The isoperimetric problem

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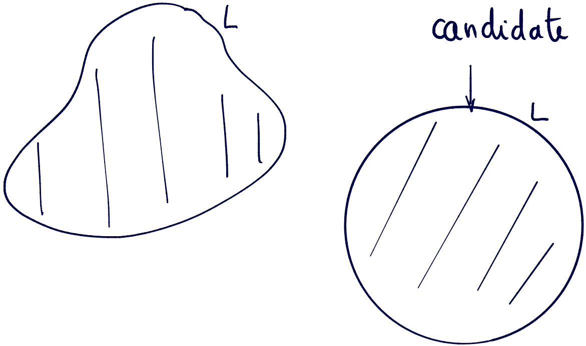
Mathematics Sin Fronteras

Motivation

The **isoperimetric problem**, which dates back to the ancient Greeks, is to determine among all planar figures with fixed perimeter the one with the largest area.



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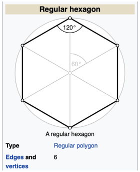
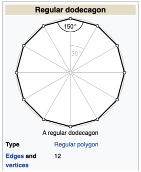
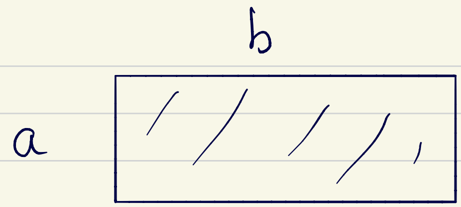
Warm up:

1. Of all rectangles with perimeter 12 which one has maximum area?
2. What is the area of a regular hexagon of side length 2?
3. What is the area of a regular dodecagon of side length 1?





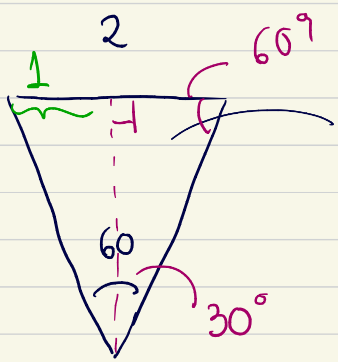
(1)



$$\begin{cases} 2a + 2b = 12 = \text{perimeter} \\ A = ab \end{cases}$$

↑ turn into a function of a then maximize

(2)



find $A_{1/6}$

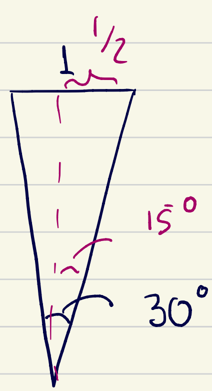
$A = 6A_{1/6}$

equilateral in this case!

$A_{1/6} = \sqrt{3}/2 \times 2$

$A = 6\sqrt{3}$

(3)



$A_{1/12}$ — one triangles

$12 A_{1/12} = A$

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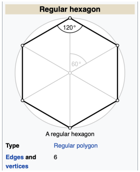

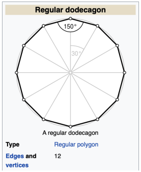

4. What properties of polygons with fixed perimeter seem to maximize area?

The isoperimetric quotient

- The isoperimetric quotient for a given a planar figure of area A and perimeter P is
$$IQ = \frac{4\pi A}{P^2}$$
Note that to maximize the area A of a planar figure with given perimeter P it is enough to maximize IQ.

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why is maximizing A with fixed P the same as minimizing P with fixed A

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- Explain why this is equivalent to minimize the perimeter P among all planar figures of area A.

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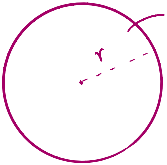
- Explain why this is equivalent to minimize the perimeter P among all planar figures of area A.
- Compute IQ for a disc of area A and perimeter P.



$A = \pi r^2$ $\qquad$ $P = 2\pi r$

$$IQ = \frac{4\pi(\pi r^2)}{(2\pi r)^2} = \frac{4\pi^2 r^2}{4\pi^2 r^2} = 1$$

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- Find a formula for the area for an $2m$-gon with perimeter 12. Compute the corresponding IQ. What happens to the formula as $m \to \infty$?

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- Explain why this is equivalent to minimize the perimeter P among all planar figures of area A.
- Compute IQ for a disc of area A and perimeter P. $IQ(D) = 1$
- Find a formula for the area for an $2m$-gon with perimeter 12. Compute the corresponding IQ. What happens to the formula as $m \to \infty$?
- Do you have a conjecture involving IQ which would prove that our guess is correct? $IQ(\Omega) \leq 1$ & if $IQ(\Omega) = 1 \Rightarrow \Omega = D$!

  planar region

The isoperimetric inequality

Theorem: Given a planar figure of area A and perimeter P

$$4\pi A \leq P^2 \qquad (IQ \leq 1)$$

Equality occurs if and only if the figure is a disc (that is the disc is the solution to the isoperimetric problem).

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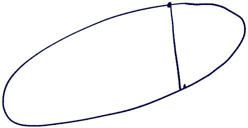
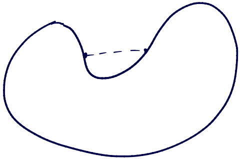
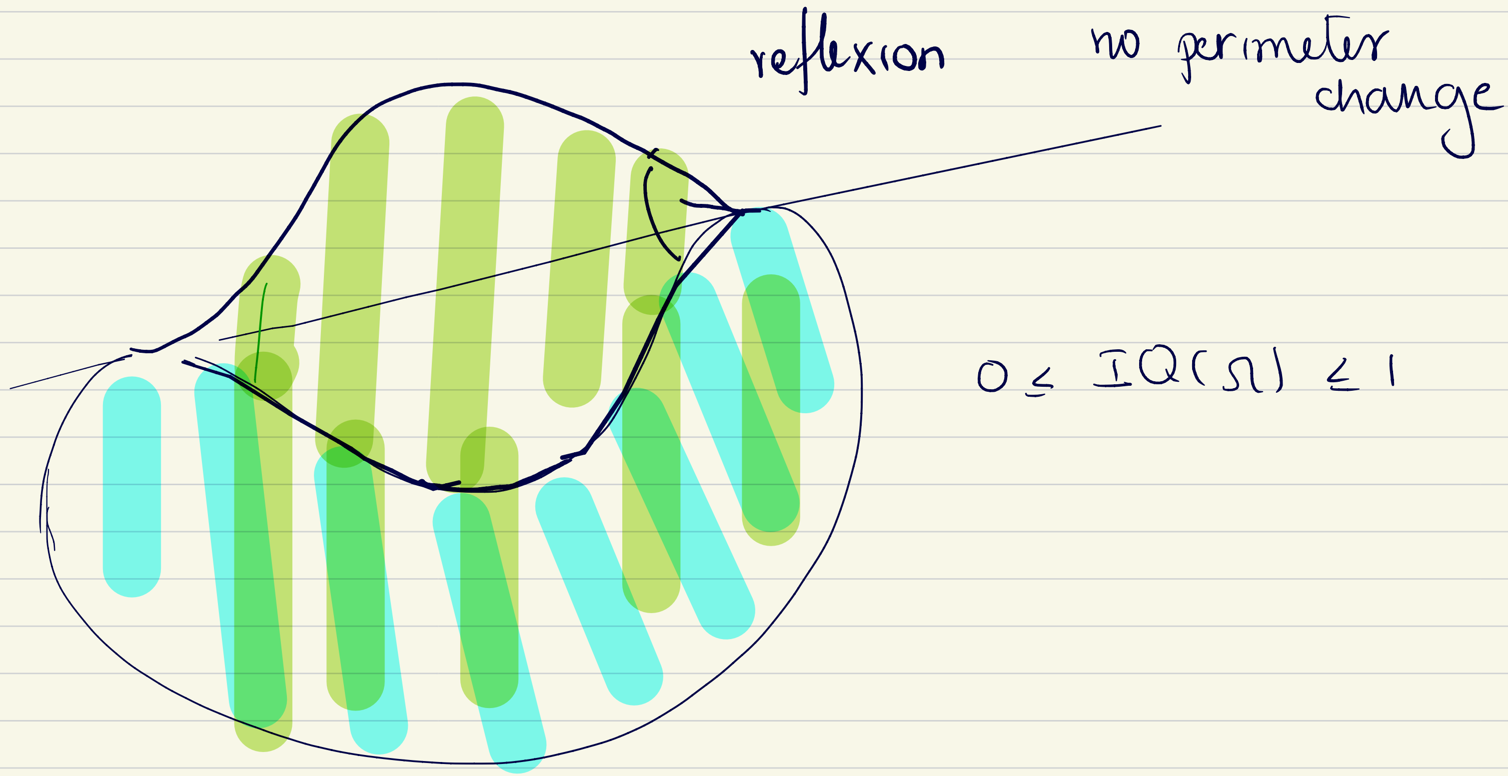
Explain why the shape (which maximizes IQ) should be convex (every line connecting two points on the boundary is contained in the shape).

convex

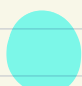
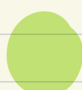


not convex



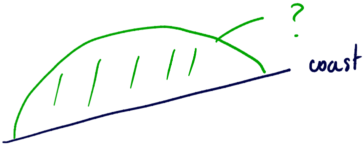


$0 \leq IQ(\Omega) \leq 1$

IQ  $\leq IQ$ 

An application from antiquity

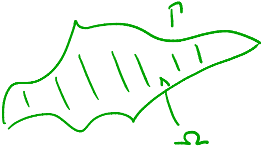
Assuming that we have solved the isoperimetric problem what would you do if you found yourself in Princess Dido's situation? Princess Dido, daughter of a Tyrian king and future founder of Carthage purchased from the North African natives an amount of land along the coastline *not larger than what an oxhide can surround*. She cut the oxhide into strips and made a very long string of length L. And then she faced the geometrical problem of finding the region of maximal area enclosed by a curve, given that she is allowed to use the shoreline as part of the region boundary. In the interior of the continent the answer would be the circle, but on the seashore the problem is different.



Hurwitz's proof using the Wirtinger's inequality

Green's theorem: If p and q are differentiable functions on the plane and Γ is a piecewise C^1 curve bounding the region Ω then

$$\underbrace{\oint_\Gamma p\,dx + q\,dy}_{\text{boundary}} = \underbrace{\int\int_\Omega}_{\text{solid integral}} (q_x - p_y)\,dx\,dy.$$



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If we take $q = x$ and $p = 0$ then Green's theorem says

$$\oint_\Gamma x\,dy = \int\int_\Omega dx\,dy = \text{Area}(\Omega).$$

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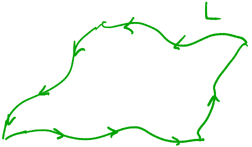
Suppose that the boundary curve Γ has length L and is parameterized by arclength. Thus there are two piecewise C^1 and L periodic functions $x, y : [0, L] \to \mathbb{R}^2$ that satisfy

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1.$$



We convert $x(\cdot)$ and $y(\cdot)$ to 2π periodic functions:

$$f(\theta) = x\left(\frac{L\theta}{2\pi}\right), \qquad g(\theta) = y\left(\frac{L\theta}{2\pi}\right)$$

$$f'(\theta) = \frac{L}{2\pi} x'\left(\frac{L\theta}{2\pi}\right) \qquad g'(\theta) = \frac{L}{2\pi} y'\left(\frac{L\theta}{2\pi}\right)$$

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f, g periodic of period 2π

$s = \frac{L\theta}{2\pi}$ $\qquad ds = \frac{L}{2\pi} d\theta \qquad g'(\theta) = \frac{L}{2\pi} y'(s)$

$$\left(\frac{df}{d\theta}\right)^2 + \left(\frac{dg}{d\theta}\right)^2 = (f')^2 + (g')^2 = \frac{L^2}{4\pi^2}$$

Recall

$$\text{Area}(\Omega) = \oint_\Gamma x\, dy = \int_0^L x(s)\, y'(s)\, ds \underset{s = \frac{L\theta}{2\pi}}{=} \int_0^{2\pi} f(\theta) g'(\theta) d\theta$$

remark $\int_0^{2\pi} g'(\theta)\, d\theta = g(2\pi) - g(0) = 0$

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Recall

$$\left(\int_0^{2\pi} g'(\theta)\,d\theta = 0 \right)$$

$$\begin{aligned} \text{Area}(\Omega) &= \oint_\Gamma x\,dy = \\ &= \int_0^{2\pi} fg'\,d\theta = \int_0^{2\pi} (f(\theta) - \bar{f})\,g'(\theta)\,d\theta \end{aligned}$$

$$\bar{f} = \frac{1}{2\pi}\int_0^{2\pi} f \cdot$$

$$ab = \frac{1}{2}\left(a^2 + b^2 - (a-b)^2\right)$$

$$(f-\bar{f})\,g' = \frac{1}{2}\left((f-\bar{f})^2 + (g')^2 - (f-\bar{f}-g')^2\right)$$

$$A = \int_0^{2\pi} (f-\bar{f})\,g'\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left[(f-\bar{f})^2 + (g')^2 - (f-\bar{f}-g')^2\right] d\theta$$

$$\leq \frac{1}{2}\int_0^{2\pi} \left[(f-\bar{f})^2 + (g')^2\right] d\theta$$

Wirtinger inequality

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise C^1 periodic function with period 2π (i.e. $f(\theta + 2\pi) = f(\theta)$).

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$$\overline{f} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta.$$

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Then

$$\int_0^{2\pi} [f(\theta) - \overline{f}]^2\, d\theta \leq \int_0^{2\pi} [f'(\theta)]^2\, d\theta.$$

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Equality holds if and only if

$$f(\theta) = \overline{f} + a\cos\theta + b\sin\theta$$

for some constants a, b.

End of Hurwitz's proof using the Wirtinger's inequality

$$\begin{aligned} A = \mathrm{Area}(\Omega) &= \frac{1}{2}\int_0^{2\pi} \left([f(\theta)-\overline{f}]^2 + [g'(\theta)]^2 - [f(\theta)-\overline{f}-g'(\theta)]^2\right)\, d\theta \\ &\leq \frac{1}{2}\int_0^{2\pi} \left([f(\theta)-\overline{f}]^2 + [g'(\theta)]^2\right)\, d\theta \end{aligned}$$

End of Hurwitz's proof using the Wirtinger's inequality

$$A = \text{Area}(\Omega) = \frac{1}{2}\int_0^{2\pi} \left([f(\theta) - \overline{f}]^2 + [g'(\theta)]^2 - \underbrace{[f(\theta) - \overline{f} - g'(\theta)]^2}_{0}\right) d\theta$$

$$\leq \frac{1}{2}\int_0^{2\pi} \left([f(\theta) - \overline{f}]^2 + [g'(\theta)]^2\right) d\theta$$

$$\leq \frac{1}{2}\int_0^{2\pi} \left([f'(\theta)]^2 + [g'(\theta)]^2\right) d\theta = \frac{1}{2}\int_0^{2\pi} \frac{L^2}{4\pi^2}\, d\theta = \frac{L^2}{4\pi}$$

$A \leq L^2/4\pi$ $\qquad$ $IQ = \dfrac{4\pi A}{L^2} \leq 1$!

equality iff

① $f(\theta) - \overline{f} - g'(\theta) = 0$ $\quad \forall \theta \in [0, 2\pi]$

② equality Wirtinger's inequality ✓

Wirtinger's

$$f = \bar{f} + a\cos\theta + b\sin\theta$$

$$g'(\theta) = f - \bar{f} = a\cos\theta + b\sin\theta$$

$$g(\theta) = a\sin\theta - b\cos\theta + C$$

$$\bar{g} = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta = \frac{a}{2\pi}\cancel{\int_0^{2\pi}\sin\theta} - \frac{b}{2\pi}\cancel{\int_0^{2\pi}\cos\theta\,d\theta} + C$$

$$0 \qquad\qquad 0$$

$$g(\theta) = a\sin\theta - b\cos\theta + \bar{g}$$

$$f(\theta) = a\cos\theta + b\sin\theta + \bar{f}$$

$$(f'(\theta))^2 + (g'(\theta))^2 = a^2 + b^2 = L^2/4\pi^2$$

$$(f(\theta) - \bar{f})^2 + (g(\theta) - \bar{g})^2 = a^2 + b^2 = L^2/4\pi^2 \quad \blacksquare$$