

Re-Imaging the World through Linear Algebra



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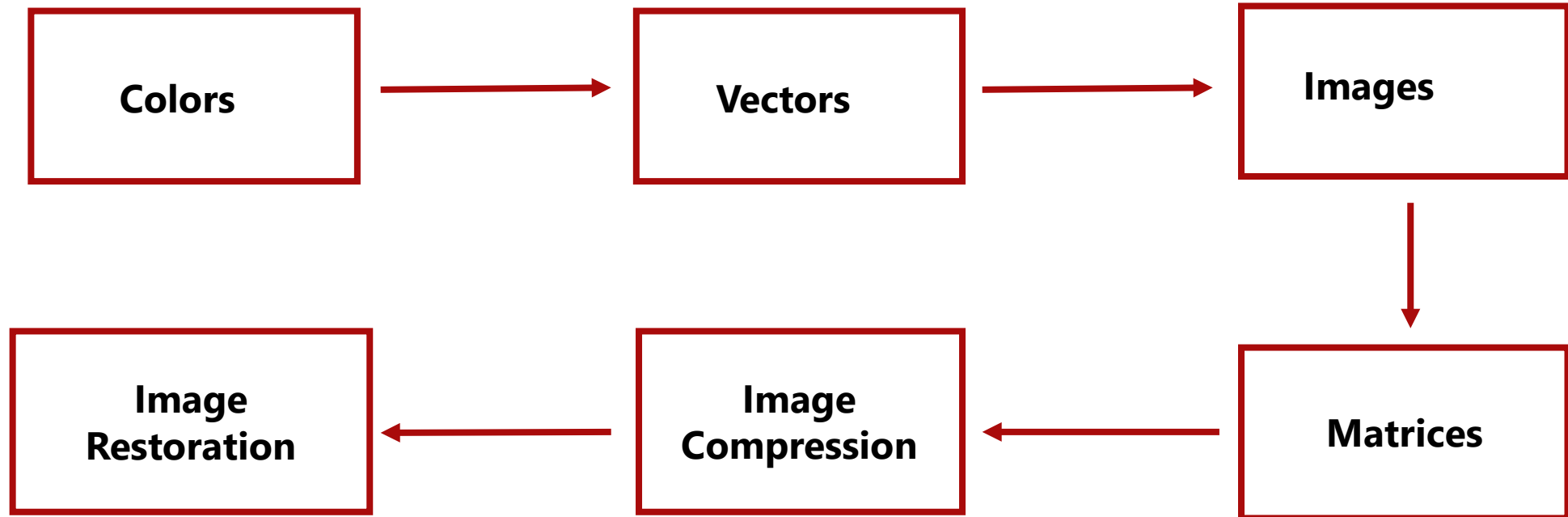
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Mathematics Sin Fronteras
March 10, 17, 24, 2021

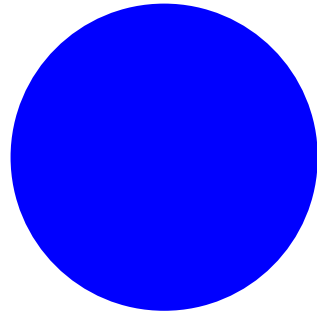


The plan

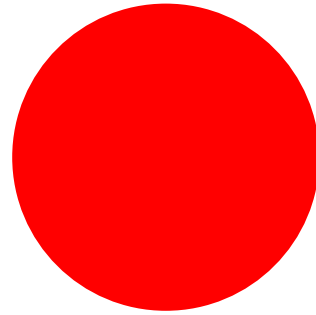


The Mathematics of Colors

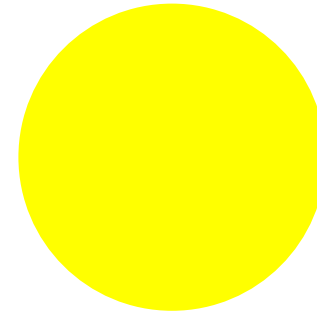
Primary Colors



Blue



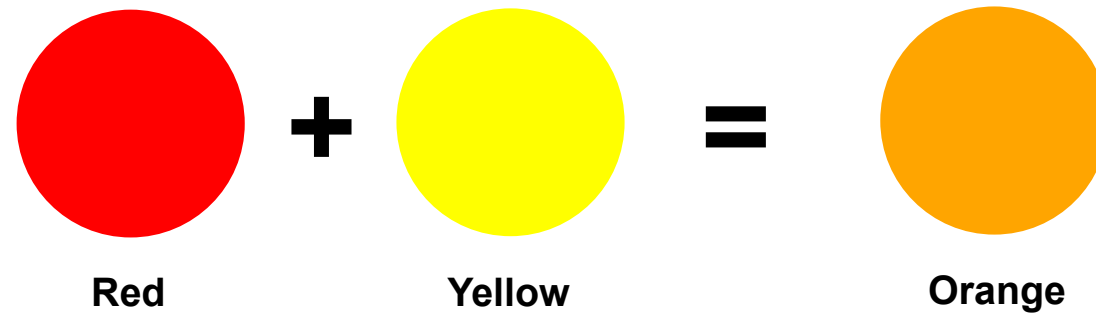
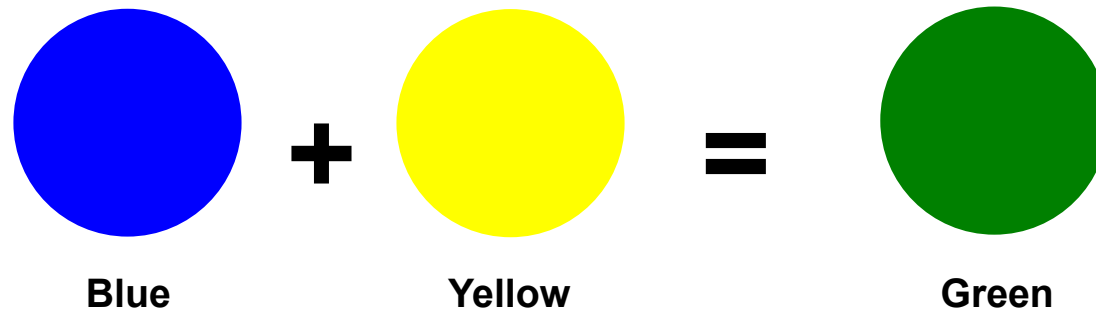
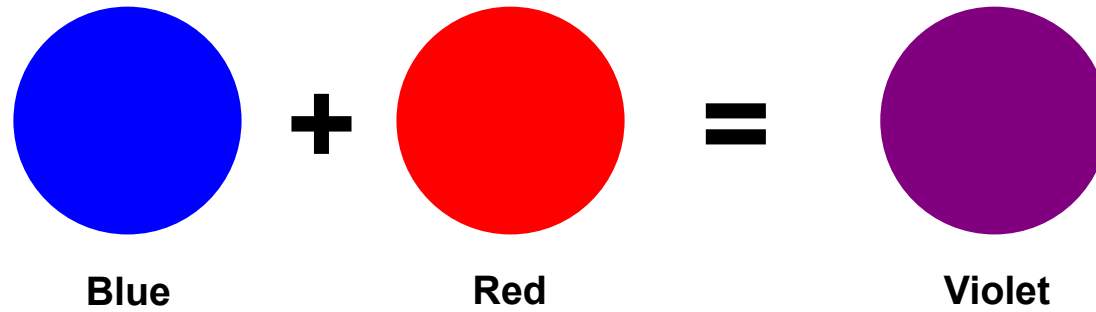
Red



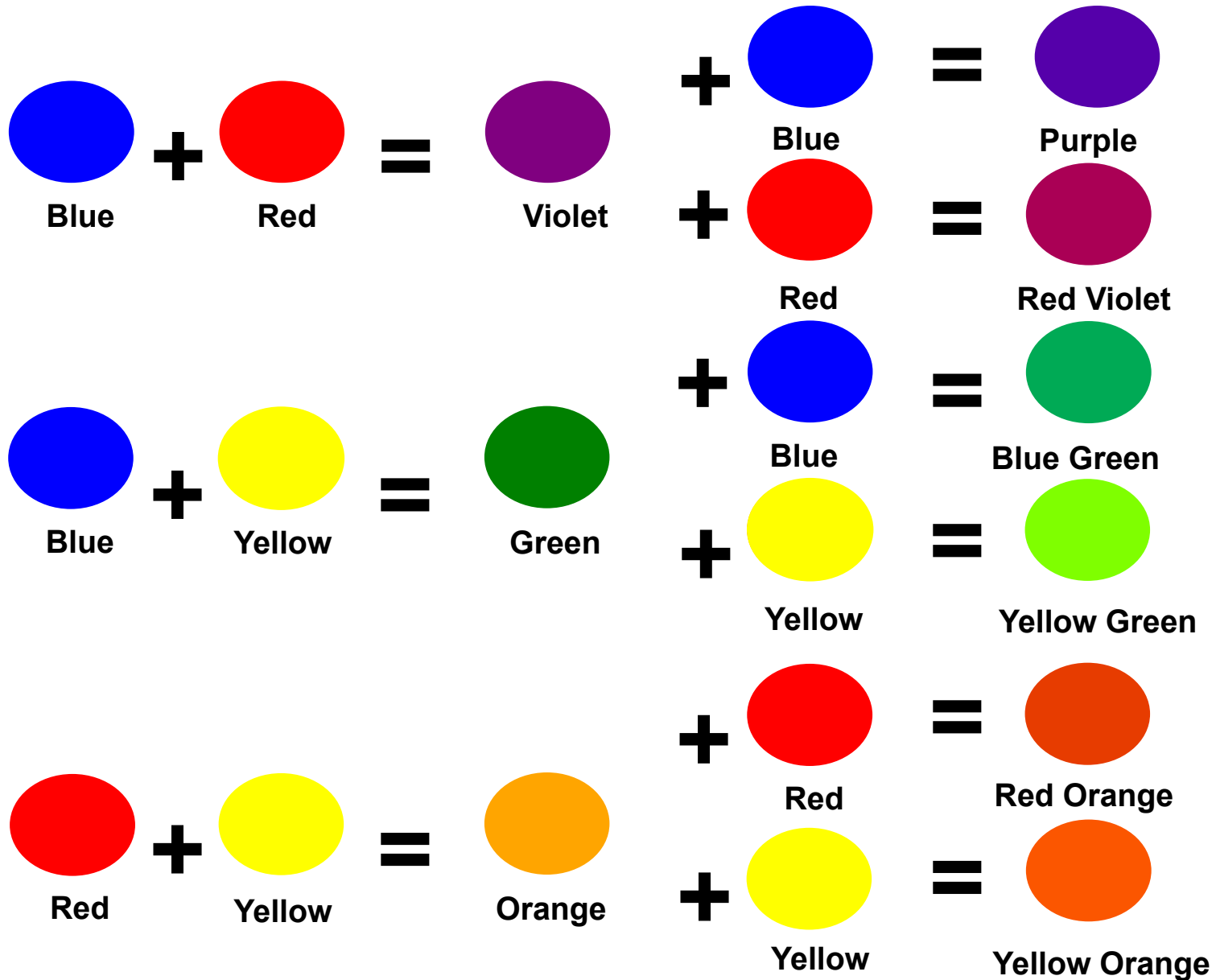
Yellow

Why do we call them that?

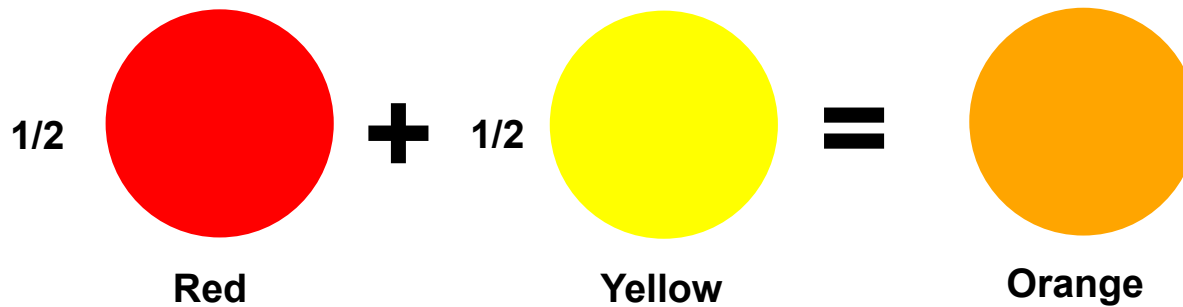
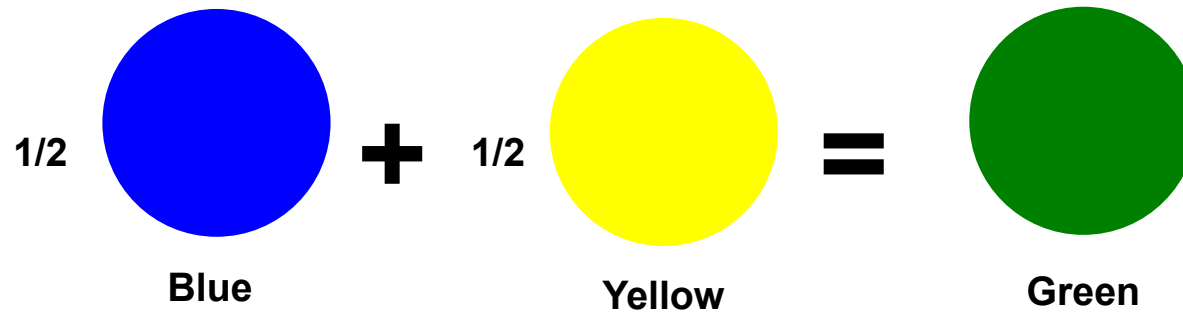
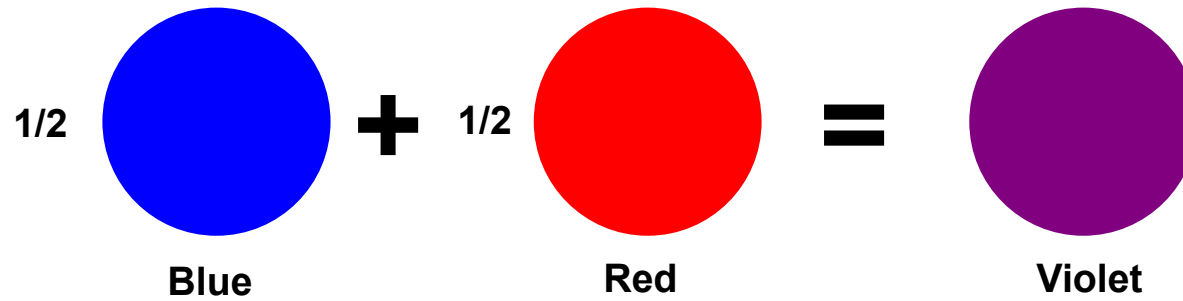
Secondary Colors



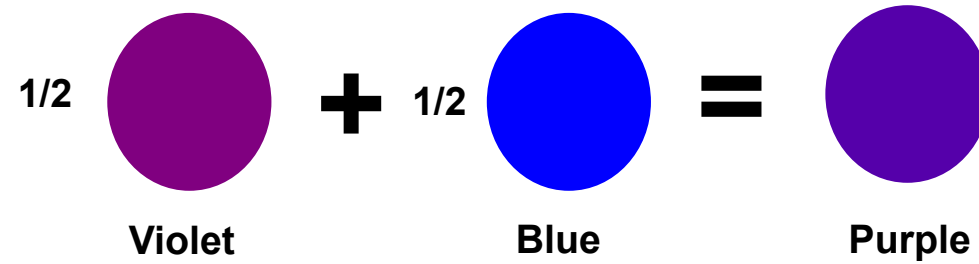
Tertiary Colors



“Recipe” for the Secondary Colors



Recipe of the Purple Color



Recipe of the Purple Color

$$\frac{1}{2} \text{ Blue} + \frac{1}{2} \text{ Red} = \text{Violet}$$

$$\frac{1}{2} \text{ Violet} + \frac{1}{2} \text{ Blue} = \text{Purple}$$

$$\frac{1}{2} \left(\frac{1}{2} \text{ Blue} + \frac{1}{2} \text{ Red} \right) + \frac{1}{2} \text{ Blue} = \text{Purple}$$

Recipe of the Purple Color

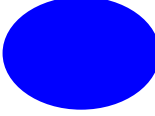

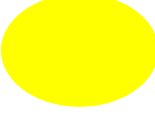
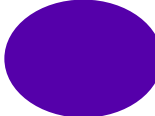
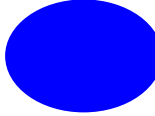
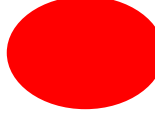
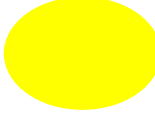

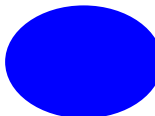
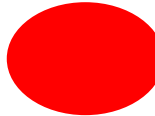
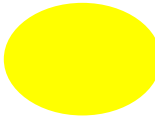

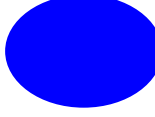
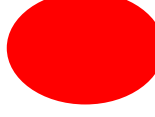
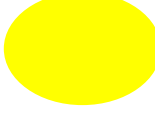
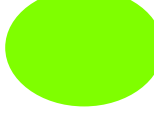
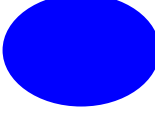

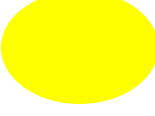

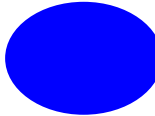
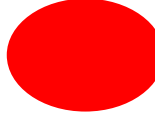
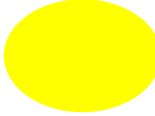

$$\frac{1}{2} \left(\frac{1}{2} \text{Blue} + \frac{1}{2} \text{Red} \right) + \frac{1}{2} \text{Blue} = \text{Purple}$$

$$\frac{1}{4} \text{Blue} + \frac{1}{4} \text{Red} + \frac{1}{2} \text{Blue} = \text{Purple}$$

$$\frac{3}{4} \text{Blue} + \frac{1}{4} \text{Red} = \text{Purple}$$

$$\frac{3}{4} \text{Blue} + \frac{1}{4} \text{Red} + 0 \text{Yellow} = \text{Purple}$$

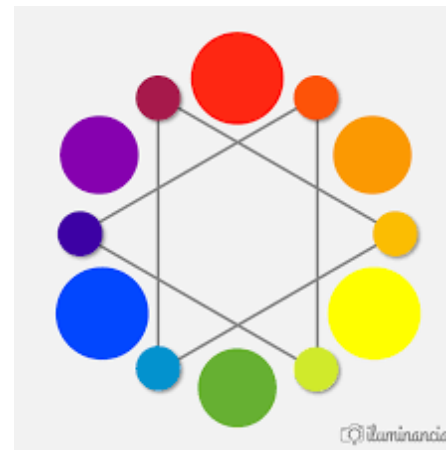
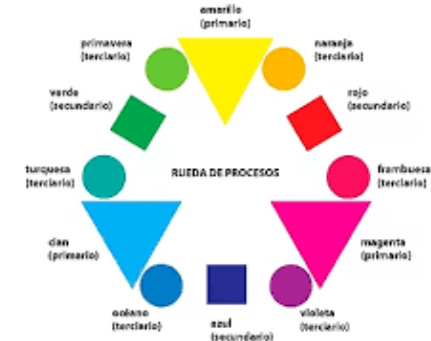
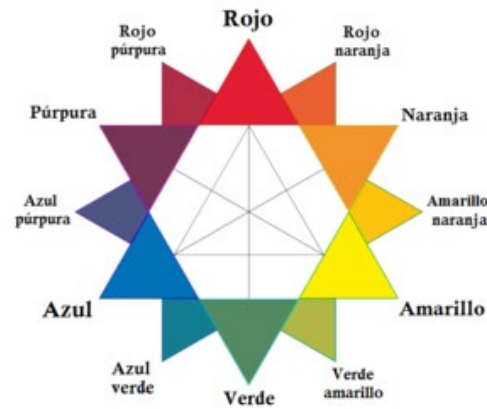
Recipes of the Tertiary Colors

3/4		+	1/4		+	0		=	
	Blue			Red			Yellow		Purple
1/4		+	3/4		+	0		=	
	Blue			Red			Yellow		Red Violet
3/4		+	0		+	1/4		=	
	Blue			Red			Yellow		Blue Green
1/4		+	0		+	3/4		=	
	Blue			Red			Yellow		Yellow Green
0		+	3/4		+	1/4		=	
	Blue			Red			Yellow		Red Orange
0		+	1/4		+	3/4		=	
	Blue			Red			Yellow		Yellow Orange

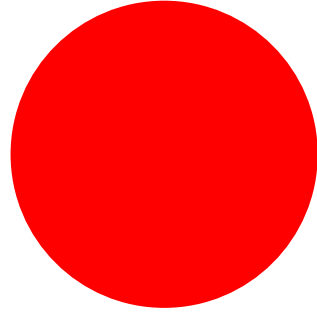
Color Wheel



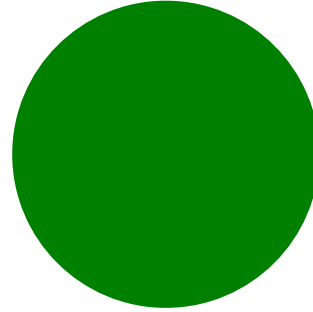
Rueda de R.J.B. Mérimée, 1830*



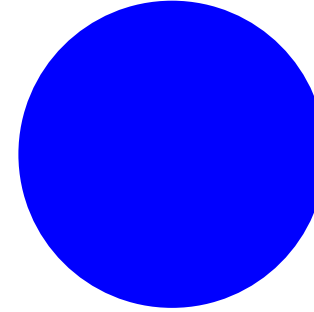
Primary Colors of the Light



Red



Green



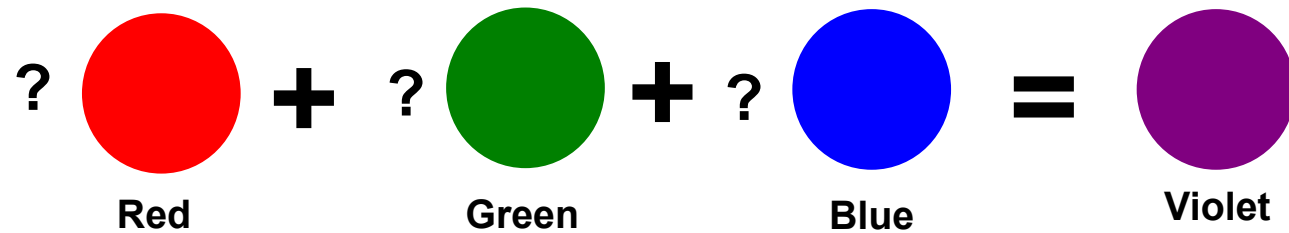
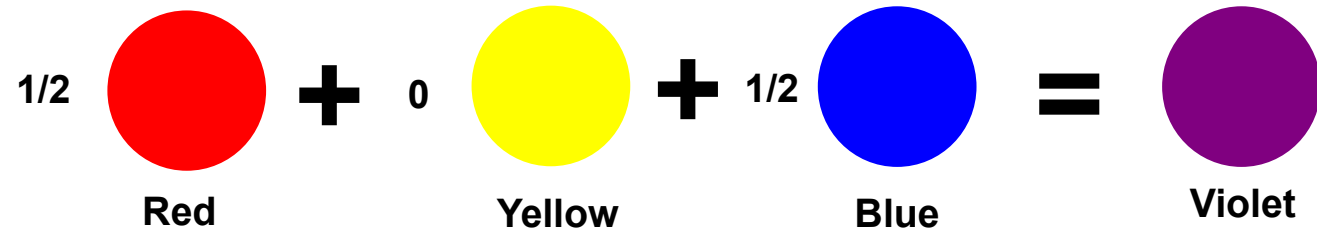
Blue

RGB

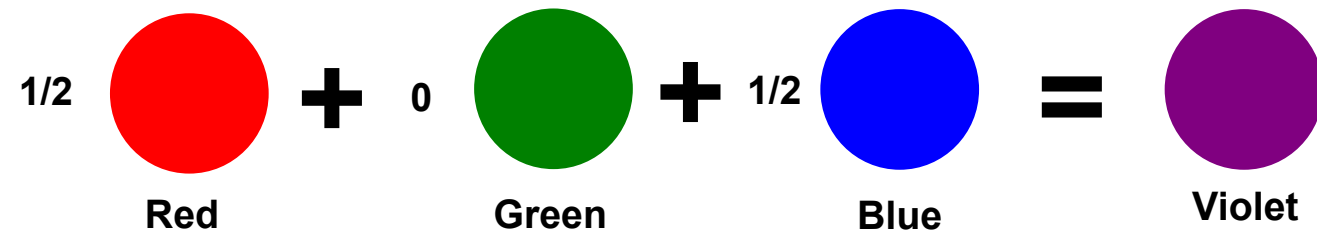
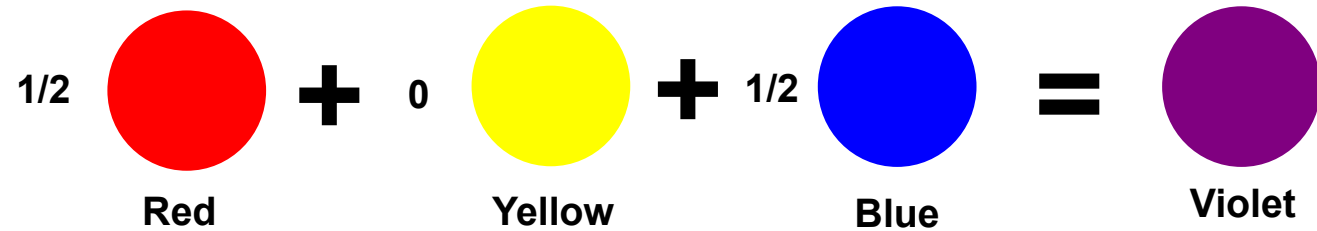
They are used in the computer screens, TVs, and movie screens

[RGB color model - Wikipedia](#)

Creating colors with RGB



Creating colors with RGB

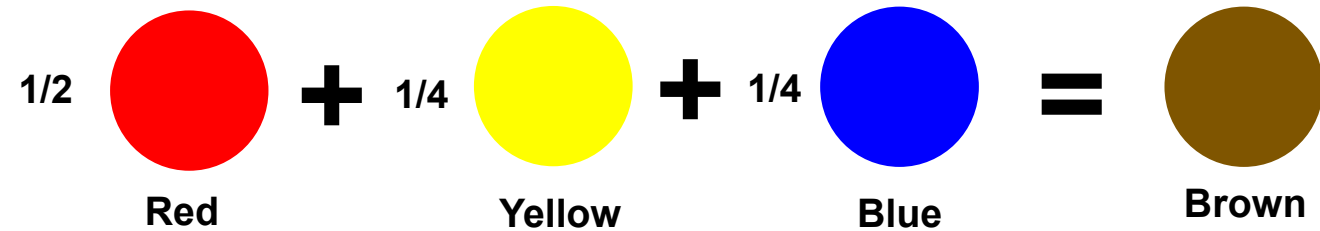


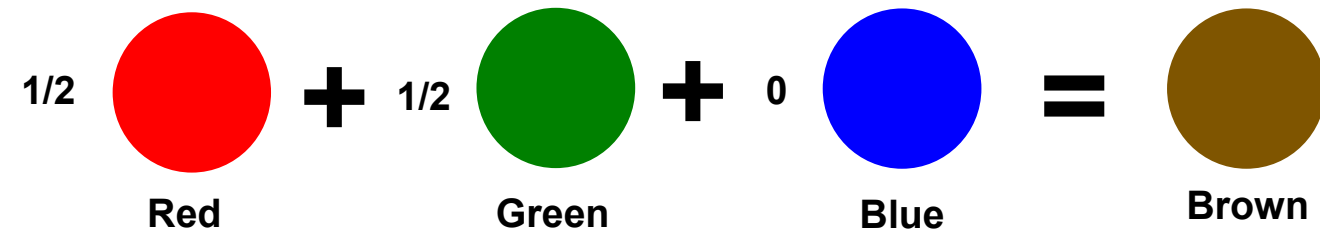
Creating colors with RGB

$$\frac{1}{2} \text{ Red} + \frac{1}{4} \text{ Yellow} + \frac{1}{4} \text{ Blue} = \text{Brown}$$

$$? \text{ Red} + ? \text{ Green} + ? \text{ Blue} = \text{Brown}$$

Creating colors with RGB

$$\frac{1}{2} \text{ Red} + \frac{1}{4} \text{ Yellow} + \frac{1}{4} \text{ Blue} = \text{Brown}$$
A diagram illustrating the creation of the color brown using RGB components. It shows three colored circles: a red circle, a yellow circle, and a blue circle. The red circle is labeled 'Red' and has a coefficient of 1/2. The yellow circle is labeled 'Yellow' and has a coefficient of 1/4. The blue circle is labeled 'Blue' and has a coefficient of 1/4. These are combined with plus signs and an equals sign to result in a brown circle labeled 'Brown'.

$$\frac{1}{2} \text{ Red} + \frac{1}{2} \text{ Green} + 0 \text{ Blue} = \text{Brown}$$
A diagram illustrating the creation of the color brown using RGB components. It shows three colored circles: a red circle, a green circle, and a blue circle. The red circle is labeled 'Red' and has a coefficient of 1/2. The green circle is labeled 'Green' and has a coefficient of 1/2. The blue circle is labeled 'Blue' and has a coefficient of 0. These are combined with plus signs and an equals sign to result in a brown circle labeled 'Brown'.

Creating colors in the Computer

$$a \text{ (Red)} + b \text{ (Green)} + c \text{ (Blue)} = ?$$

**a, b, c are numbers
that can only use
1 byte = 8 bits**



Creating colors in the Computer

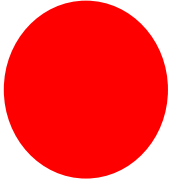
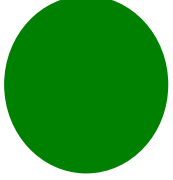
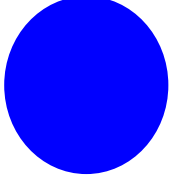
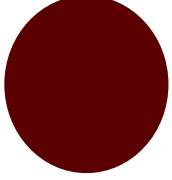
$$a \text{ (Red)} + b \text{ (Green)} + c \text{ (Blue)} = ?$$

a, b, c are integer numbers from 0 to 255

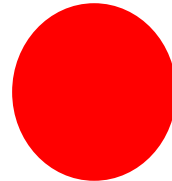
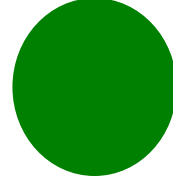
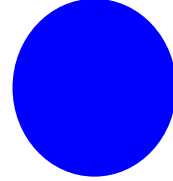
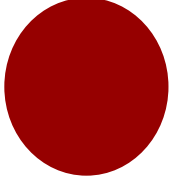
$$0 \text{ (Red)} + 0 \text{ (Green)} + 0 \text{ (Blue)} = \text{Black}$$

$$255 \text{ (Red)} + 255 \text{ (Green)} + 255 \text{ (Blue)} = \text{White}$$

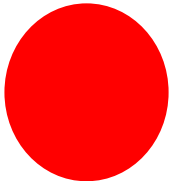
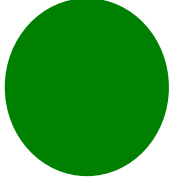
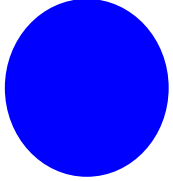
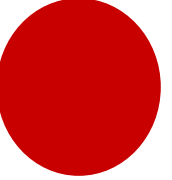
Creating colors in the Computer

90  + 0  + 0  = 

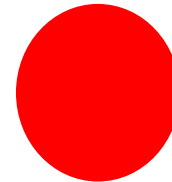
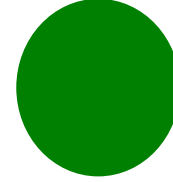
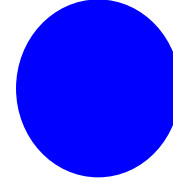
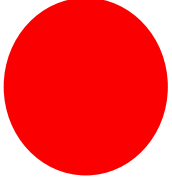
Red Green Blue

150  + 0  + 0  = 

Red Green Blue

200  + 0  + 0  = 

Red Green Blue

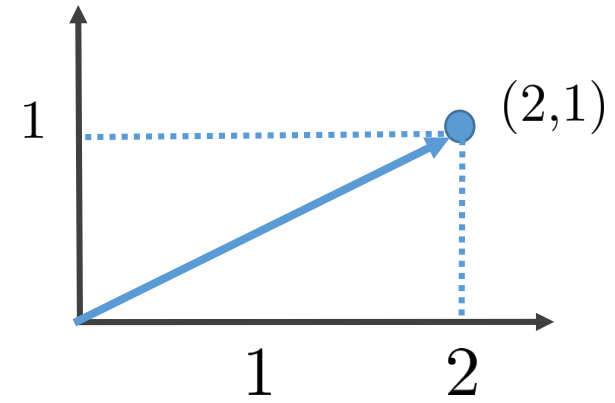
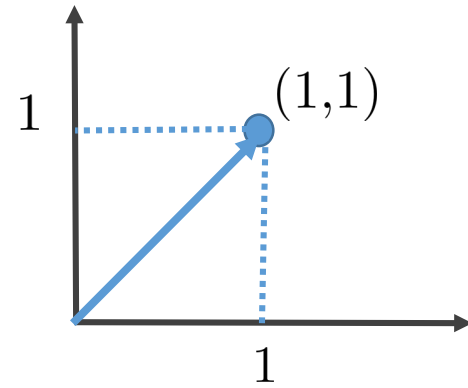
250  + 0  + 0  = 

Red Green Blue

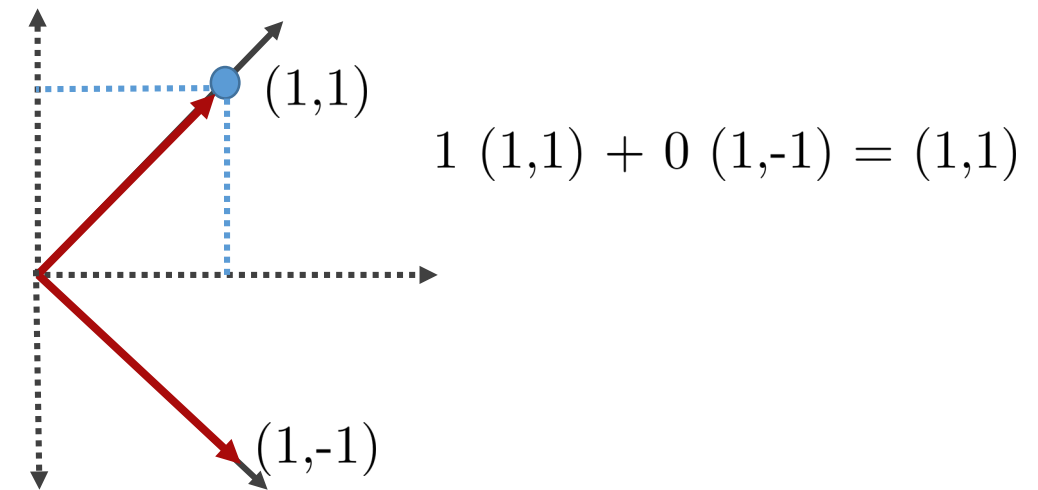
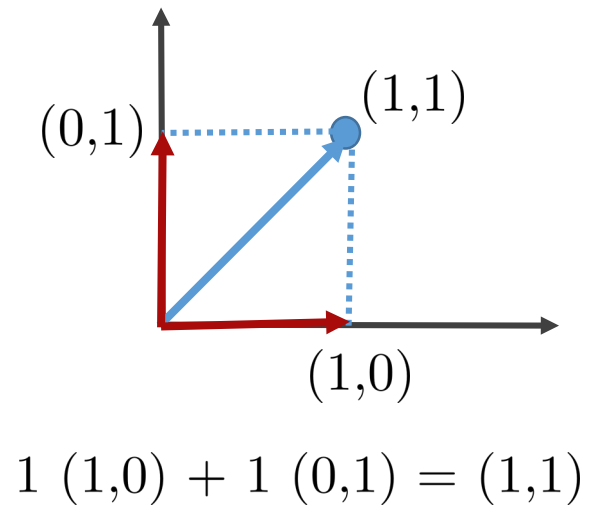
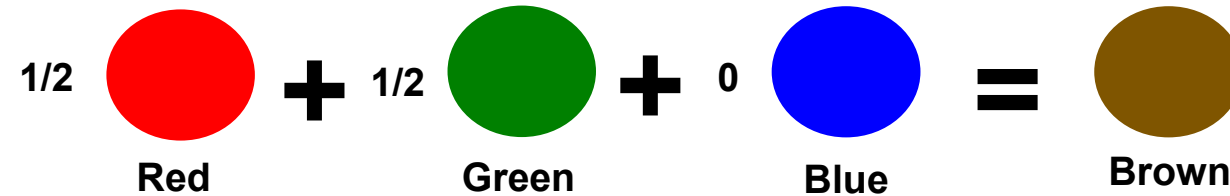
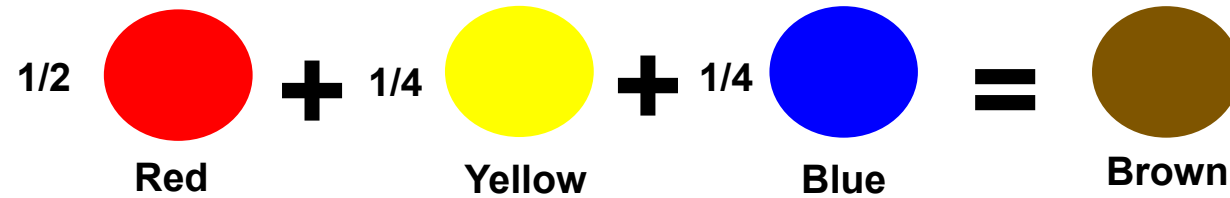
How many colors are in a computer?

$$256 \times 256 \times 256 = 16.777.216$$

Points in the Plane and Vectors



Coordinate Systems



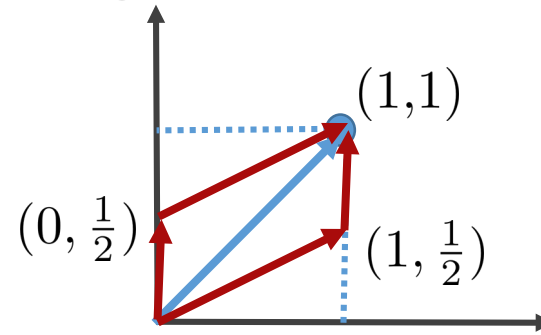
Vector Operations

Let u and v be vectors in \mathbb{R}^n , that is, $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$.

The sum of u and v is a vector obtained by adding the corresponding components:

$$u + v = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n).$$

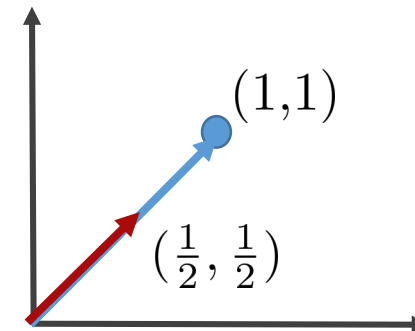
Example: $(0, \frac{1}{2}) + (1, \frac{1}{2}) = (1, 1)$



The product of a real number k times a vector u , is a vector obtained by multiplying each component of u by k :

$$ku = (ku_1, ku_2, \dots, ku_n)$$

Example: $\frac{1}{2}(1, 1) = (\frac{1}{2}, \frac{1}{2})$



Properties of Vectors

For vectors $u, v, w \in \mathbb{R}^n$ and scalar $k, k' \in \mathbb{R}$ we have that

1. $(u + v) + w = u + (v + w)$

2. $u + 0 = u$

3. $u + (-u) = 0$

4. $u + v = v + u$

5. $k(u + v) = ku + kv$

6. $(k + k')u = ku + k'u$

7. $(kk')u = k(k')u$

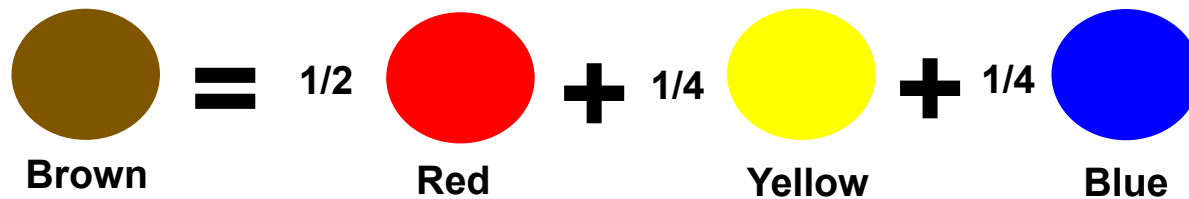
8. $1u = u$

Linear Combination






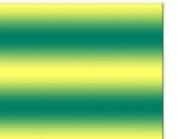
Definition: A vector v is a *linear combination* of vectors u_1, u_2, \dots, u_n if there exist scalars k_1, k_2, \dots, k_n such that




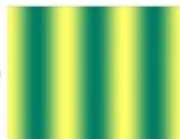


$$v = k_1 u_1 + k_2 u_2 + \dots + k_n u_n$$

Ejemplo:
$$\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 7 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



Linear Combination of Images

$= a_1$  $+ a_2$  $+ a_3$  $+ a_4$  $+ a_5$  $+ a_6$ 

$+ a_7$  $+ a_8$  $+ a_9$  $+ a_{10}$  $+ a_{11}$  $+ a_{12}$  $+ \dots$

Linear Combination of Eigenfaces

$$s = \alpha_1 \cdot \text{img}_1 + \alpha_2 \cdot \text{img}_2 + \alpha_3 \cdot \text{img}_3 + \alpha_4 \cdot \text{img}_4 + \dots$$

$$t = \beta_1 \cdot \text{img}_1 + \beta_2 \cdot \text{img}_2 + \beta_3 \cdot \text{img}_3 + \beta_4 \cdot \text{img}_4 + \dots$$

This is used to identify faces.

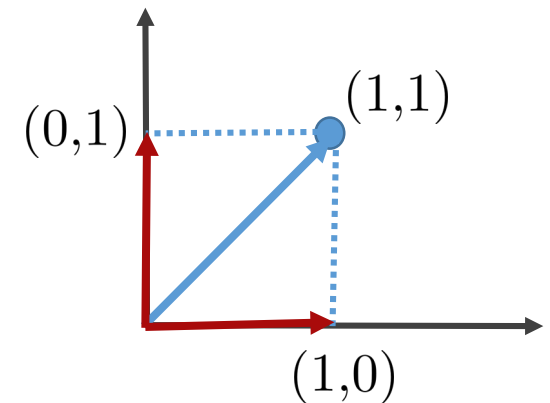
Inner Product

Definition: Let u and v be vectors in \mathbb{R}^n , that is, $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$. The *inner product* of u and v is defined by

$$u \cdot v = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

The vectors u and v are called *orthogonals* (or perpendiculars) if their inner product is zero, that is $u \cdot v = 0$.

Example: $(1, 0) \cdot (0, 1) = 0$, $(1, 0) \cdot (1, 1) = 1$

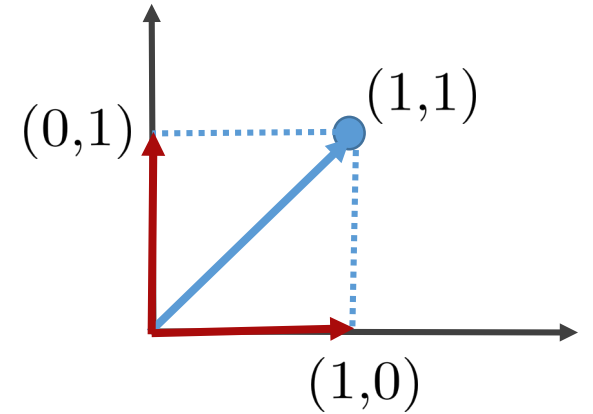


Norms y Distances

Definition: The *Euclidean norm* of a vector $u = (u_1, u_2, \dots, u_n)$ is defined by

$$\|x\|_2 = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

Examples: $\|(1, 0)\|_2 = \|(0, 1)\|_2 = 1$ y $\|(1, 1)\|_2 = \sqrt{2}$

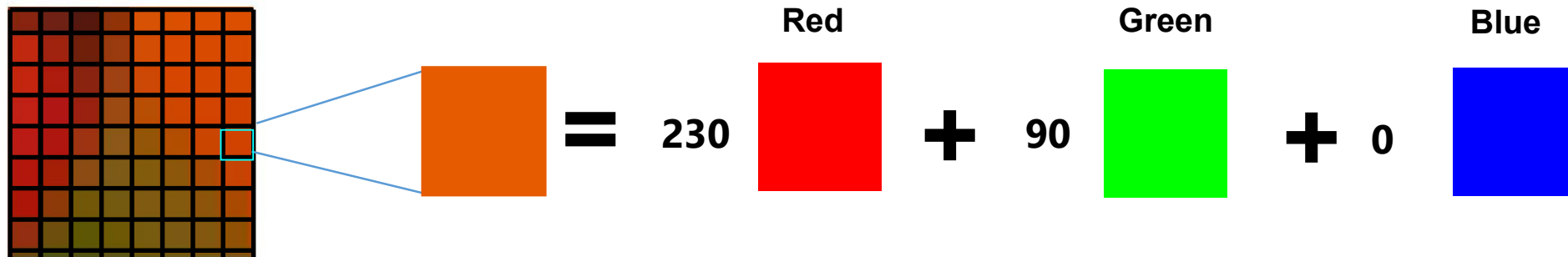
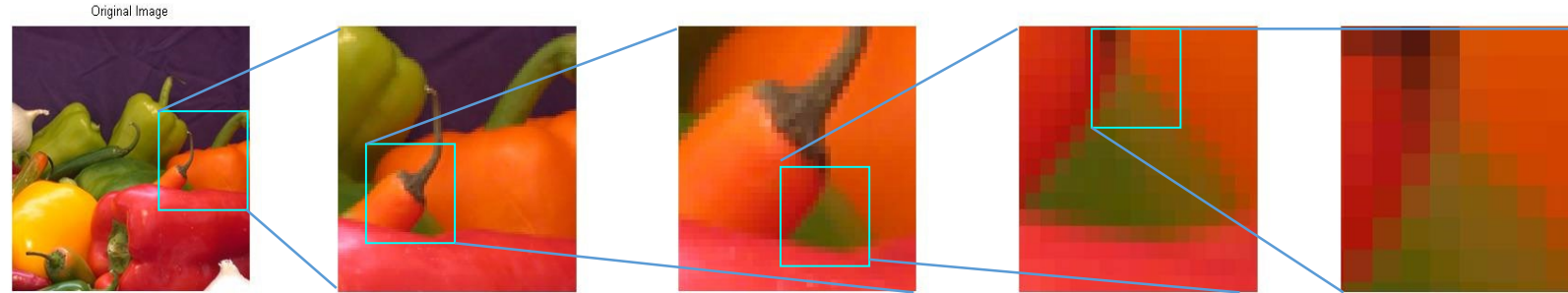


Definition: The *Euclidean distance* between two vectors u and v is defined by

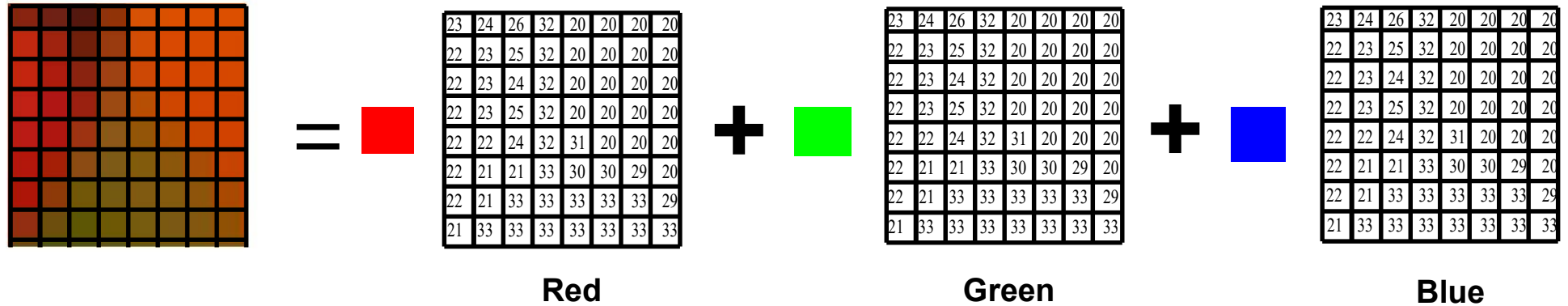
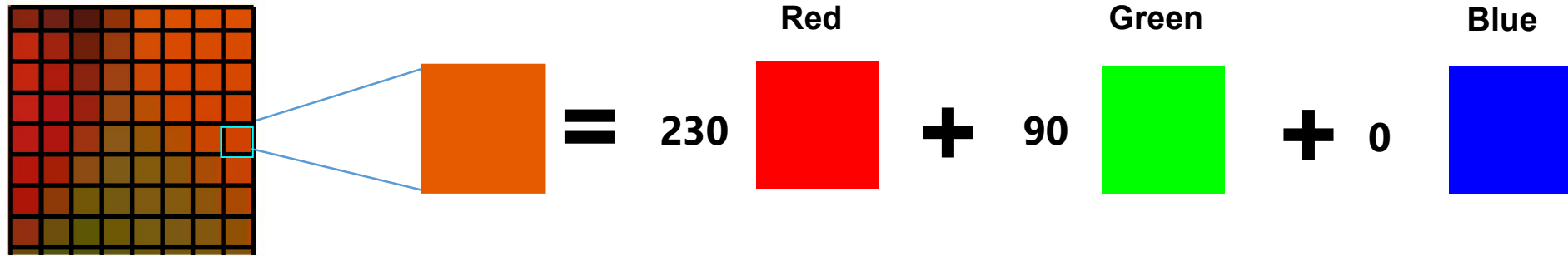
$$d(u, v) = \|u - v\|_2 = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2}$$

Example: $d((1, 1), (1, 0)) = \|(1, 1) - (1, 0)\|_2 = \|(0, 1)\|_2 = 1$

Digital Images



Digital Images



Matrices

Definition: A matrix is an array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Examples: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{2 \times 3}$ $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3}$

Sum of Matrices

Definition: The sum of two matrices of equal size is defined by

$$\begin{aligned} A + B &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix} \end{aligned}$$

Product by a escalar

Definition: The product of a scalar k by a matrix A is defined by

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

Example: $2 \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix} + 3 \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 7 & 10 \\ 5 & 7 & 12 \\ 5 & 9 & 14 \end{pmatrix}$

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