Re-Imaging the World through Linear Algebra

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Mathematics Sin Fronteras
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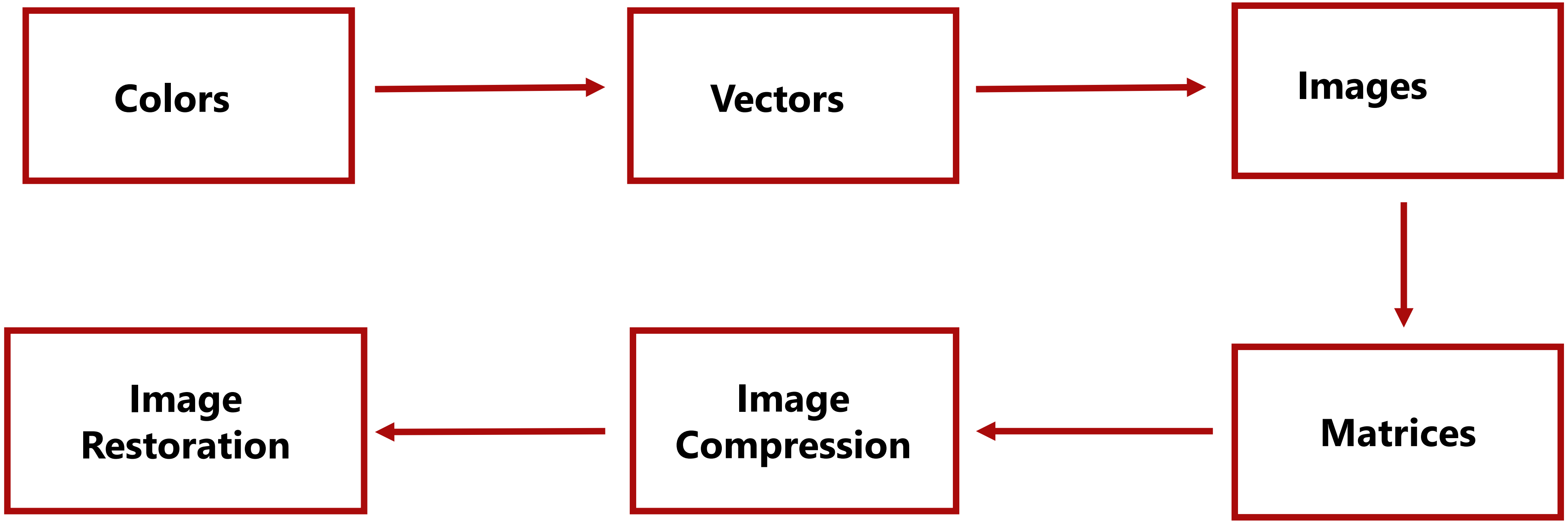
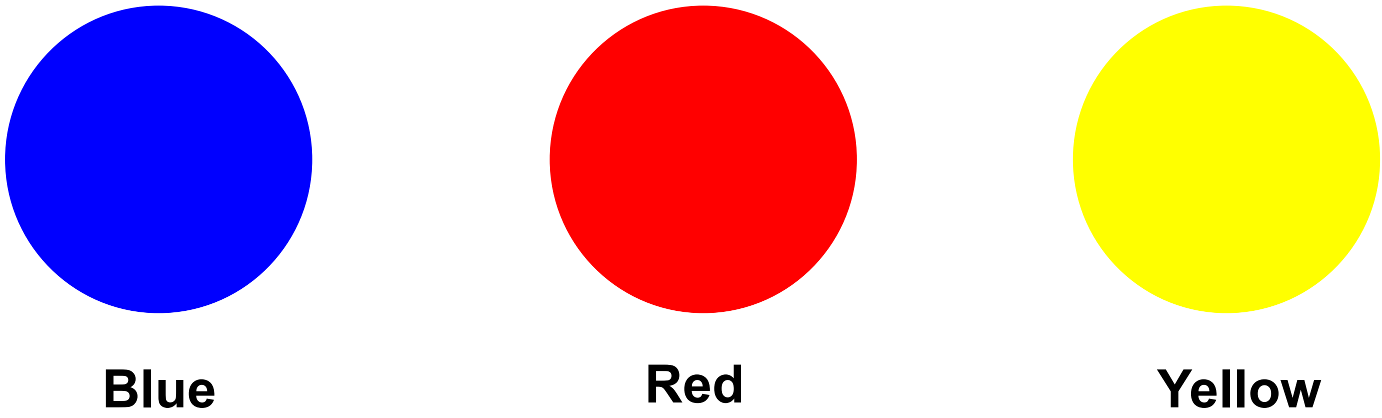
The plan


Colors
Vectors
Images
Matrices
Image Compression
Image Restoration

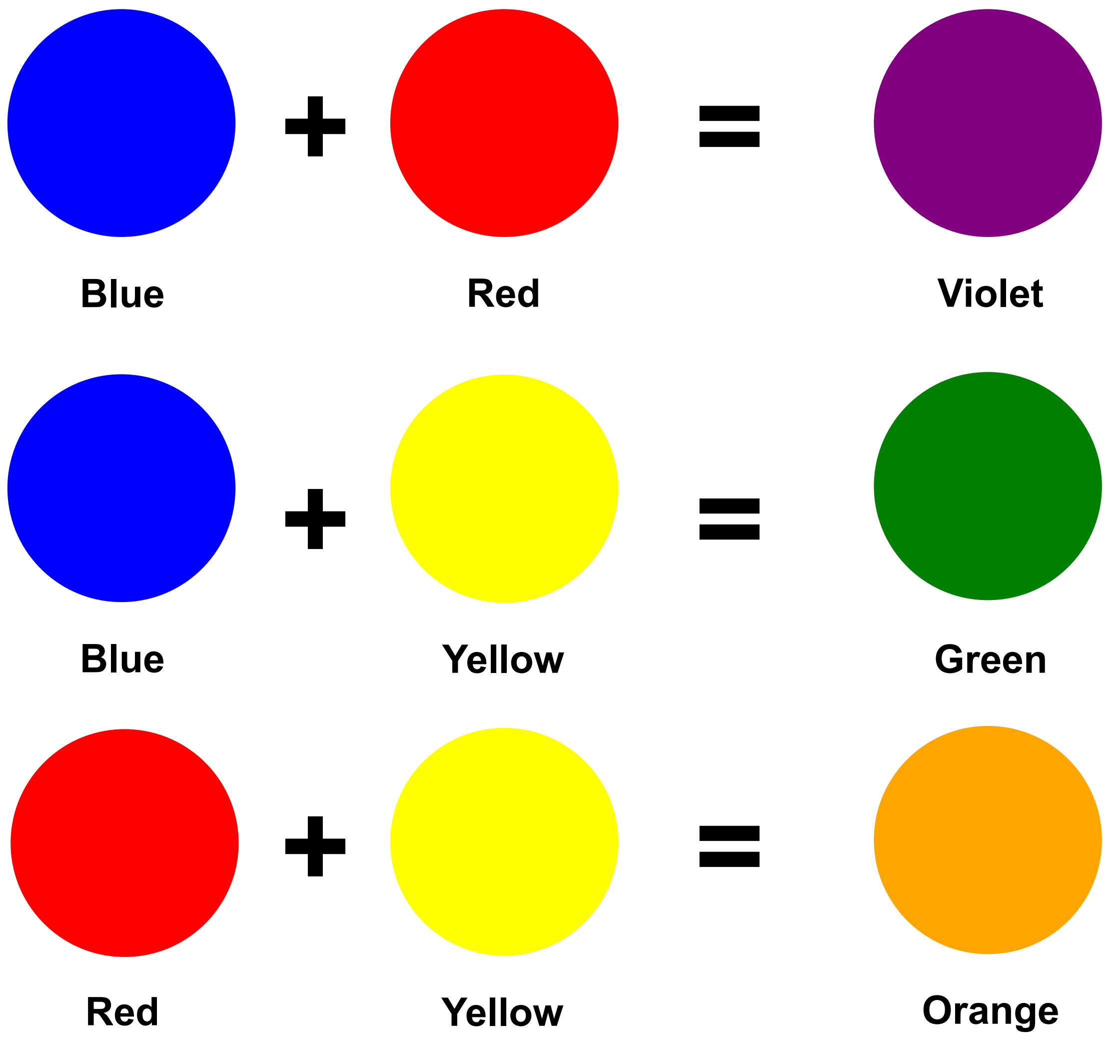
The Mathematics of Colors

Primary Colors

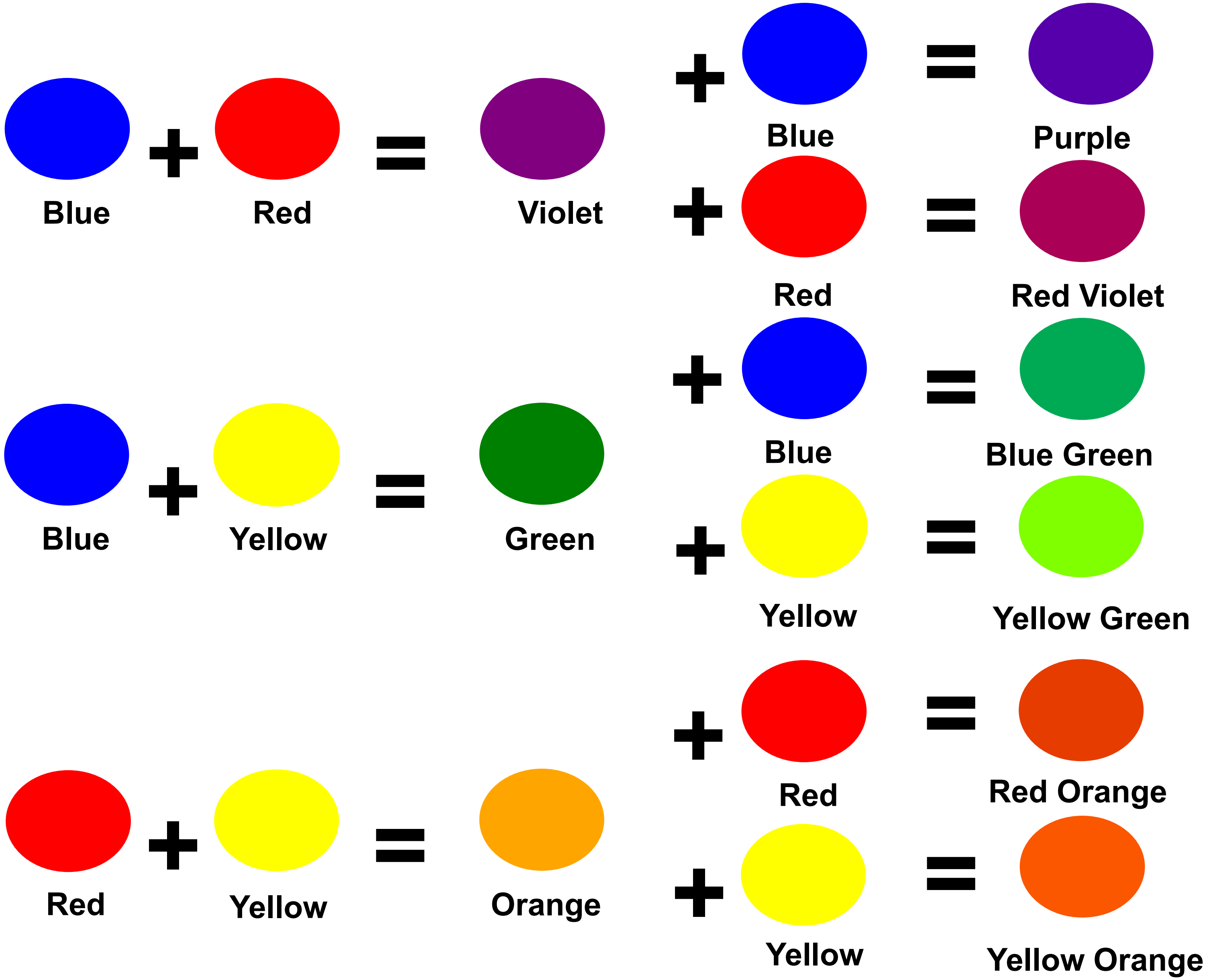


Why do we call them that?

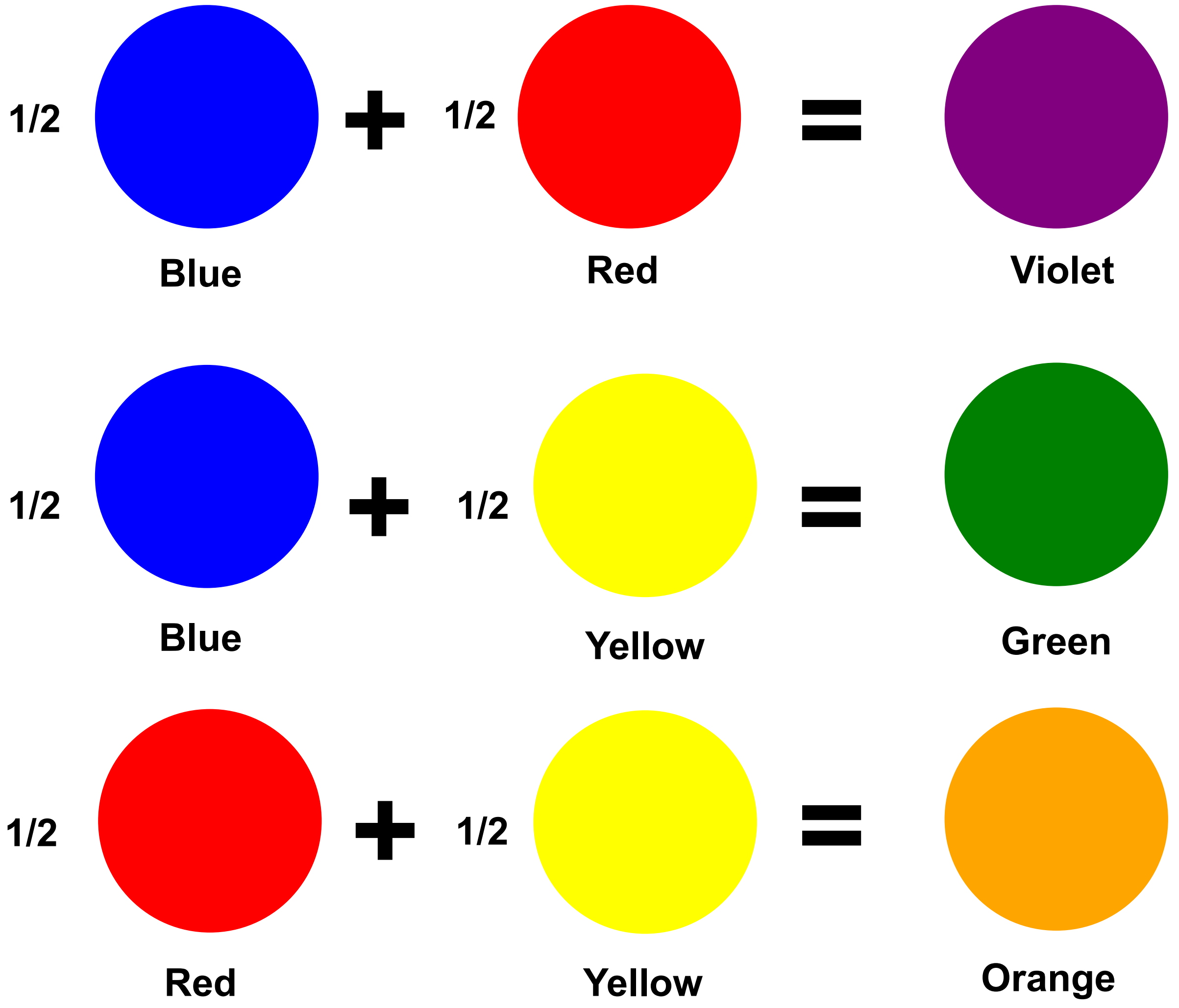
Secondary Colors



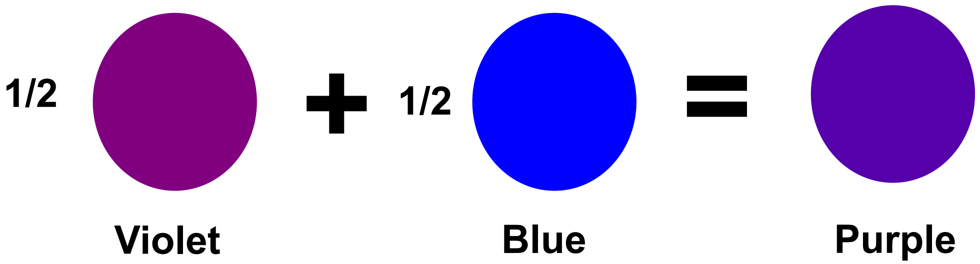
Tertiary Colors



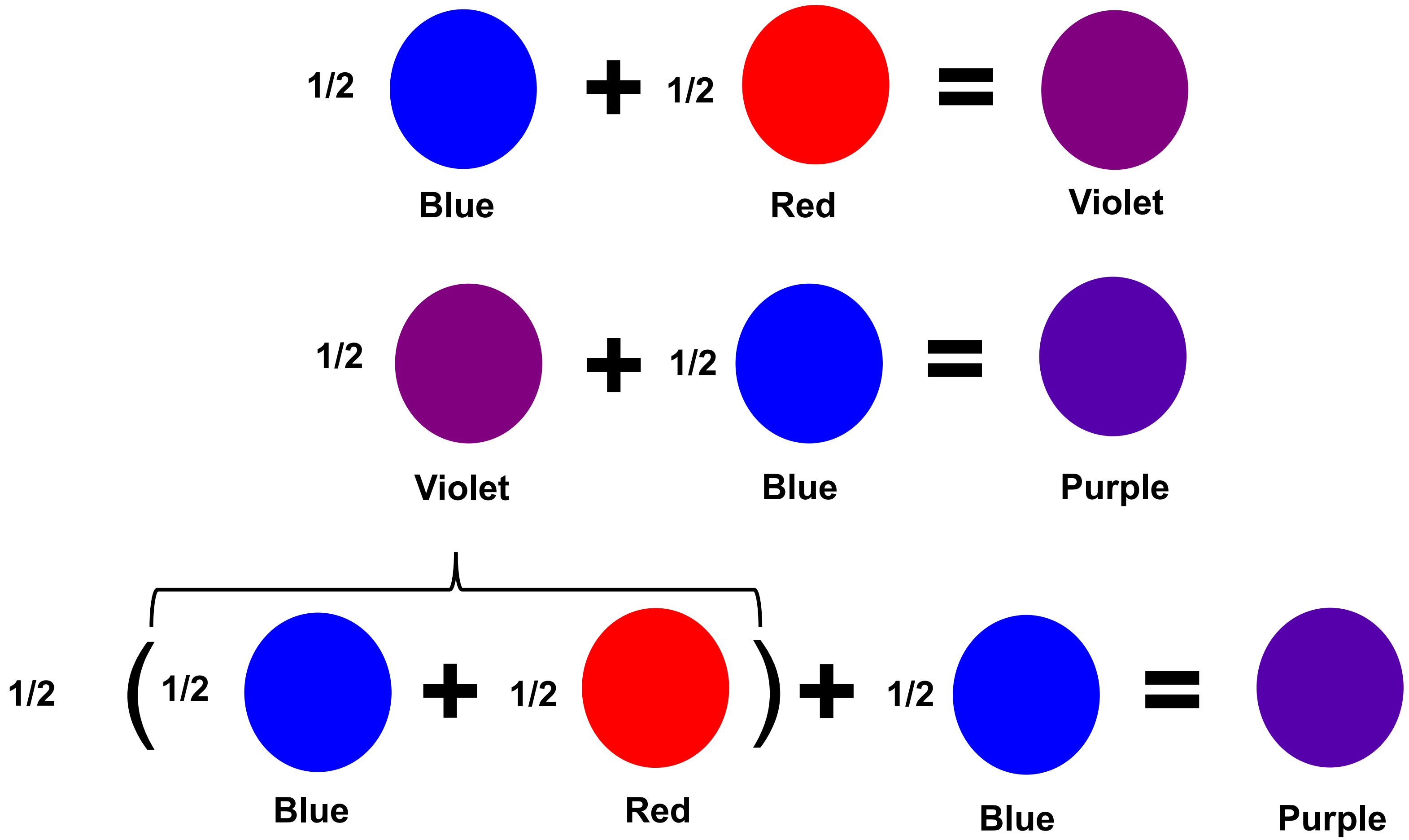
“Recipe” for the Secondary Colors



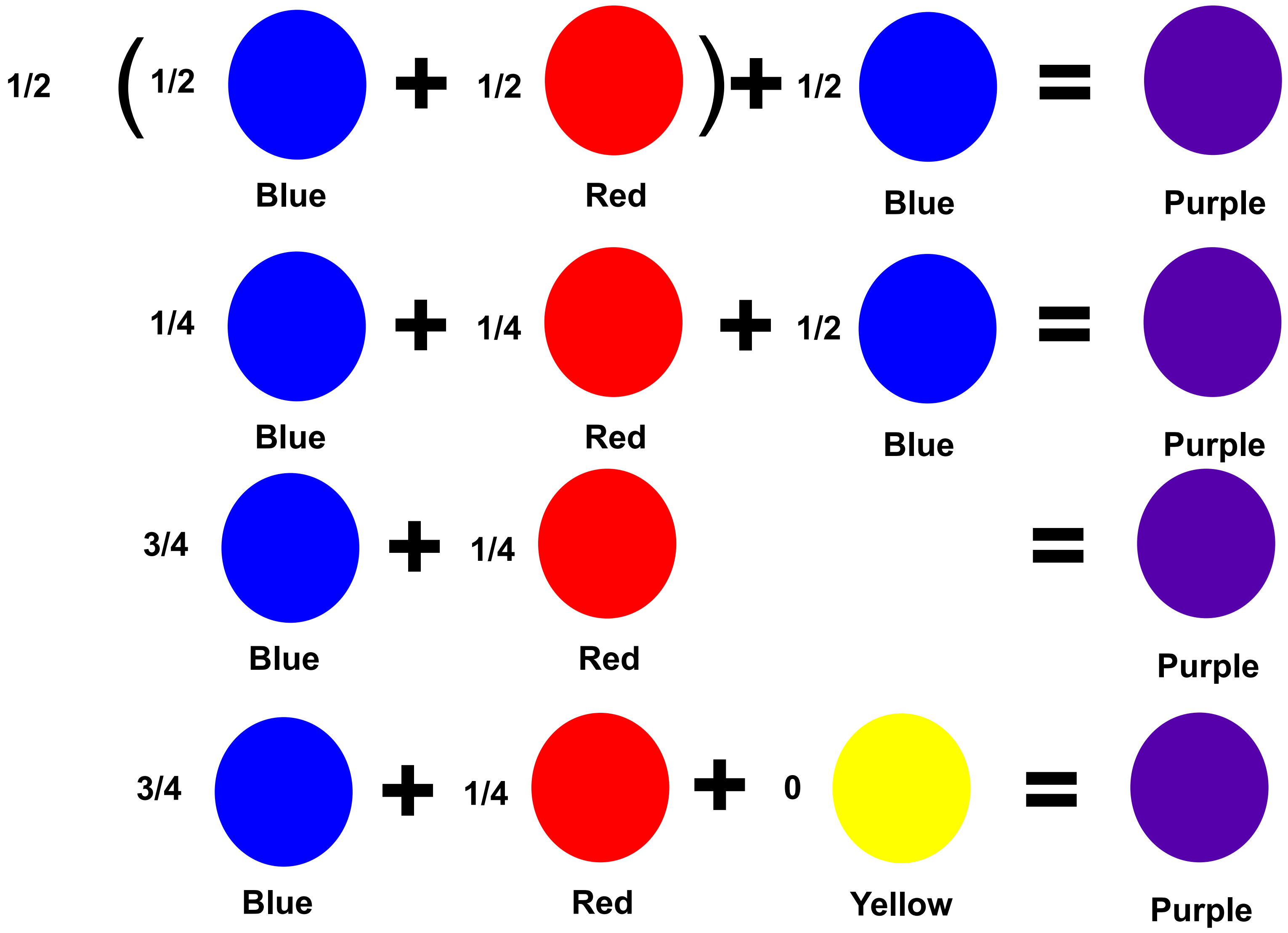
Recipe of the Purple Color



Recipe of the Purple Color



Recipe of the Purple Color

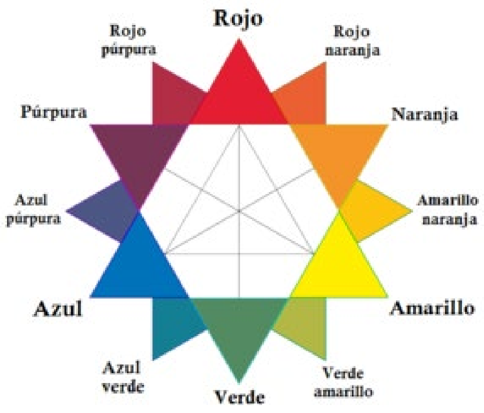


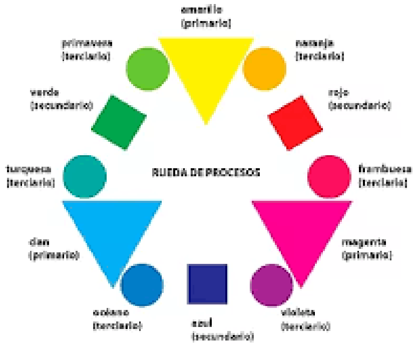
Recipes of the Tertiary Colors

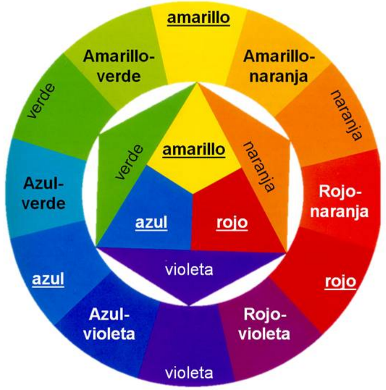
3/4 Blue	+	1/4 Red	+	0 Yellow	=	Purple
1/4 Blue	+	3/4 Red	+	0 Yellow	=	Red Violet
3/4 Blue	+	0 Red	+	1/4 Yellow	=	Blue Green
1/4 Blue	+	0 Red	+	3/4 Yellow	=	Yellow Green
0 Blue	+	3/4 Red	+	1/4 Yellow	=	Red Orange
0 Blue	+	1/4 Red	+	3/4 Yellow	=	Yellow Orange

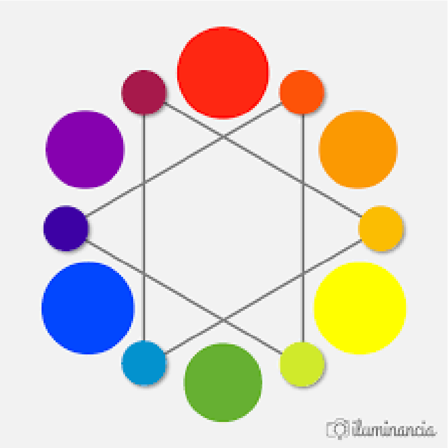
Color Wheel

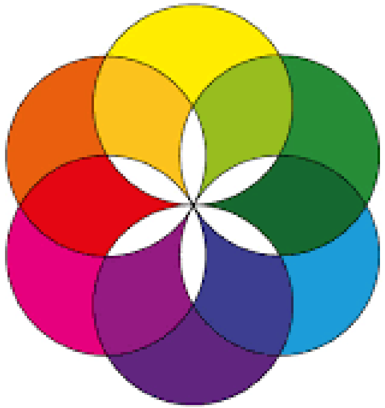
Rueda de R.J.B. Mérimée, 1830*





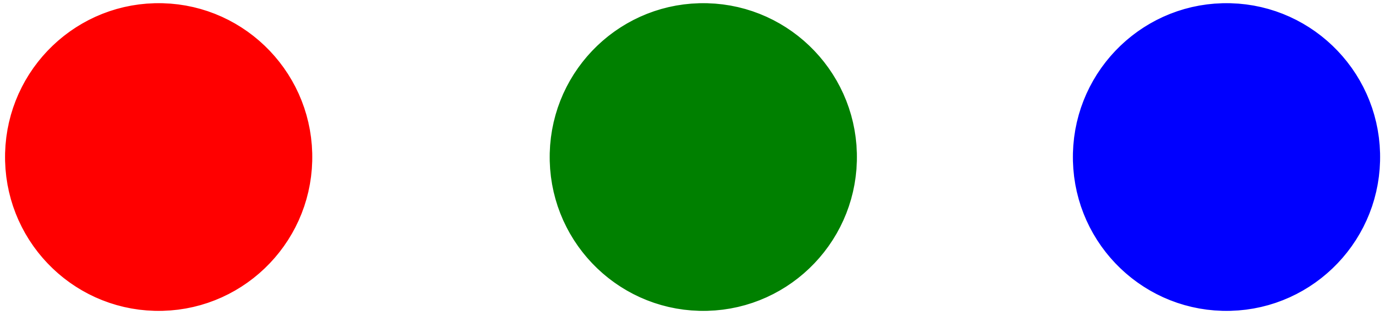






[Color wheel - Wikipedia]

Primary Colors of the Light



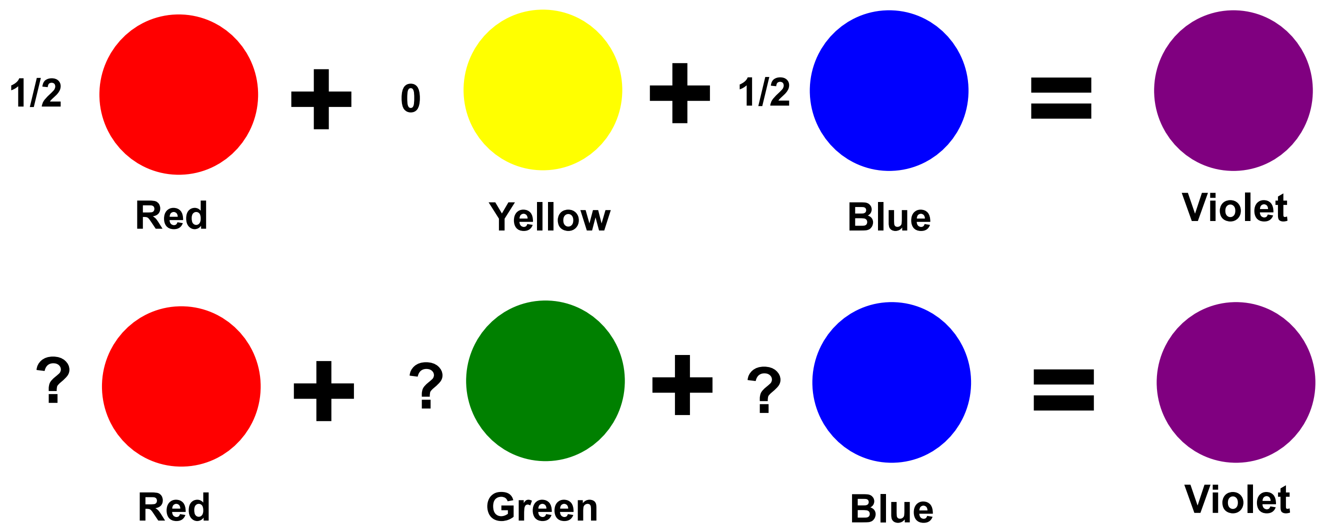
Red Green Blue

RGB

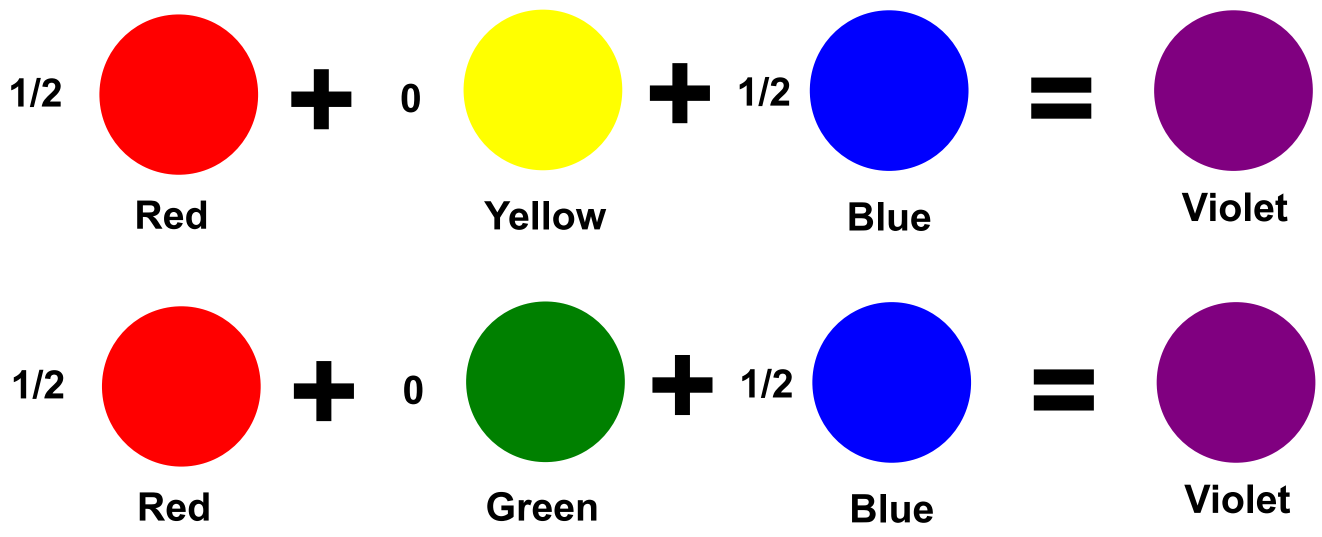
They are used in the computer screens, TVs, and movie screens

[RGB color model - Wikipedia]

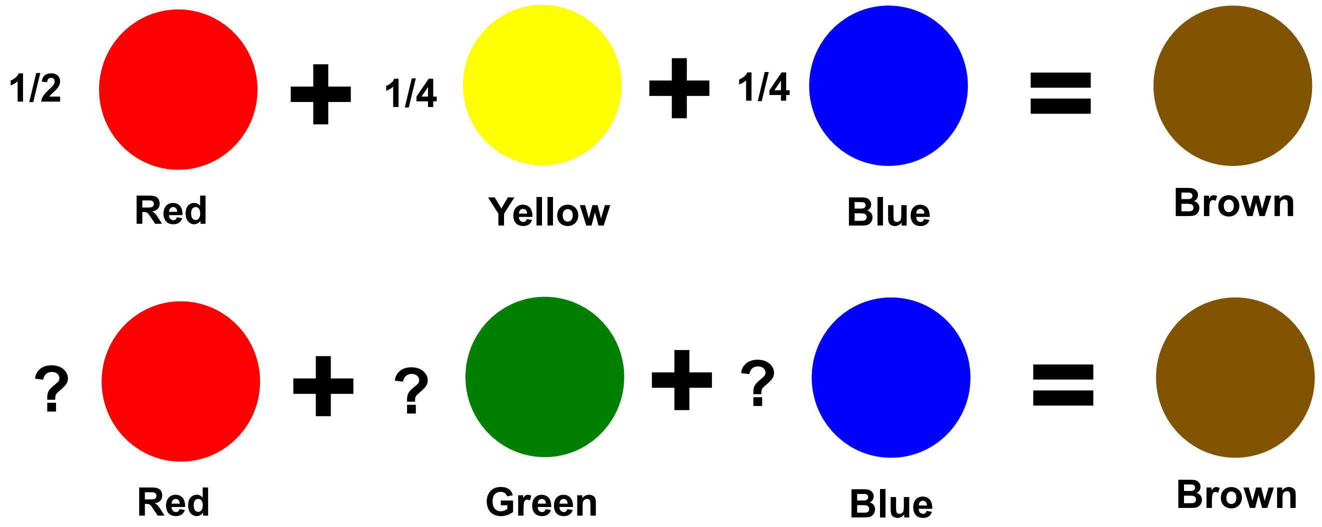
Creating colors with RGB



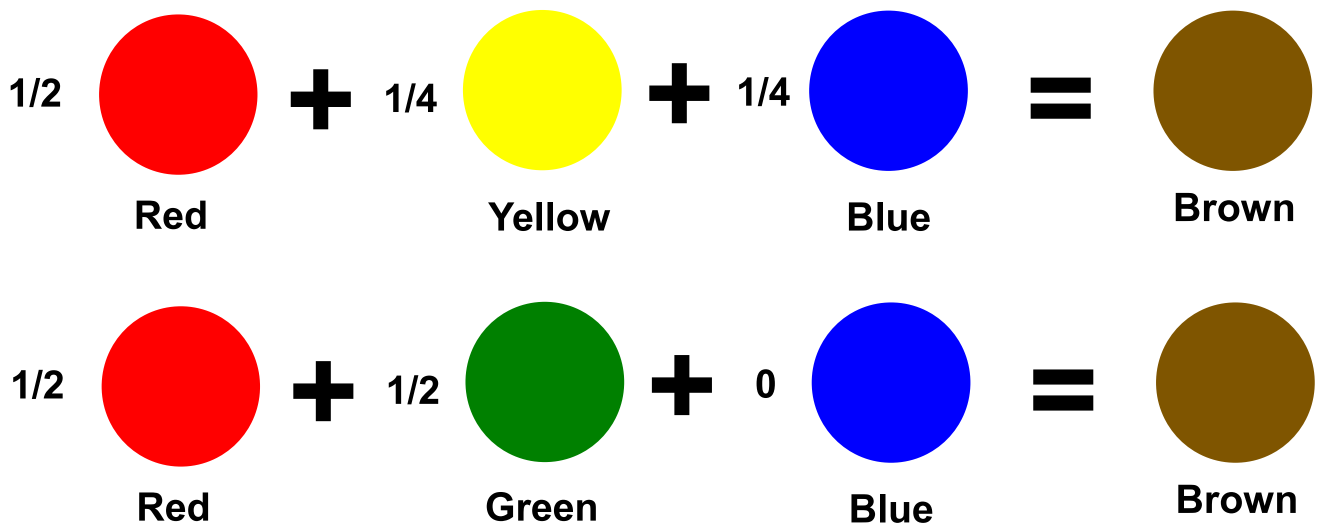
Creating colors with RGB



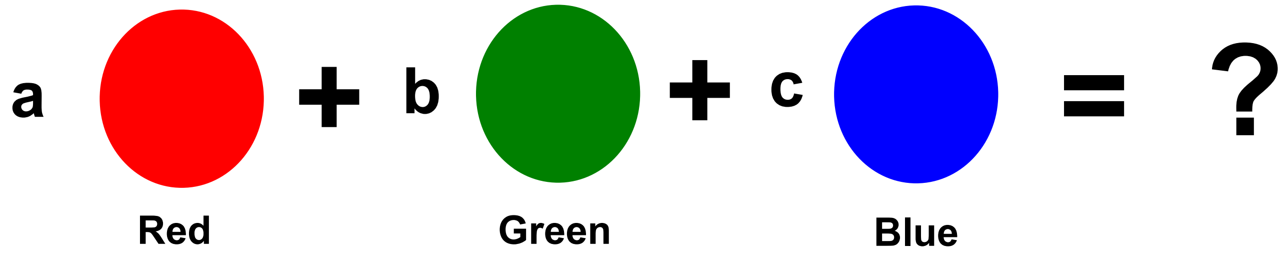
Creating colors with RGB



Creating colors with RGB



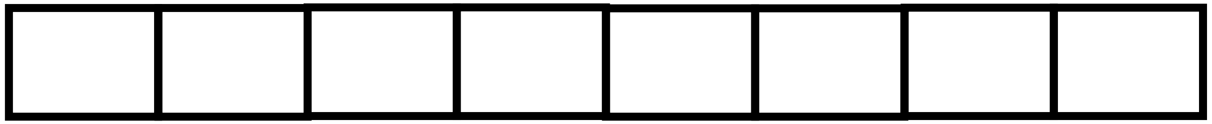
Creating colors in the Computer



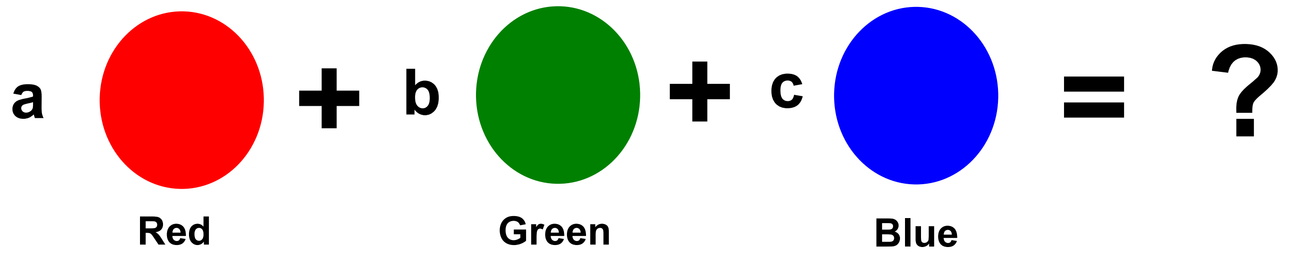
a, b, c are numbers

that can only use

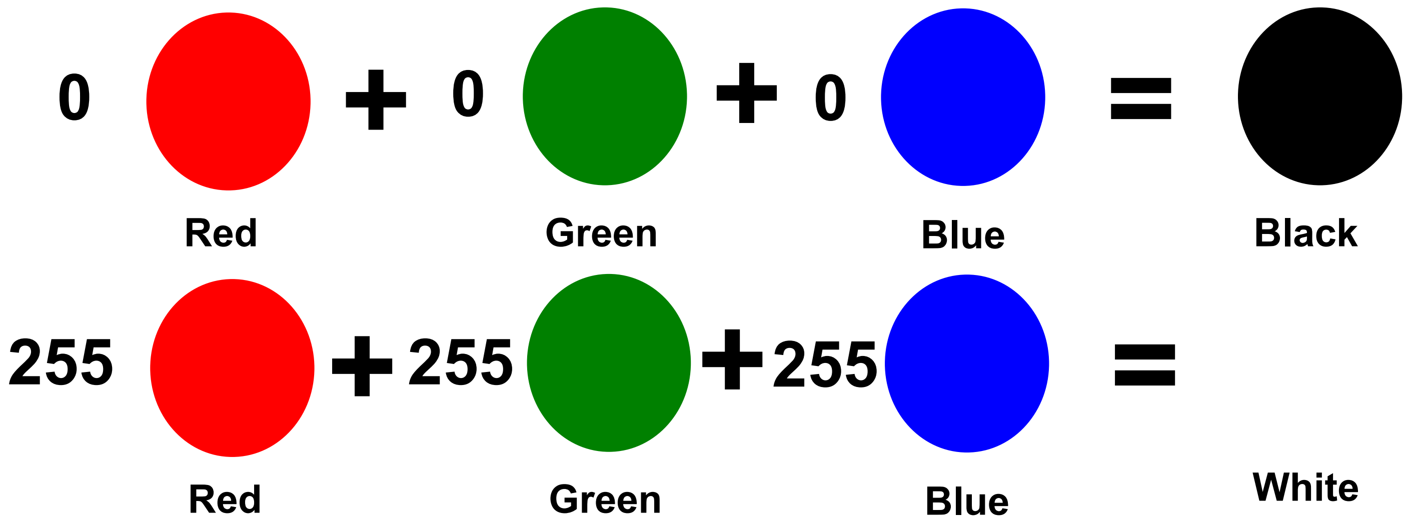
1 byte = 8 bits



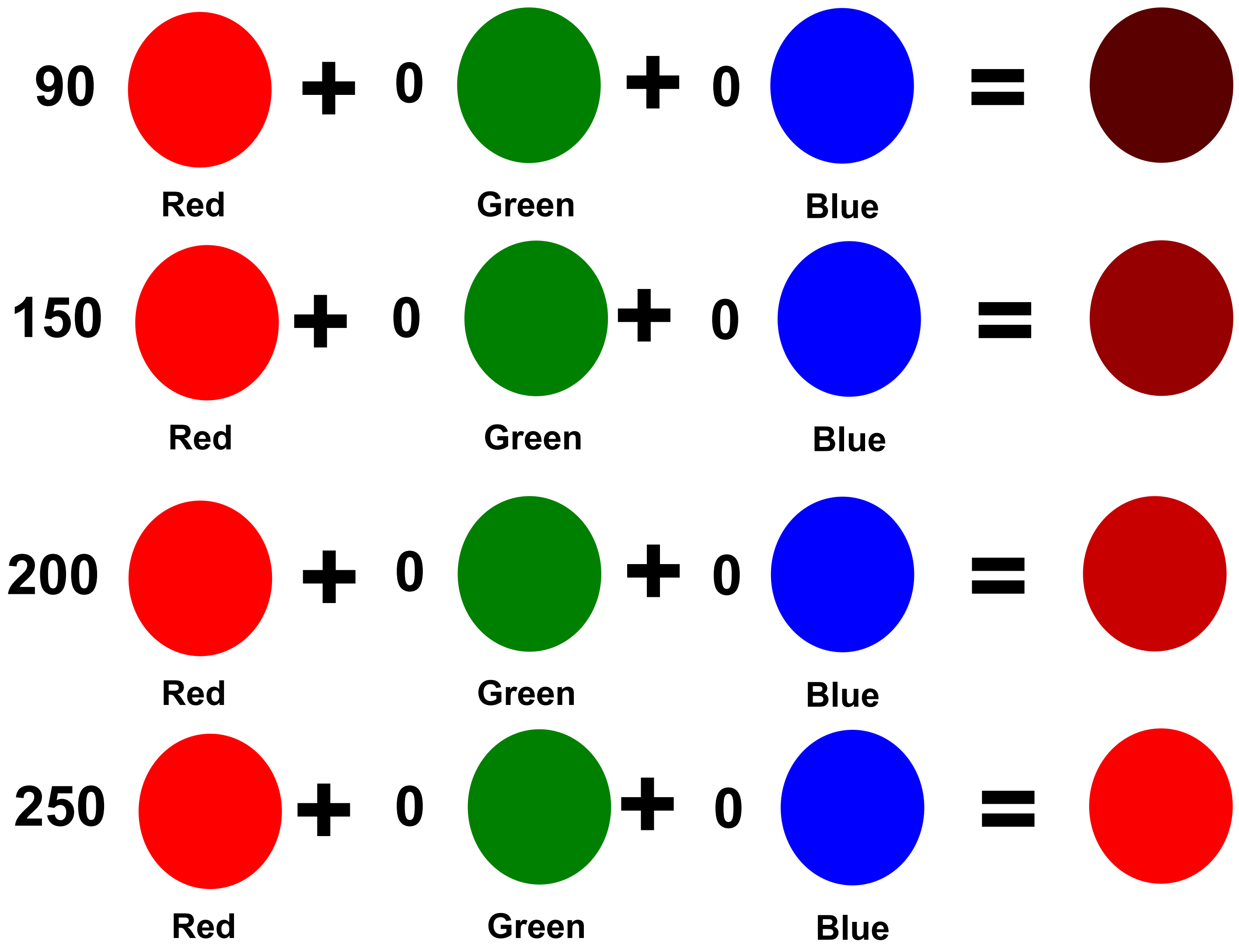
Creating colors in the Computer



a, b, c are integer numbers from 0 to 255



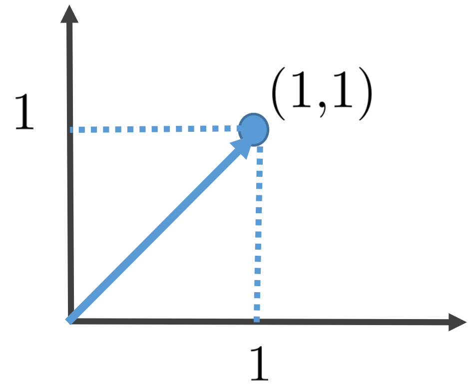
Creating colors in the Computer

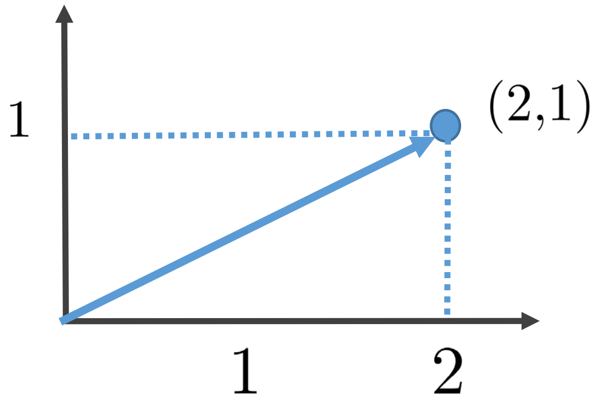


How many colors are in a computer?

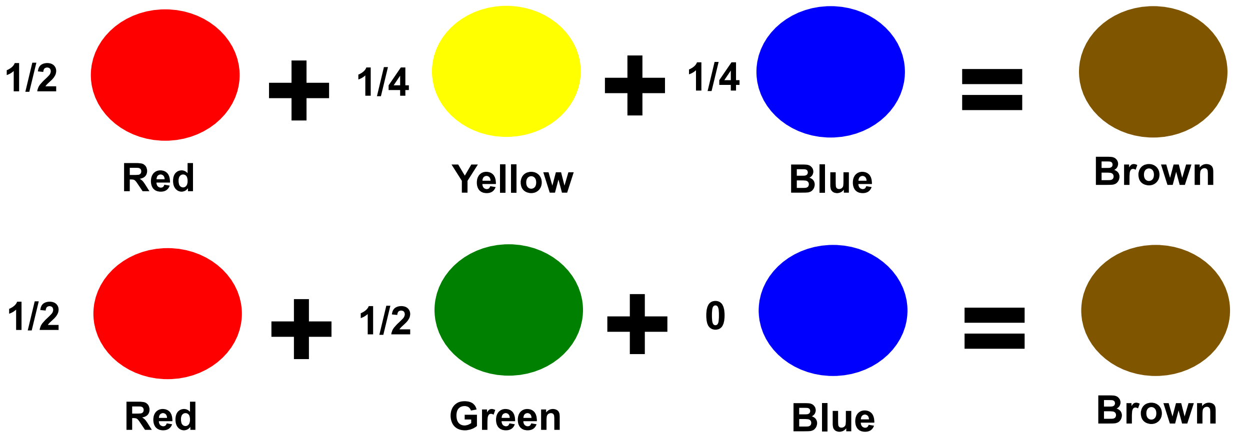
256 x 256 x 256 = 16.777.216

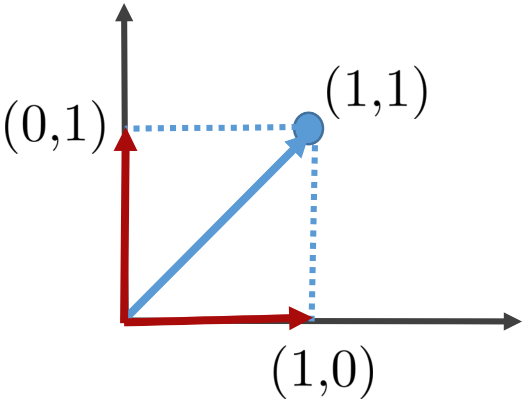
Points in the Plane and Vectors



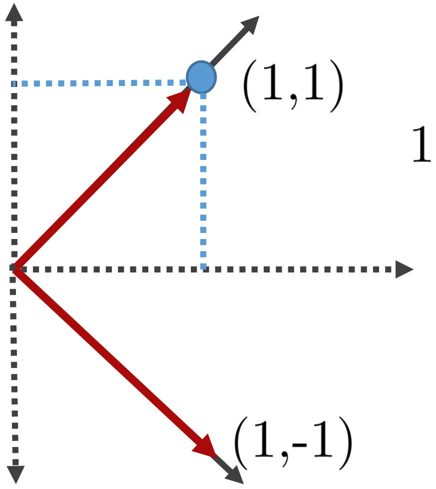


Coordinate Systems





$$1\ (1,0) + 1\ (0,1) = (1,1)$$

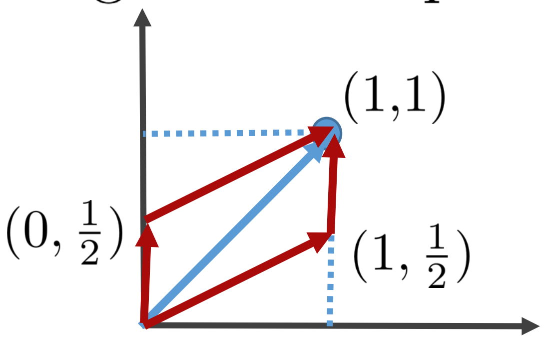


$$1\ (1,1) + 0\ (1,-1) = (1,1)$$

Vector Operations

Let u and v be vectors in $\mathbb{R}^n$, that is, $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$.

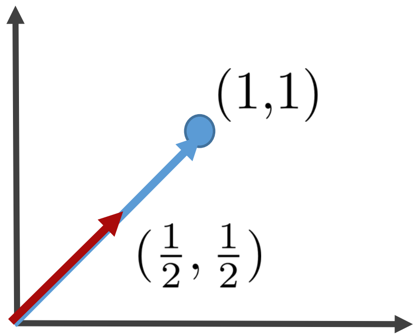
The sum of u and v is a vector obtained by adding the corresponding components:



$$u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

Example: $(0, \frac{1}{2}) + (1, \frac{1}{2}) = (1, 1)$

The product of a real number k times a vector u, is a vector obtained by multiplying each component of u by k:



$$ku = (ku_1, ku_2, \ldots, ku_n)$$

Example: $\frac{1}{2}(1, 1) = (\frac{1}{2}, \frac{1}{2})$

Properties of Vectors

For vectors $u, v, w \in \mathbb{R}^n$ and scalar $k, k' \in \mathbb{R}$ we have that

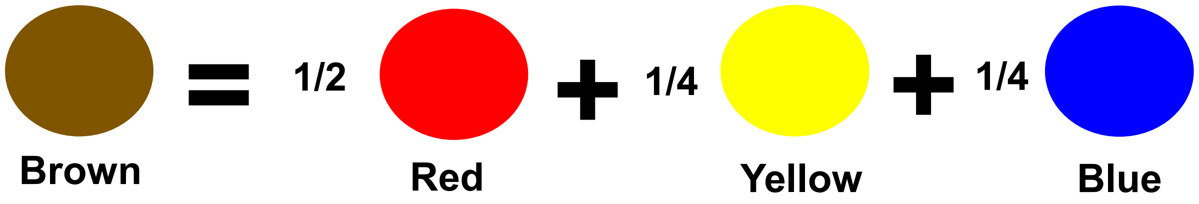
1. $(u + v) + w = u + (v + w)$
2. $u + 0 = u$
3. $u + (-u) = 0$
4. $u + v = v + u$
5. $k(u + v) = ku + kv$
6. $(k + k')u = ku + k'u$
7. $(kk')u = k(k')u$
8. $1u = u$

Linear Combination

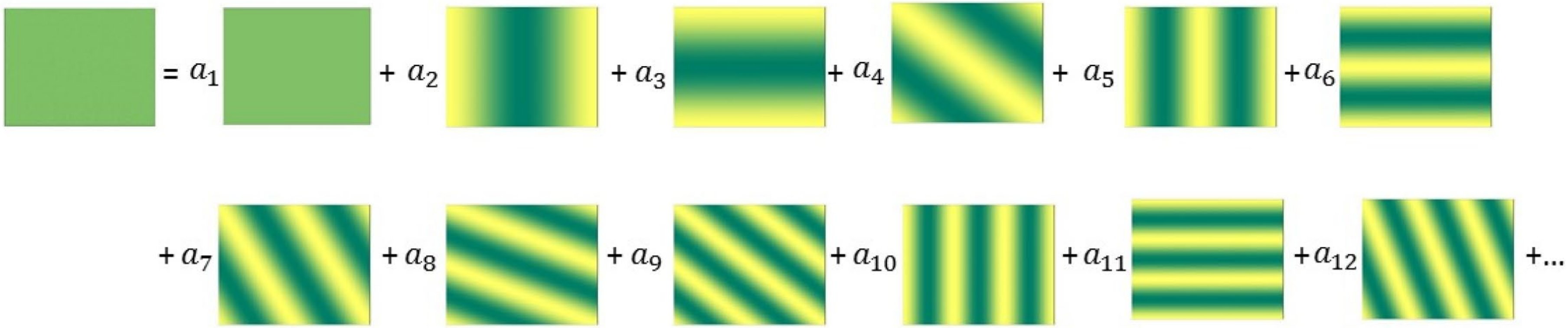
Definition: A vector v is a *linear combination* of vectors $u_1, u_2, \ldots, u_n$ if there exist scalars $k_1, k_2, \ldots, k_n$ such that

$$v = k_1 u_1 + k_2 u_2 + \cdots + k_n u_n$$

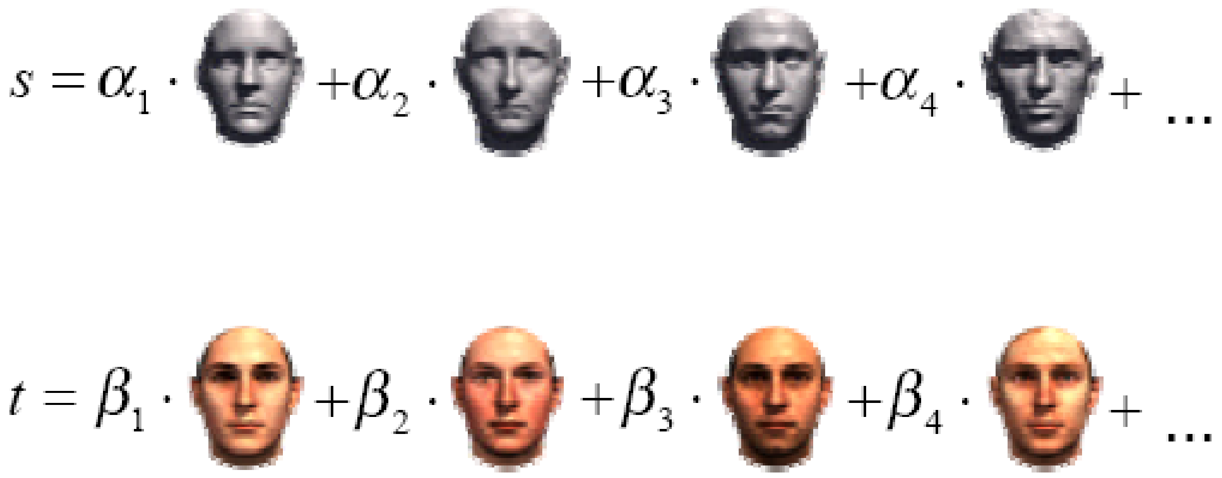
Ejemplo: $$\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 7 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



Linear Combination of Images



Linear Combination of Eigenfaces



This is used to identify faces.

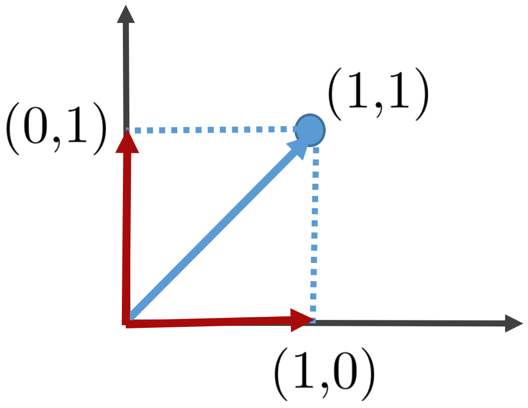
Inner Product

Definition: Let u and v be vectors in $\mathbb{R}^n$, that is, $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. The *inner product* of u and v is defined by

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

The vectors u and v are called *orthogonals* (or perperdiculars) if their inner product is zero, that is $u \cdot v = 0$.

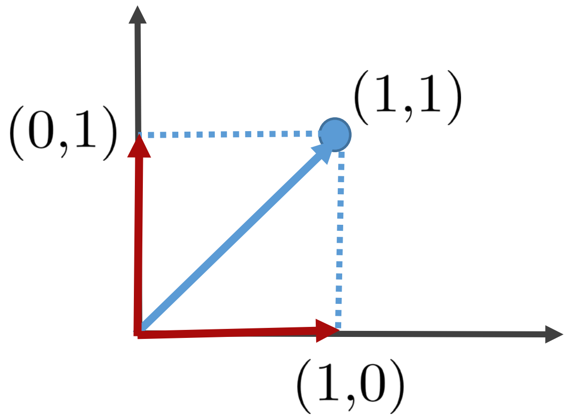
Example: $(1, 0) \cdot (0, 1) = 0$, $(1, 0) \cdot (1, 1) = 1$



Norms y Distances

Definition: The *Euclidean norm* of a vector $u = (u_1, u_2, \ldots, u_n)$ is defined by

$$\|x\|_2 = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$



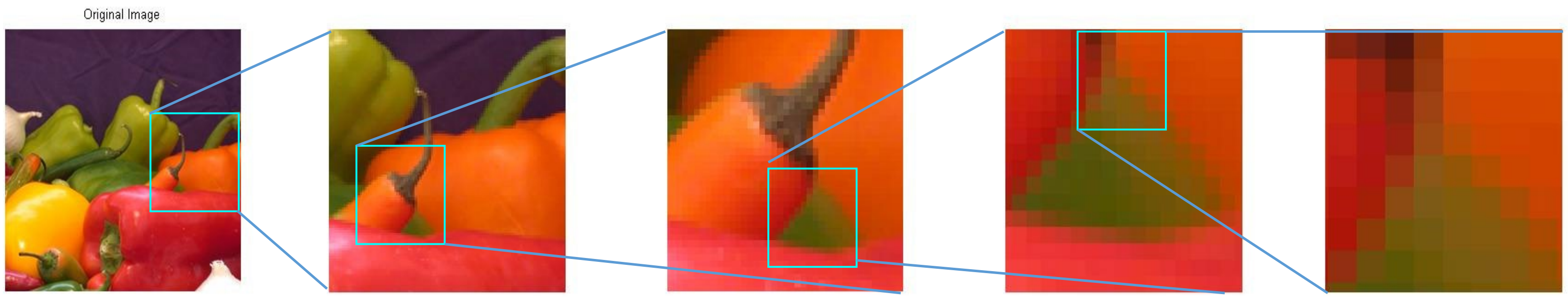
Examples: $\|(1,0)\|_2 = \|(0,1)\|_2 = 1$ y $\|(1,1)\|_2 = \sqrt{2}$

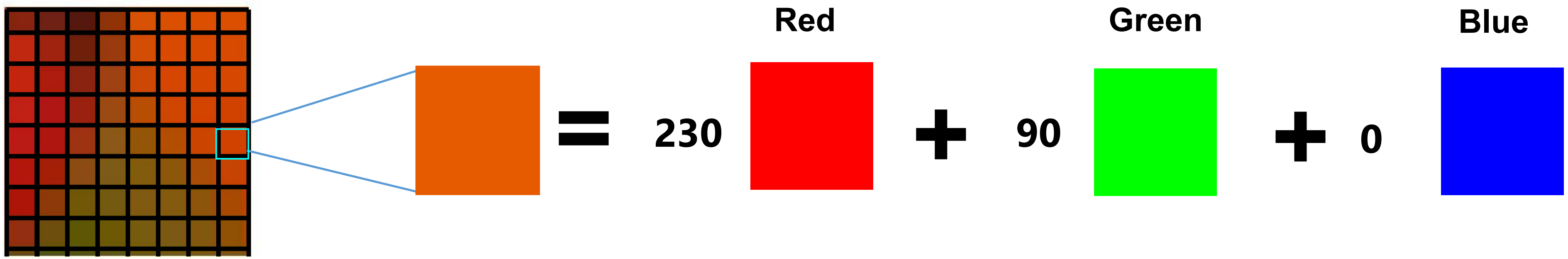
Definition: The *Euclidean distance* between two vectors u and v is defined by

$$d(u,v) = \|u - v\|_2 = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

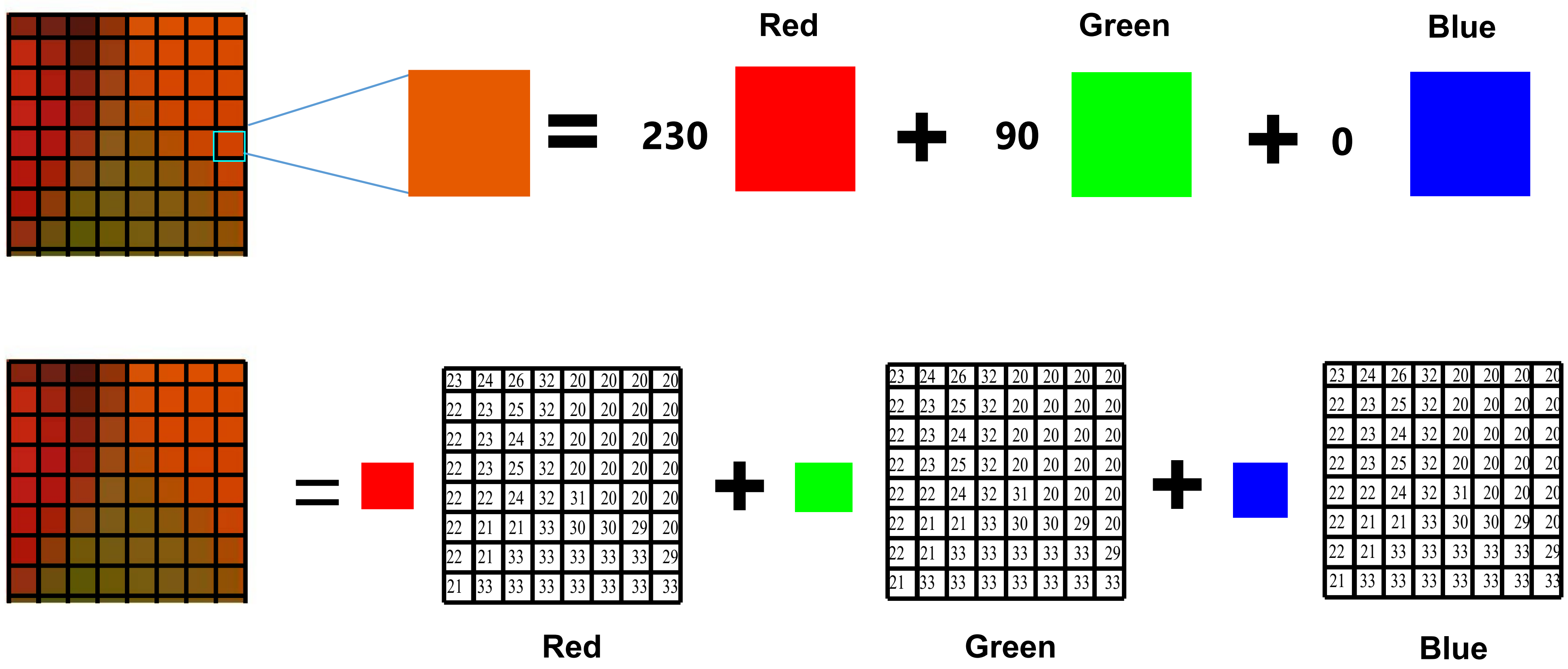
Example: $d((1,1),(1,0)) = \|(1,1) - (1,0)\|_2 = \|(0,1)\|_2 = 1$

Digital Images





Digital Images



Matrices

Definition: A matrix is an array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Examples: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{2\times 3}$ $\qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}$

Sum of Matrices

Definition: The sum of two matrices of equal size is defined by

$$A + B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

Product by a escalar

Definition: The product of a scalar k by a matrix A is defined by

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

Example: $2\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix} + 3\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 7 & 10 \\ 5 & 7 & 12 \\ 5 & 9 & 14 \end{pmatrix}$

Linear Combination of Images