The isoperimetric problem

Tatiana Toro

University of Washington

Mathematics Sin Fronteras

The isoperimetric inequality

Theorem: Given a planar figure of area A and perimeter P

$$4\pi A \leq P^2$$

Equality occurs if and only if the figure is a disc.

The isoperimetric inequality

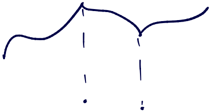
Theorem: Given a planar figure of area A and perimeter P

$$4\pi A \leq P^2$$

Equality occurs if and only if the figure is a disc.

Theorem (Wirtinger inequality): Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π (i.e. $f(\theta + 2\pi) = f(\theta)$).
Let $\overline{f}$ denote the mean value of f

$$\overline{f} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta.$$



Then

$$\int_0^{2\pi} [f(\theta) - \overline{f}]^2\, d\theta \leq \int_0^{2\pi} [f'(\theta)]^2\, d\theta.$$

Equality holds if and only if

$$f(\theta) = \overline{f} + a\cos\theta + b\sin\theta$$

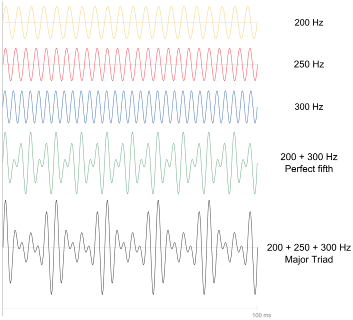
for some constants a, b.

Fourier analysis

The central idea of Fourier analysis is to decompose a function into a combination of simpler functions. The simpler functions are the building blocks. Sine and cosine functions are examples of building blocks.

Fourier analysis

The central idea of Fourier analysis is to decompose a function into a combination of simpler functions. The simpler functions are the building blocks. Sine and cosine functions are examples of building blocks.



https://upload.wikimedia.org/wikipedia/commons/thumb/d/d1/Major_triad.svg/1200px-Major_triad.svg.png

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π (i.e. $f(\theta + 2\pi) = f(\theta)$). Can f be expanded as a series of the form

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) \ ? \tag{1}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π (i.e. $f(\theta + 2\pi) = f(\theta)$). Can f be expanded as a series of the form

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) \ ? \tag{1}$$

$x \in \mathbb{R}$

Recall that $e^{ix} = \cos x + i \sin x$. Thus

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2} \text{ and } \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}.$$

$$e^{in\theta} = \cos n\theta + i \sin n\theta$$

$$+ \quad e^{-in\theta} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i \sin n\theta$$

$$e^{in\theta} + e^{-in\theta} = 2\cos n\theta$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π (i.e. $f(\theta + 2\pi) = f(\theta)$). Can f be expanded as a series of the form

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) \; ? \tag{1}$$

Recall that $e^{ix} = \cos x + i \sin x$. Thus

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2} \text{ and } \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}.$$

Thus (1) can be rewritten as

$$f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} \tag{2}$$

where for $n \in \mathbb{N}$

$$c_0 = \frac{1}{2}a_0; \quad c_n = \frac{1}{2}(a_n - ib_n); \quad c_{-n} = \frac{1}{2}(a_n + ib_n) \tag{3}$$

equivalently

$$a_0 = 2c_0; \quad a_n = c_n + c_{-n}; \quad b_n = i(c_n - c_{-n}). \tag{4}$$

Assume f admits a series expansion of the form (2), how can we compute c_n in terms of f?

$$\to f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}, \quad \text{who is } c_n ?$$

$$e^{in\theta} = \cos n\theta + i \sin n\theta \qquad k \in \mathbb{Z}$$

$$\int_{-\pi}^{\pi} f(\theta) e^{-ik\theta}\, d\theta = \sum_{-\infty}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} e^{in\theta} e^{-ik\theta}\, d\theta}$$

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}\, d\theta = \begin{cases} \int_{-\pi}^{\pi} d\theta = 2\pi & n = k \\ \frac{1}{i(n-k)} e^{i(n-k)\theta} \Big|_{-\pi}^{\pi} & \text{if } n \neq k \end{cases}$$

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}\, d\theta = \begin{cases} \displaystyle\int_{-\pi}^{\pi} d\theta = 2\pi \\ \underbrace{\overset{0}{\cancel{\dfrac{1}{i(n-k)}\, e^{i(n-k)\theta}\Big|_{-\pi}^{\pi}}}} & \text{if } n \neq k \end{cases}$$

$$\frac{1}{i(n-k)}\left(\underbrace{e^{i(n-k)\pi}} - e^{-i(n-k)\pi}\right)$$

$$\cos(n-k)\pi + i\sin \overset{0}{\cancel{(n-k)\pi}}$$

$$-\left(\cos-(n-k)\pi + i\sin \overset{0}{\cancel{(-(n-k)\pi)}}\right)$$

$$= \cos(n-k)\pi - \cos(n-k)\pi = 0$$

$$\int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta}\, d\theta = 2\pi\, c_k$$

$$\boxed{c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta}\, d\theta}$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta = \bar{f}$$

Fourier series

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π, the numbers a_n, b_n in (1) and c_n in (2) are called the Fourier coefficients of f. The corresponding series

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} \qquad \text{or} \qquad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

is called the Fourier series of f.

Fourier series

Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π, the numbers a_n, b_n in (1) and c_n in (2) are called the Fourier coefficients of f. The corresponding series

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} \qquad \text{or} \qquad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

is called the Fourier series of f. Here

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\zeta)\cos n\zeta \, d\zeta \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\zeta)\sin n\zeta \, d\zeta \tag{5}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\zeta)e^{in\zeta} \, d\zeta \tag{6}$$

Special cases

f even	$f(-\theta) = f(\theta)$	$a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(\theta)\cos n\theta\, d\theta$	$b_n = 0$
f odd	$f(-\theta) = -f(\theta)$	$a_n = 0$	$b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(\theta)\sin n\theta\, d\theta$

$$f(-\theta) = f(\theta) \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(\theta)\cos n\theta\, d\theta + \int_0^{\pi} f(\theta)\cos n\theta\, d\theta\right]$$

f f cos even

$$u = -\theta \qquad du = -d\theta$$

$$a_n = \frac{1}{\pi}\left[\int_{\pi}^{0} \overbrace{f(-u)}^{f(u)}\, \overbrace{\cos n(-u)}^{\cos nu}\,(-du) + \int_0^{\pi} f(\theta)\cos n\theta\, d\theta\right] = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta$$

Special cases

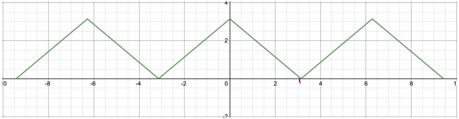
f even	$f(-\theta) = f(\theta)$	$a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(\theta)\cos n\theta \, d\theta$	$b_n = 0$
f odd	$f(-\theta) = -f(\theta)$	$a_n = 0$	$b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(\theta)\sin n\theta \, d\theta$

Compute the Fourier series for the following functions:

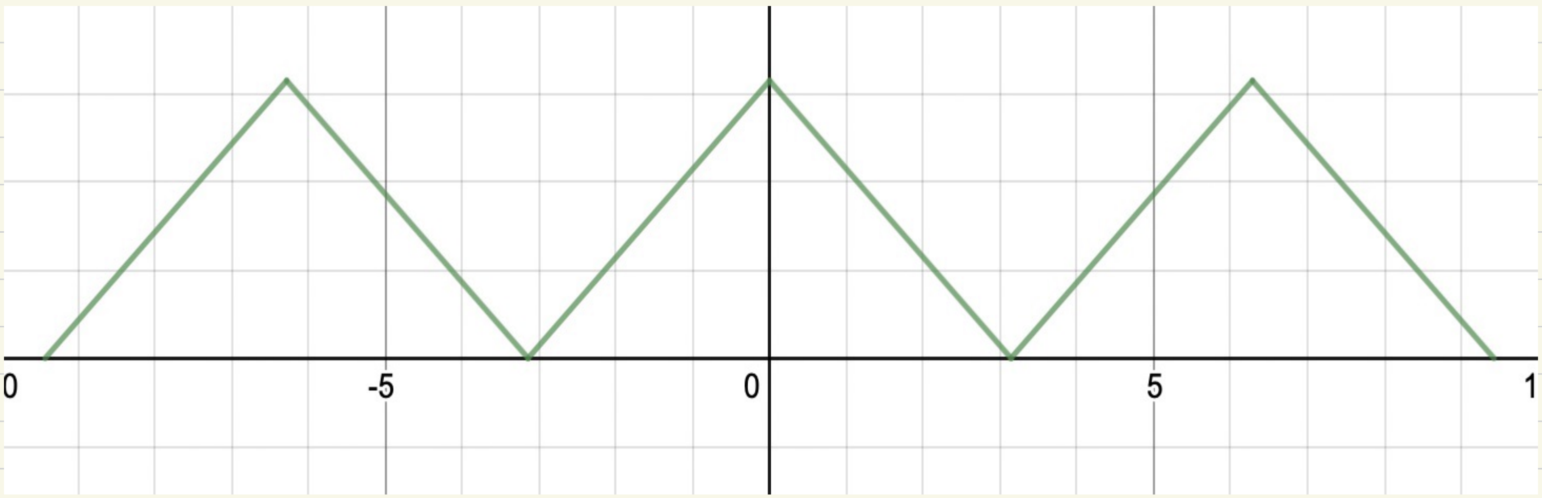
$$f(\theta) = \begin{cases} \pi - \theta & 0 \leq \theta \leq \pi \\ \pi + \theta & -\pi \leq \theta < 0 \end{cases} \qquad f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & -\pi < \theta < 0 \end{cases}$$

Example 1

$$f(\theta) = \begin{cases} \pi - \theta & 0 \leq \theta \leq \pi \\ \pi + \theta & -\pi \leq \theta < 0 \end{cases}$$



① $$f(\theta) = \begin{cases} \pi - \theta & 0 \le \theta \le \pi \\ \pi + \theta & -\pi \le \theta \le 0 \end{cases}$$



<u>Properties</u> : ① $f(-\theta) = f(\theta)$ f even

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \cos n\theta \, d\theta \qquad n = 0, 1, 2$$

$$\int_0^{\pi} f(\theta)\cos n\theta \, d\theta = \int_0^{\pi} (\pi - \theta)\cos n\theta \, d\theta$$

$$= \pi \int_0^{\pi} \cos n\theta \, d\theta - \int_0^{\pi} \theta \cos n\theta \, d\theta$$

If $\boxed{n=0}$ $\quad \pi \int_0^{\pi} d\theta - \int_0^{\pi} \theta \, d\theta = \pi\theta - \frac{\theta^2}{2}\Big|_0^{\pi} = \boxed{\frac{\pi^2}{2}}$

$n \neq 0$ $\quad \cancel{\frac{\pi}{n} \sin n\theta \Big|_0^{\pi}}^{\,0} - \int_0^{\pi} \theta \cos n\theta \, d\theta$ $\quad$ integration by parts

$$= -\cancel{\theta \frac{\sin n\theta}{n}\Big|_0^{\pi}}^{\,0} + \int_0^{\pi} \frac{\sin n\theta}{n} \, d\theta = -\frac{\cos n\theta}{n^2}\Big|_0^{\pi}$$

$n \neq 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta = \frac{2}{\pi}\left(- \frac{\cos n\theta}{n^2}\Big|_0^{\pi} \right)$$

$$a_n = \frac{2}{\pi} \cdot \frac{-(-1)^n + 1}{n^2} = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{\pi n^2} & \text{if } n \text{ odd} \end{cases}$$

$n = 0$ $$a_0 = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi$$

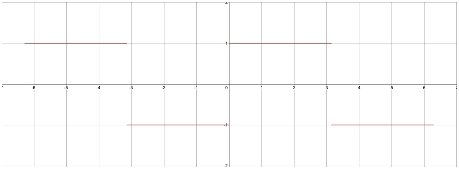
The Fourier series of f is

$$\frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5} \frac{\cos n\theta}{n^2}$$

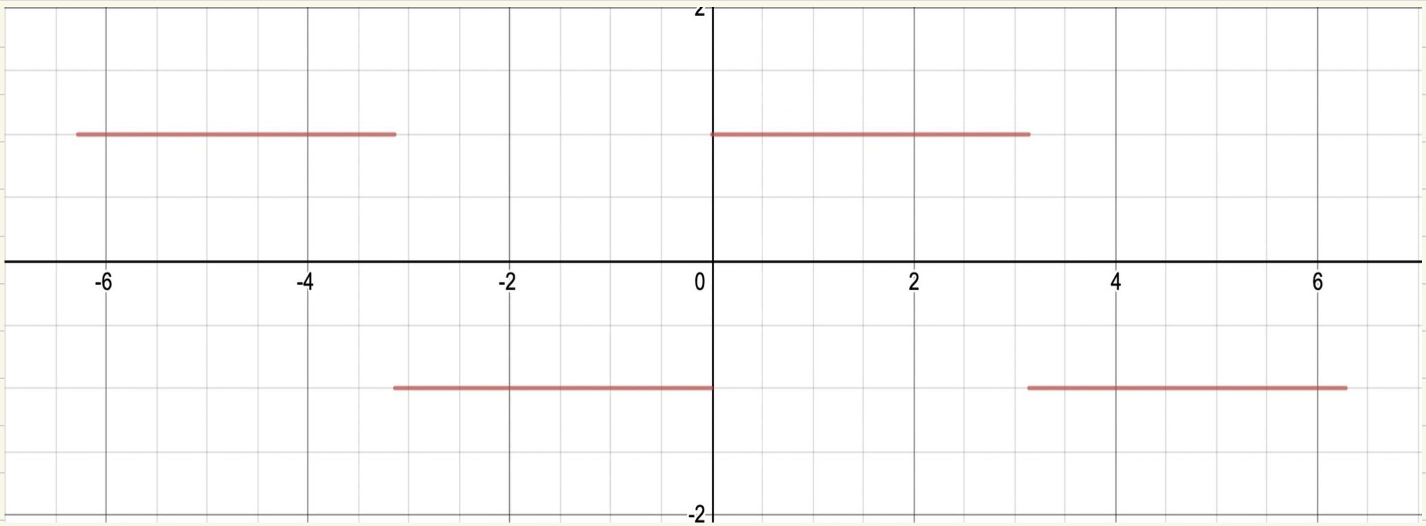
$$\boxed{= \pi/2 + 4/\pi \sum_{k=0}^{\infty} \frac{\cos (2k+1)\theta}{(2k+1)^2}}$$

Example 2

$$f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & -\pi < \theta < 0 \end{cases}$$



(2) $$f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}$$



Properties : $f(-\theta) = -f(\theta)$ $\quad$ f odd

$a_n = 0$ $\quad$ & $\quad$ $b_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(\theta) \sin n\theta \, d\theta$ $\quad$ $n = 1, 2, 3$

$$\int_0^\pi f(\theta)\,\sin n\theta\,d\theta = \int_0^\pi \sin n\theta\,d\theta = -\frac{\cos n\theta}{n}\Big|_0^\pi$$

$$= -\frac{(-1)^n - 1}{n} = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{2}{n} & \text{if } n \text{ odd} \end{cases}$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd} \end{cases}$$

the Fourier series is $\displaystyle \frac{4}{\pi} \sum_{n=1,3,5} \frac{\sin n\theta}{n}$

$$\boxed{\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin (2k+1)\theta}{2k+1}}$$

Does the Fourier series of a periodic function f converge to f?

For $N \in \mathbb{N}$ let

$$S_N^f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos n\theta + b_n \sin n\theta) = \sum_{-N}^{N} c_n e^{in\theta} \tag{7}$$