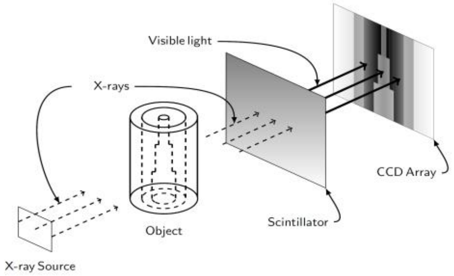
Arizona State University

The plan


Colors
Vectors
Images
Matrices
Image Compression
Image Restoration

What do I do?

Applied Mathematician



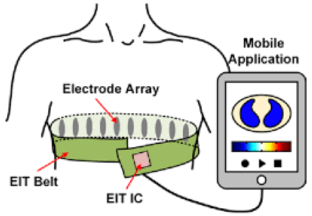
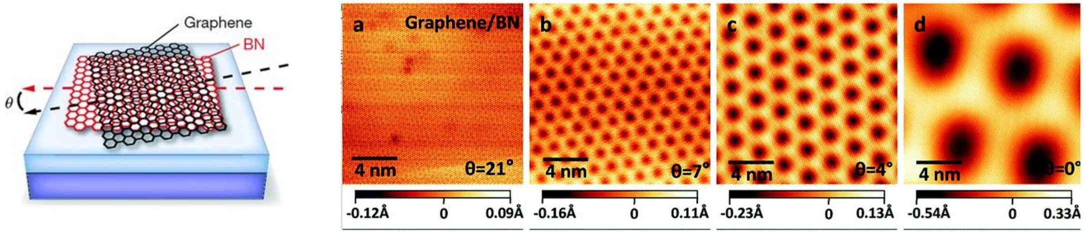
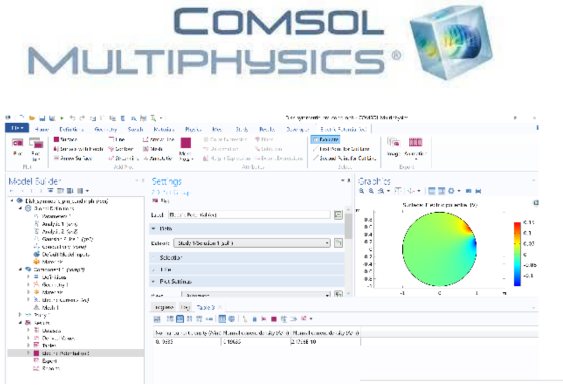
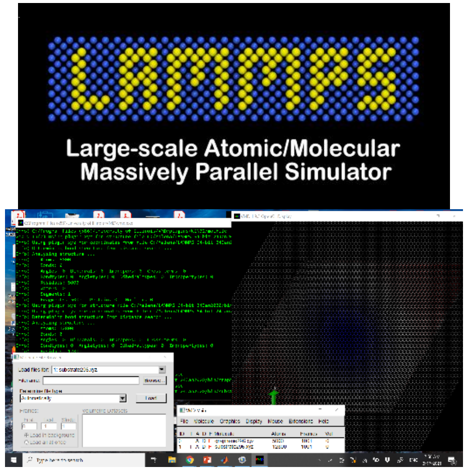


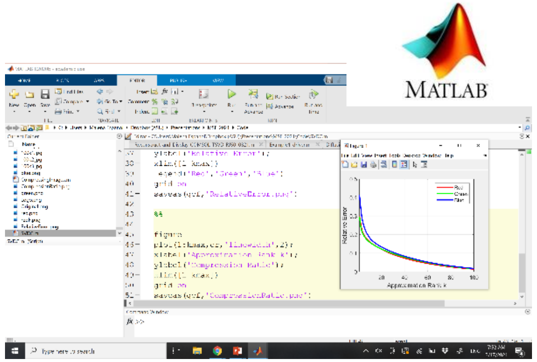
Fig. 1. Proposed wearable lung-health monitoring system.



Computational Mathematician

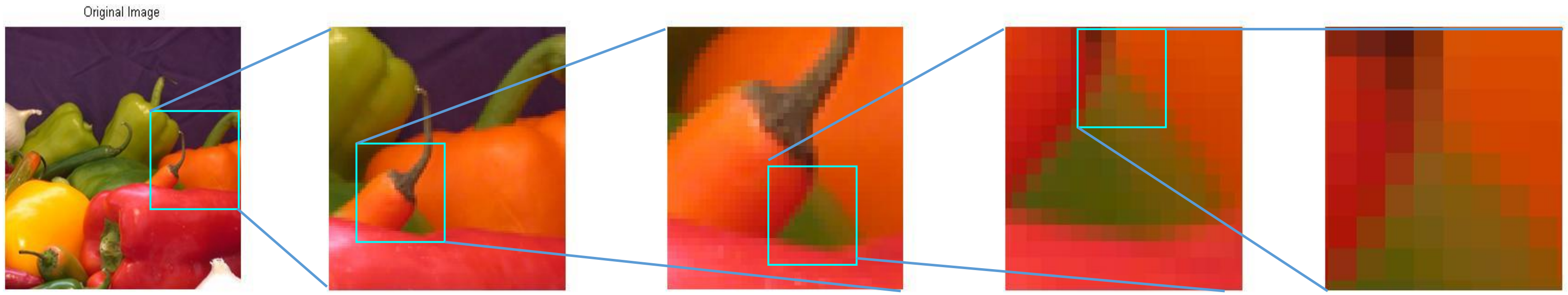


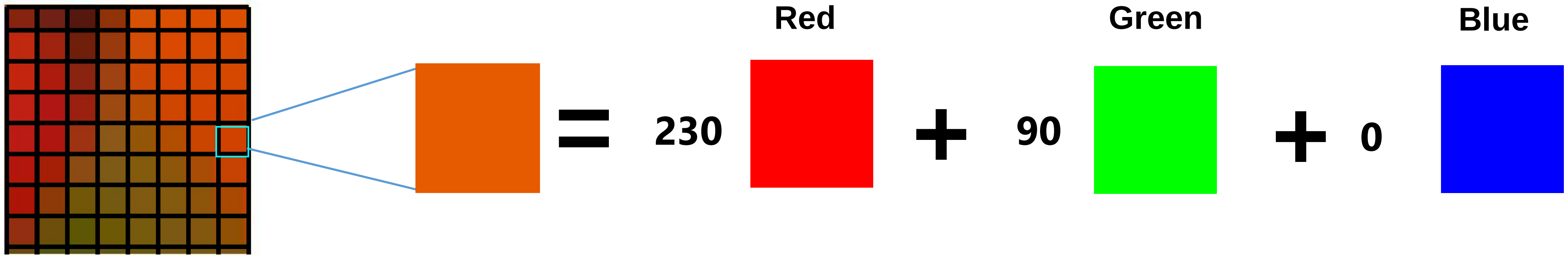




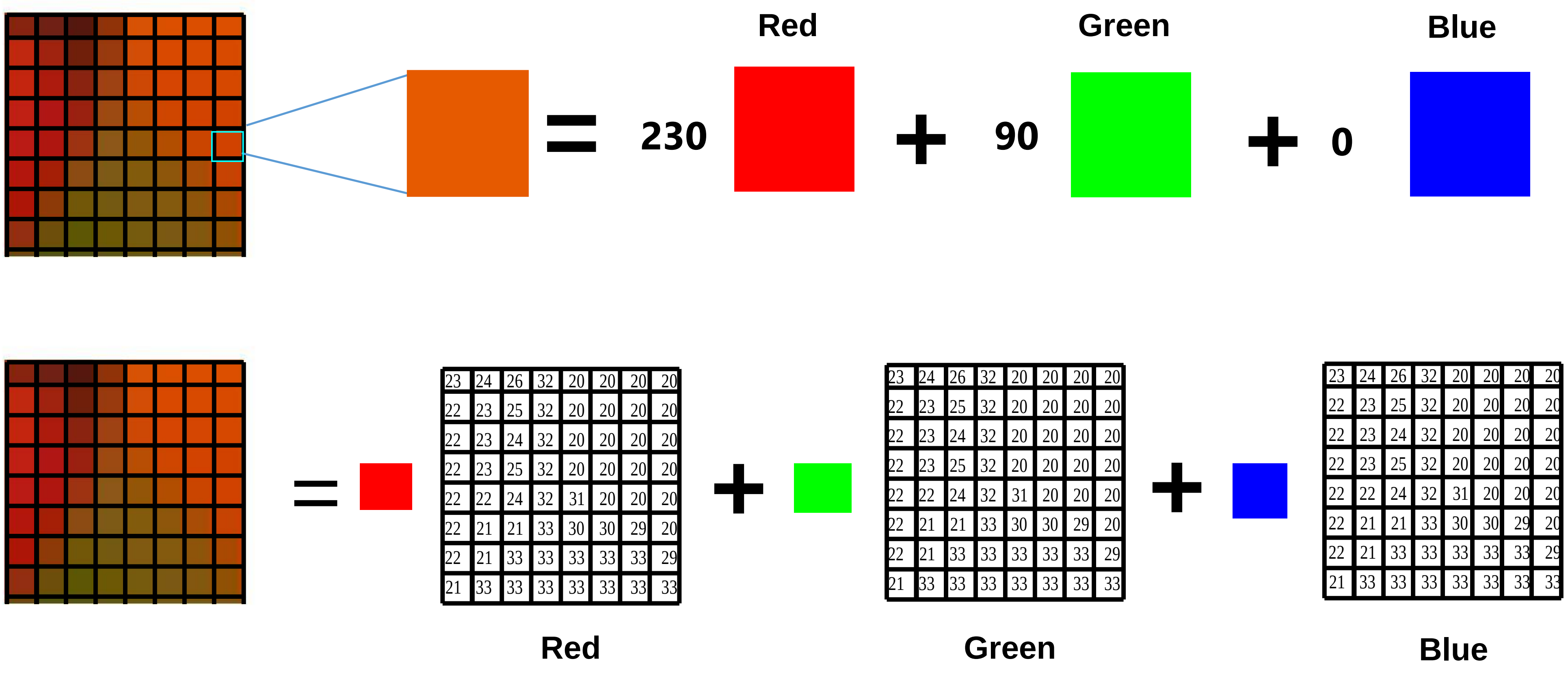
1. Storage
2. Computational Cost
3. Accuracy

Digital Images





Digital Images



Red

23	24	26	32	20	20	20	20
22	23	25	32	20	20	20	20
22	23	24	32	20	20	20	20
22	23	25	32	20	20	20	20
22	22	24	32	31	20	20	20
22	21	21	33	30	30	29	20
22	21	33	33	33	33	33	29
21	33	33	33	33	33	33	33

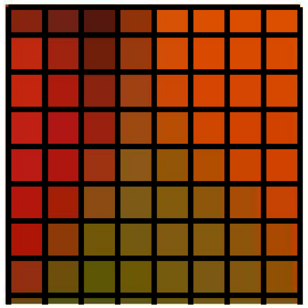
Green

23	24	26	32	20	20	20	20
22	23	25	32	20	20	20	20
22	23	24	32	20	20	20	20
22	23	25	32	20	20	20	20
22	22	24	32	31	20	20	20
22	21	21	33	30	30	29	20
22	21	33	33	33	33	33	29
21	33	33	33	33	33	33	33

Blue

23	24	26	32	20	20	20	20
22	23	25	32	20	20	20	20
22	23	24	32	20	20	20	20
22	23	25	32	20	20	20	20
22	22	24	32	31	20	20	20
22	21	21	33	30	30	29	20
22	21	33	33	33	33	33	29
21	33	33	33	33	33	33	33

Storing a picture of size mxn



=

23	24	26	32	20	20	20	20
22	23	25	32	20	20	20	20
22	23	24	32	20	20	20	20
22	23	25	32	20	20	20	20
22	22	24	32	31	20	20	20
22	21	21	33	30	30	29	20
22	21	33	33	33	33	33	29
21	33	33	33	33	33	33	33

Red

+

23	24	26	32	20	20	20	20
22	23	25	32	20	20	20	20
22	23	24	32	20	20	20	20
22	23	25	32	20	20	20	20
22	22	24	32	31	20	20	20
22	21	21	33	30	30	29	20
22	21	33	33	33	33	33	29
21	33	33	33	33	33	33	33

Green

+

23	24	26	32	20	20	20	20
22	23	25	32	20	20	20	20
22	23	24	32	20	20	20	20
22	23	25	32	20	20	20	20
22	22	24	32	31	20	20	20
22	21	21	33	30	30	29	20
22	21	33	33	33	33	33	29
21	33	33	33	33	33	33	33

Blue

We need to store $m \times n$ numbers between 0 and 255 for each color channel red, green, and blue. So, to store an image we need 3mn bytes.

What can we do to reduce storage?

Matrices

Definition: A matrix is an array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Examples: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{2\times 3}$ $\qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}$

The transpose of a Matrix

Definition: The transpose of a matrix $A \in \mathbb{R}^{m\times n}$, noted $A^T \in \mathbb{R}^{n\times m}$ is given by $A^T_{ij} = A_{ji}$, that is,

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{2\times 3}$. Then $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in \mathbb{R}^{3\times 2}$.

Symmetric Matrices

Definition: A matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if $A^T = A$, that is,

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

Examples: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & -1 \\ 3 & -1 & 6 \end{pmatrix}$

What is the cost of storing symmetric matrices?

Toeplitz Matrices

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix}$$

What is the cost of storing Toeplitz matrices?

Circulant Matrices

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

What is the cost of storing circulant matrices?

Sum of Matrices

Definition: The sum of two matrices of equal size is defined by

$$A + B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

What is the cost of computing a sum?

Sum of Structured Matrices

1) Is the sum of two symmetric matrices symmetric?
2) Is the sum of two Toeplitz matrices Toeplitz?
3) Is the sum of two circulant matrices circulant?

Product by a scalar

Definition: The product of a scalar k by a matrix A is defined by

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

Example: $2\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix} + 3\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 7 & 10 \\ 5 & 7 & 12 \\ 5 & 9 & 14 \end{pmatrix}$

What is the cost of computing this matrix?

Matrix-Vector Product – Way 1

Definition: Let $A \in \mathbb{R}^{m\times n}$ and $u \in \mathbb{R}^n$. The *matrix-vector product* of A and u is defined by

$$Au = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \text{---} & a_1 & \text{---} \\ \text{---} & a_2 & \text{---} \\ & \vdots & \\ \text{---} & a_m & \text{---} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} a_1 \cdot u \\ a_2 \cdot u \\ \vdots \\ a_n \cdot u \end{pmatrix} \in \mathbb{R}^m$$

Example:

$$Au = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \times 5 + 2 \times 10 \\ 3 \times 5 + 4 \times 10 \end{pmatrix} = \begin{pmatrix} 25 \\ 55 \end{pmatrix}$$

Matrix-Vector Product – Way 2

Definition: Let $A \in \mathbb{R}^{m\times n}$ and $u \in \mathbb{R}^n$. The *matrix-vector product* of A and u is defined by

$$Au = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

$$= u_1 \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} + u_2 \begin{pmatrix} | \\ a_2 \\ | \end{pmatrix} + \cdots + u_n \begin{pmatrix} | \\ a_n \\ | \end{pmatrix}$$

Example: $Au = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 10 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 25 \\ 55 \end{pmatrix}$

Matrix-Matrix Product – Way 1

Definition: Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$. The *matrix-matrix product* of A and B is defined by

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_p \\ | & | & & | \end{pmatrix}$$

Example: $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 10 & 6 \end{pmatrix} = \begin{pmatrix} 25 & 12 \\ 55 & 24 \end{pmatrix}$

Matrix-Matrix Product – Way 2

Definition: Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$. The *matrix-matrix product* of A and B is defined by

$$AB = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} \begin{pmatrix} — & b_1 & — \\ & \vdots & \\ — & b_n & — \end{pmatrix}$$

$$= \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} \begin{pmatrix} — & b_1 & — \end{pmatrix} + \cdots + \begin{pmatrix} | \\ a_n \\ | \end{pmatrix} \begin{pmatrix} — & b_n & — \end{pmatrix}$$

Example: $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 10 & 6 \end{pmatrix} = \begin{pmatrix} 25 & 12 \\ 55 & 24 \end{pmatrix}$

Singular Value Decomposition (SVD)

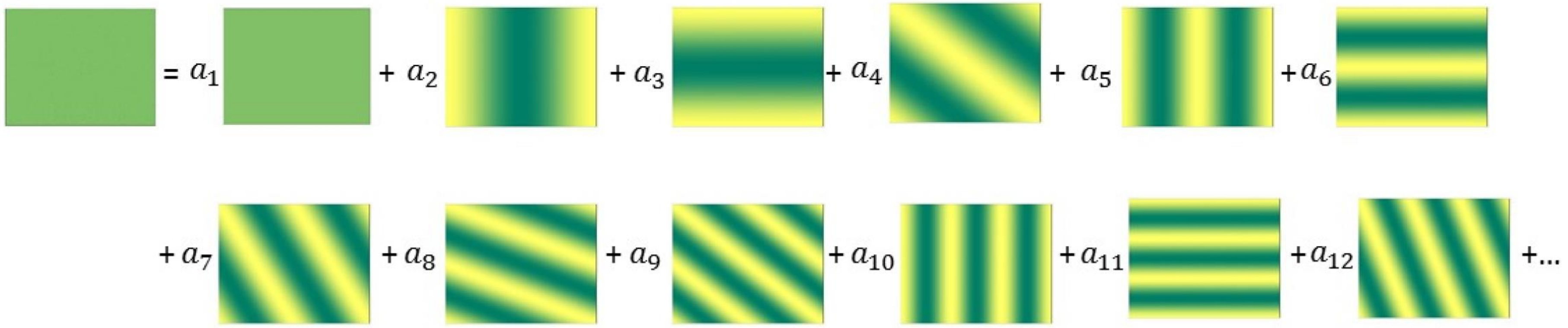
Theorem: Suppose $A \in \mathbb{R}^{n\times n}$. Then, there exist matrices $U \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{n\times n}$ and $\Sigma \in \mathbb{R}^{n\times n}$, with $U^TU = V^TV = I_n$, and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_\mathrm{n})$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, such that $A = U\Sigma V^T$. The columns of U are called the left singular vectors, the columns of V the right singular vectors, and σ_i are called the singular values.

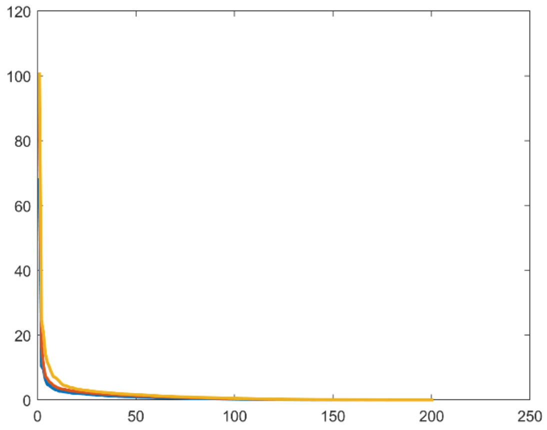
$$A = U\Sigma V^T = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{pmatrix} \begin{pmatrix} — & v_1^T & — \\ & \vdots & \\ — & v_n^T & — \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} — & \sigma_1 v_1^T & — \\ & \vdots & \\ — & \sigma_n v_n^T & — \end{pmatrix} = \Sigma_{i=1}^n \sigma_i u_i v_i^T$$

SVD of an Image

$$A = U\Sigma V^T = \Sigma_{i=1}^{n} \sigma_i u_i v_i^T$$





Truncated SVD

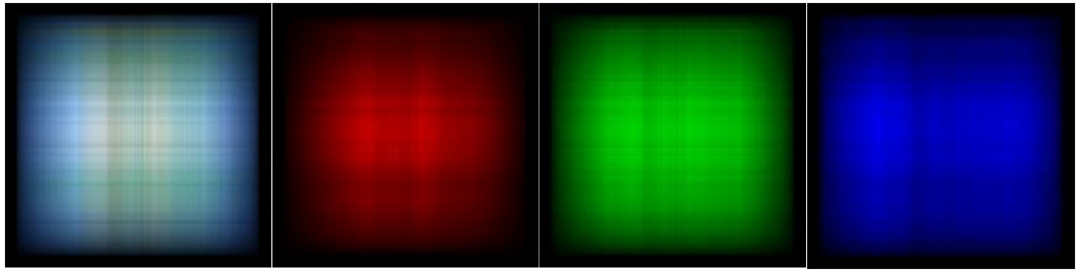
$$A_k = U_k \Sigma_k V_k^T = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_k \end{pmatrix} \begin{pmatrix} — & v_1^T & — \\ & \vdots & \\ — & v_k^T & — \end{pmatrix} = \Sigma_{i=1}^k \sigma_i u_i v_i^T$$

What is the cost of storing these matrices?

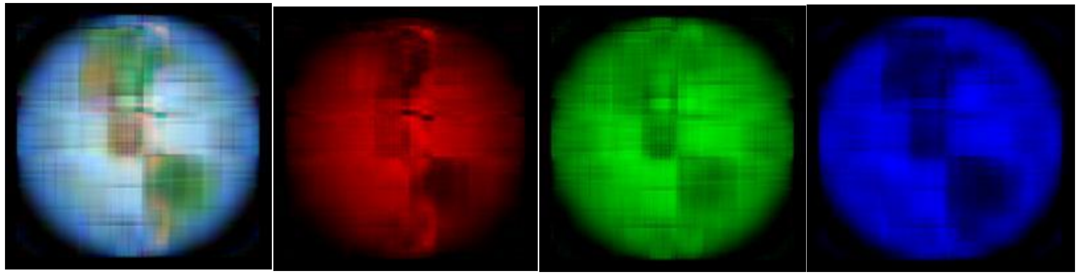
What is the cost of computing the matrix A_k?

Image Compression using SVD

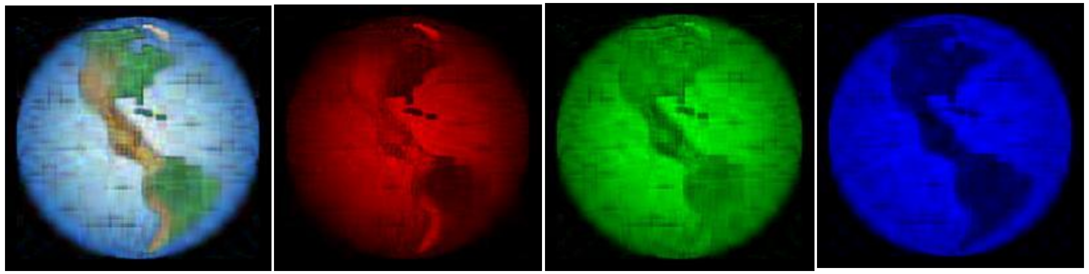
k=1



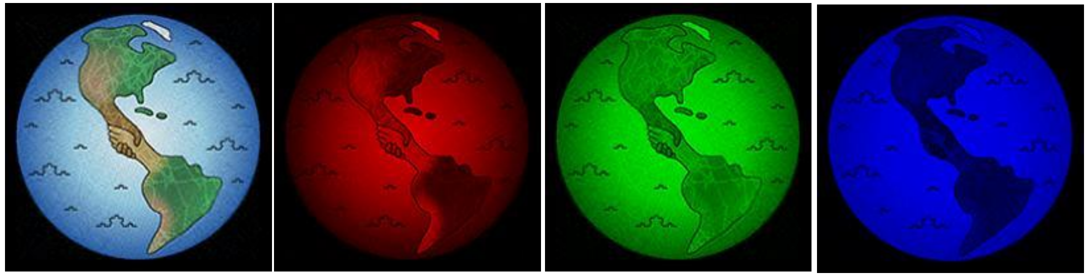
k=5



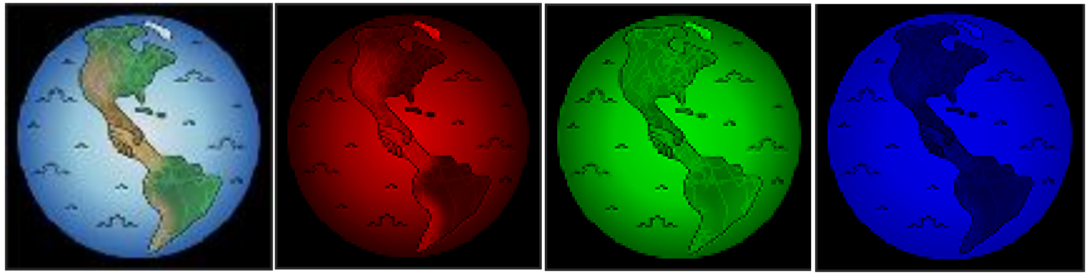
k=10



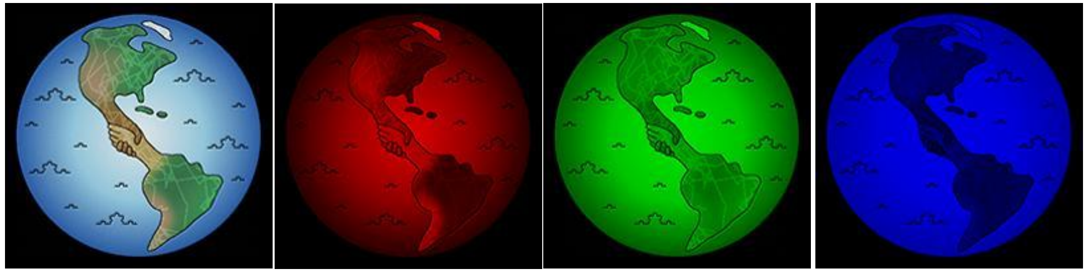
k=50



k=100



k=256



Matrix Norms and Distances

Definition: The *Frobenius norm* of a matrix A is defined by

$$\|A\|_F = \sqrt{a_{11}^2 + a_{12}^2 + \cdots + a_{mn}^2}$$

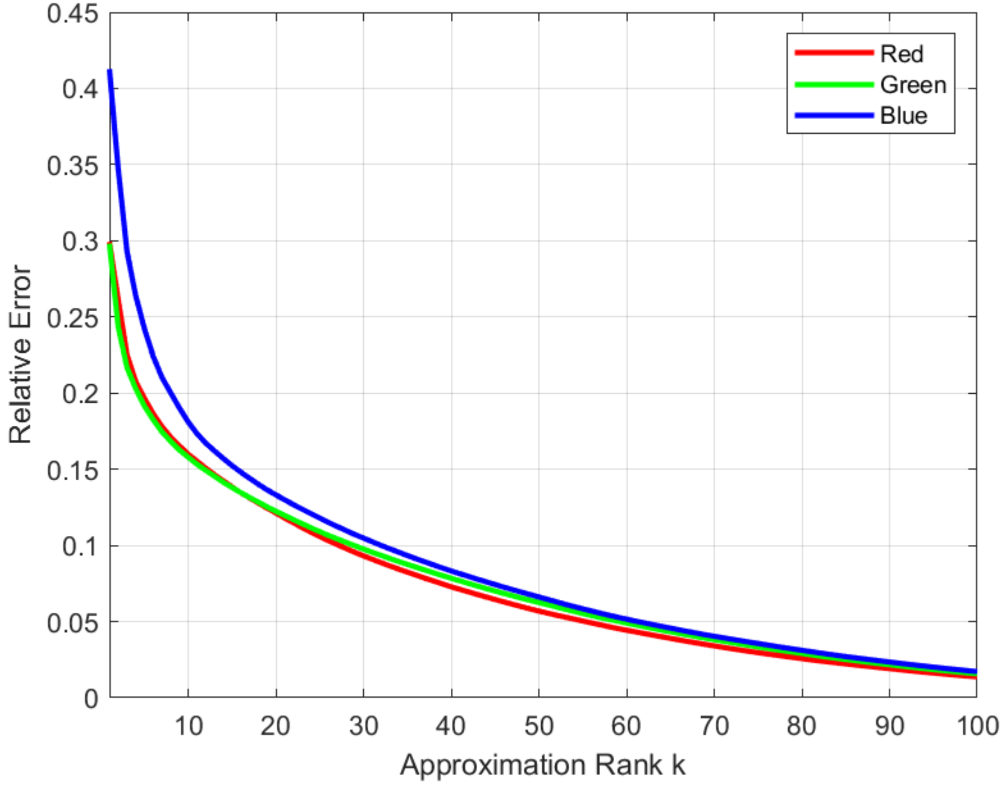
Examples: $\left\|\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right\|_F = \sqrt{30}$ y $\left\|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\|_F = \sqrt{2}$

Definition: The *Frobenius distance* between two matrices A and B is defined by

$$d(A, B) = \|A - B\|_F = \sqrt{(a_{11} - b_{11})^2 + (a_{12} - b_{12})^2 + \cdots + (a_{mn} - b_{mn})^2}$$

Example: $d\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \left\|\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\|_F = \left\|\begin{pmatrix} 0 & 2 \\ 3 & 3 \end{pmatrix}\right\|_F = \sqrt{22}$

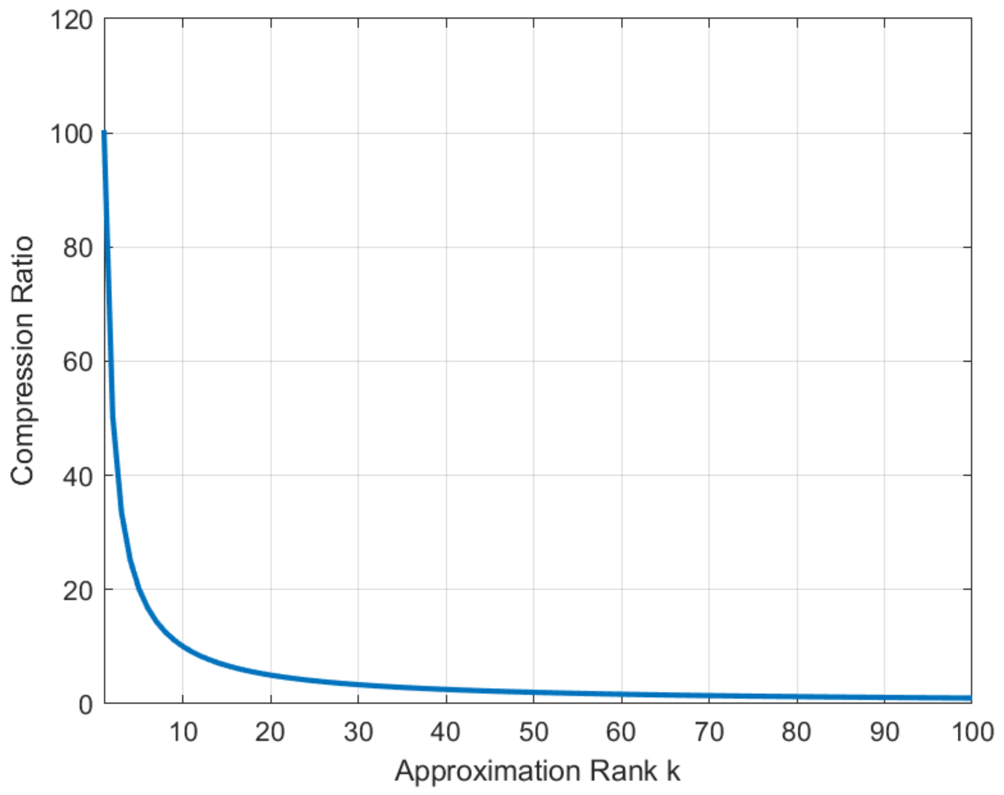
Image Compression: Relative Error



Relative Error:

$$\frac{\|A - A_k\|_F}{\|A\|_F}$$

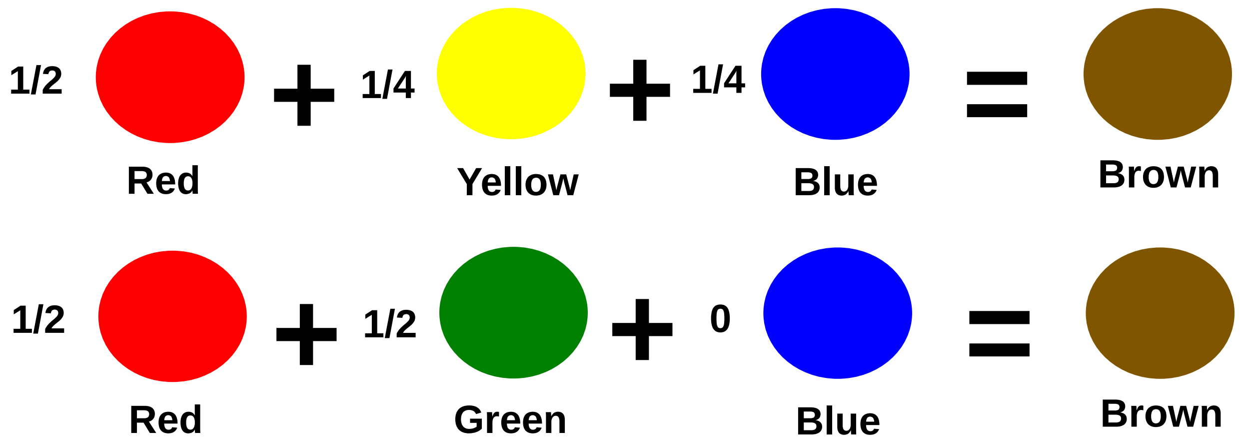
Image Compression: Compression Ratio

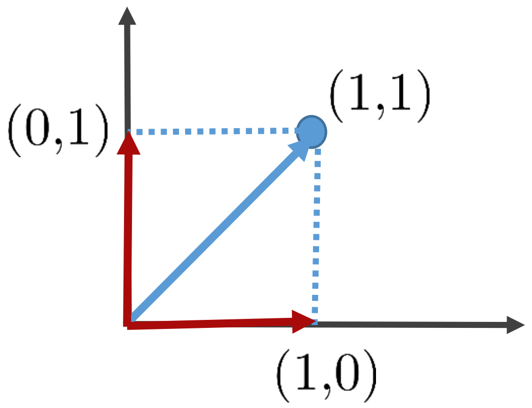


Compression Ration:

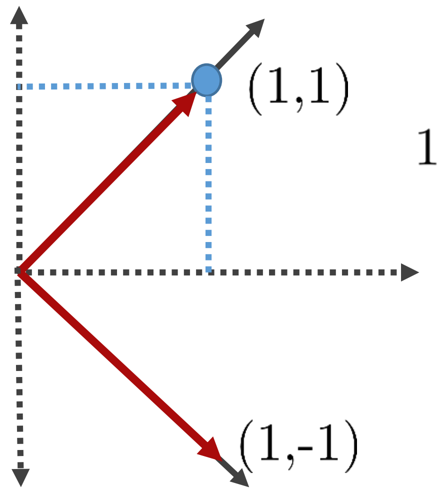
$$\frac{mn}{k(m+n+1)}$$

Coordinate Systems





$$1\ (1,0) + 1\ (0,1) = (1,1)$$



$$1\ (1,1) + 0\ (1,-1) = (1,1)$$

Ingrid Daubechies

She has developed mathematical tools called Daubechies Wavelets, which have arrived to our homes! They are used to compressed images, videos, and music.