Now we return to CSF:

Assume that $(x, u_1(x,t))$ and $(x, u_2(x,t))$ are solutions to CSF.

Assume in addition that

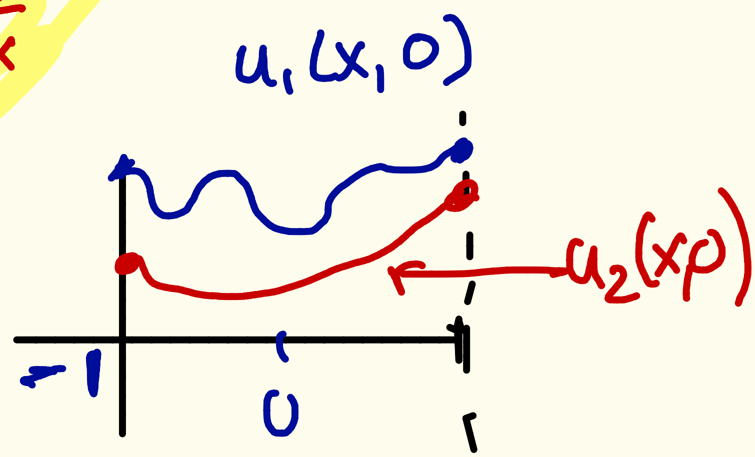
1) $u_1(x,0) > u_2(x,0)$, $x \in [-1,1]$
2) $u_1(\pm 1, t) > u_2(\pm 1, t)$

Then $u_1(x,t) > u_2(x,t)$ for every $x \in [-1,1]$ and $t \in [0,T)$

Exercise: Show that

$$(u_i)_t = \frac{(u_i)_{xx}}{1+(u_i)_x^2}$$

Proof:



$$(u_1)_t = \frac{(u_1)_{xx}}{1+(u_1)_x^2}$$

$$(u_2)_t = \frac{(u_2)_{xx}}{1+(u_2)_x^2}$$

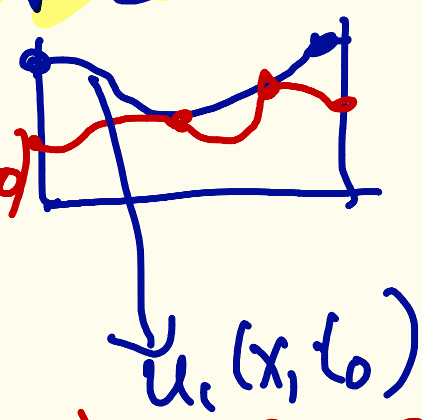
We prove the result by contradiction: we assume there is a first time $t_0>0$ where

$u_1(x_0,t_0) = u_2(x_0,t_0)$

$x_0 \in (-1,1)$

① $u_1(x_0,t) > u_2(x_0,t)$, $\forall\ 0 < t < t_0$

② $u_1(x, t_0) \geq u_2(x,t_0)$



Consider the function

$v(x,t) = u_1(x,t) - u_2(x,t) \geq 0$ for $x \in [-1,1]$, $t \in [0,t_0]$

For t_0 fixed, x_0 is a minimum of $v(\cdot, t_0)$:

$v_x(x_0,t_0)=0 \rightsquigarrow (u_1)_x(x_0,t_0)=(u_2)_x(x_0,t_0)$

$v_{xx} \geq 0 \rightsquigarrow (u_1)_{xx}(x_0,t_0) \geq (u_2)_{xx}(x_0,t_0)$

$(v)_t(x_0,t_0) \leq 0 \Rightarrow (u_1)_t(x_0,t_0) \geq (u_2)_t(x_0,t_0)$

From the evolution equations

$$\underbrace{\left[u_1(x_0,t_0)-u_2(x_0,t_0)\right]_t}_{\geq 0} = \frac{(u_1)_{xx}}{1+(u_1)_x^2} - \frac{(u_2)_{xx}}{1+(u_2)_x^2} \; {+\varepsilon}$$

$$= \frac{(u_1)_{xx}-(u_2)_{xx}}{1+(u_1)_x^2}(x_0,t_0) \geq 0 \quad > 0$$

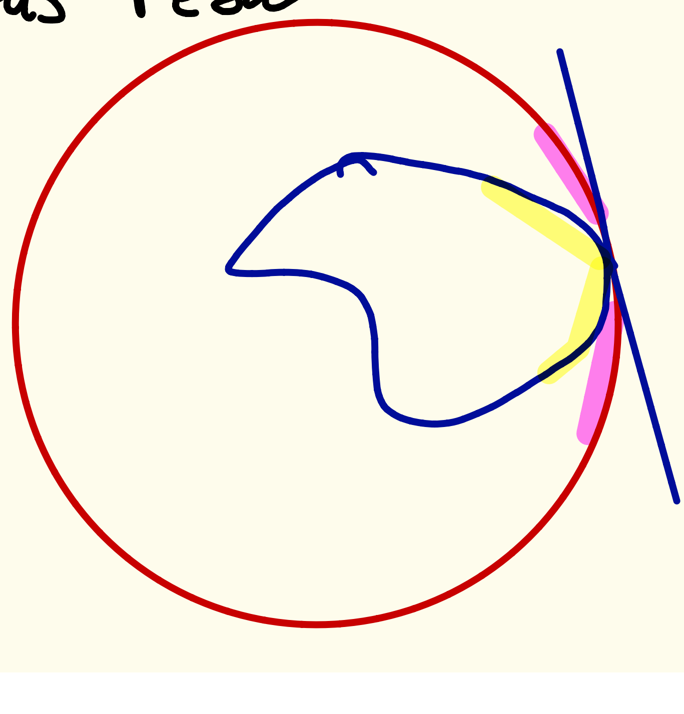
This doesn't allow us to conclude, but we can perturb v

$\rightsquigarrow v_\varepsilon = u_1 - u_2 + \varepsilon t > 0$

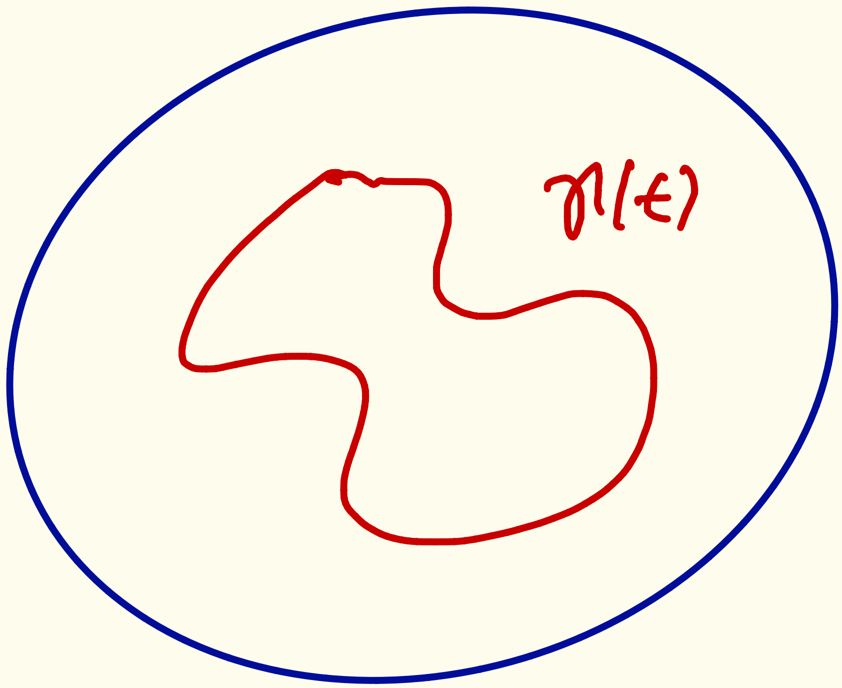
Taking $\varepsilon \to 0 \Rightarrow v(x,t) \geq 0$ //

Theorem If $\gamma_1(x,t)$ and $\gamma_2(x,t)$ are two bounded closed curves such that $\gamma_1(x,0) \cap \gamma_2(x,0) = \phi$. Then $\gamma_1(x,t) \cap \gamma_2(x,t) = \phi$ while the solutions are defined.

Proof: If they touch at some point, we can locally (after rotation and translation) write them as graphs and apply the previous result.



Corollary: If $\gamma(x,t)$ is a compact (bounded) curve the solution can exist at most for a finite point.



Lecture III

Theorem:

Every compact, embedded, C^2 curve converges to a point in finite time

This result was first proved in the mid 80's by a combination of results by Gage & Hamilton and Grayson.

A few alternative proofs have been published later. Here we will analyze one provided by G. Huisken

There is another proof by B. Andrews - P. Bryan

We first check the following simpler result:

Theorem

Let $\gamma : I \times [0,T] \to \mathbb{R}^2$ be an open curve that evolves under curve shortening flow (in its interior). Let

$$d(x,y,t) = |\gamma(x,t) - \gamma(y,t)|$$

$$\ell(x,y,t) = \int_x^y |\gamma'(\lambda,t)|\, d\lambda$$

Assume that $\frac{d}{\ell}$ attains an infimum in the interior at time t_0 then

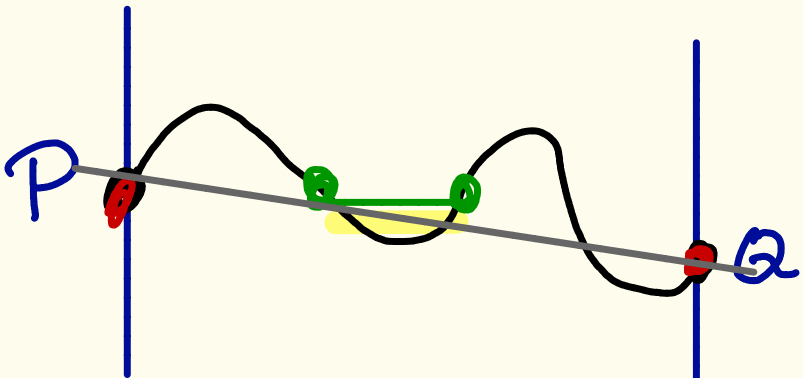
$$\frac{d}{dt}\frac{d}{\ell}(x,y,t_0) \geq 0$$

with equality when γ is a straight line

Remark: In general, $\frac{d}{\ell} \leq 1$, since ℓ is the shortest distance.

The "isoperimetric quantity" $\frac{d}{\ell}$ gives a quantitative measure of how γ differs from a straight line.

Some Remarks



For an open curve we need to specify the behavior at the boundary.

Two standard choices to have a well defined problem

1) To prescribe the points

2) To prescribe an angle with fixed lines

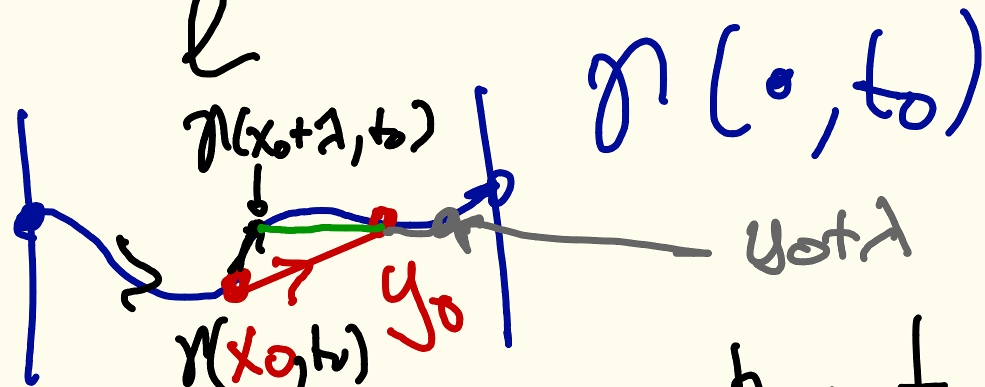
Note that if we fix the end points, then

$$\frac{d}{dt}\left(\frac{d}{\ell}(P,Q,t)\right) = -\frac{d}{\ell^2}\frac{d\ell}{dt} = \frac{d}{\ell^2}\int_P^Q k^2\,ds \geq 0$$

With equality only for a straigth line

Ideas of the proofs

Fix t_0 and pick x_0, y_0 such that $\frac{d}{\ell}(x_0, y_0, t_0)$ is minimum



We can assume without loss of generality that γ is parametrized by arc-length.

Consider

$$f_1(\lambda) = \frac{|\gamma(x_0+\lambda, t_0) - \gamma(y_0, t_0)|}{\ell(x_0+\lambda, y_0, t_0)}$$

$$= \frac{d}{\ell}(x_0+\lambda, y_0, t_0)$$

f_1 tiene un minimo en $\lambda = 0 \Rightarrow (f_1)_\lambda(0) = 0$

$$\frac{d}{d\lambda} |\gamma(x_0+\lambda, t_0) - \gamma(y_0, t_0)|^2 \Big|_{\lambda=0}$$

$$= \frac{d}{d\lambda} \langle \gamma(x_0+\lambda, t_0) - \gamma(y_0, t_0), \gamma(x_0+\lambda, t_0) - \gamma(y_0, t_0) \rangle \Big|_{\lambda=0}$$

$$= 2 \left\langle \frac{d}{d\lambda} \gamma(x_0+\lambda, t_0) \Big|_{\lambda=0}, \gamma(x_0, t_0) - \gamma(y_0, t_0) \right\rangle$$

$$= \cancel{2} \langle \tau(x_0, t_0), \gamma(x_0, t_0) - \gamma(y_0, t_0) \rangle$$

$$= (d^2)_\lambda = \cancel{2} d \, d_\lambda$$

$$\Rightarrow \boxed{d_\lambda = \langle \tau(x_0, t_0), w \rangle} \quad \text{where}$$

$$w = \frac{\gamma(x_0, t_0) - \gamma(y_0, t_0)}{|\gamma(x_0, t_0) - \gamma(y_0, t_0)|}$$

$$|w| = 1$$

On the other hand

$$\ell = \int_{x_0+\lambda}^{y_0} |\underbrace{\gamma_x(s,t)}_{=1}|\,dt = y_0 - x_0 - \lambda$$

in arclenth parameter

$$(\ell)_\lambda = -1$$

$$(f_1)_\lambda(0) = \left(\frac{d}{\ell}\right)_\lambda(0)$$

$$= \frac{(d)_\lambda}{\ell} - \frac{d}{\ell^2}(\ell)_\lambda$$

$$= \frac{\langle \tau(x_0), \omega \rangle}{\ell} + \frac{d}{\ell^2} = 0$$

We can do the same computation for $f_2(\lambda) = \frac{d}{\ell}(x_0, y_0+\lambda, t_0)$

We obtain

$$(f_2)_\lambda = -\frac{\langle w, \tau(y_0, t_0)\rangle}{\ell} - \frac{d}{\ell^2}\cdot 1 = 0$$

$$\langle w, \tau(x_0)\rangle - \langle w, \tau(y_0) = 0$$

$$\langle w, \tau(x_0) - \tau(y_0)\rangle = 0$$

$|w| = 1$, $w \neq 0$

There are two cases

1. $\tau(x_0, t_0) = \tau(y_0, t_0)$
2. $\tau(x_0, t_0) - \tau(y_0, t_0) \neq 0$ and $\tau(x_0, t_0) - \tau(y_0, t_0) \perp w$

<u>Case 1</u>: Consider

$$g_1(\lambda) = \frac{d}{\ell}(x_0+\lambda,\ y_0+\lambda,\ t_0)$$

g_1 has a minimum at $\lambda = 0$

$$\Rightarrow (g_1)_\lambda = 0 \ \wedge\ (g_1)_{\lambda\lambda} \geq 0$$

$$\underbrace{(g_1)_{\lambda\lambda}(0)}_{\leq 0} = \frac{\langle w, k\nu(x_0,t_0) - k\nu(y_0,t_0)\rangle}{\ell}$$

$$\overset{?}{=} \frac{\langle w, \frac{d}{dt}(\gamma(x_0,t) - \gamma(y_0,t))\rangle}{\ell}\Big|_{t=t_0}$$

from CSF

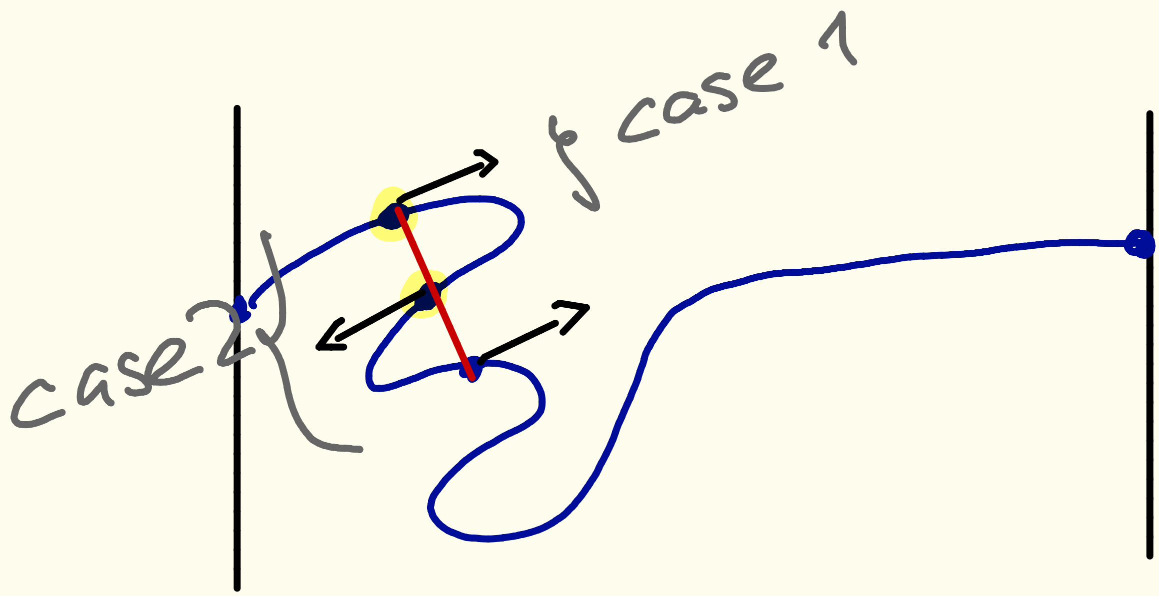
$$\left(\frac{d}{\ell}\right)_t = \frac{d_t}{\ell} - \frac{d}{\ell^2}\ell_t = \frac{\hat{d}_t}{\ell} + \frac{d}{\ell^2}\int_{x_0}^{y_0} k^2$$

$$\underset{\geq 0}{=} \frac{\langle w, k\nu(x_0,t_0) - k\nu(y_0,t_0)\rangle}{\ell} + \frac{d}{\ell^2}\int k^2$$

Case 2:

$$g_2(\lambda) = \frac{d}{\ell}(x_0 + \lambda, y_0 - \lambda, t_0)$$

and the same result follows


case 1
case 2

For our main theorem a similar idea can be used but we need to consider a different "isoperimetric profile"

Instead of $\frac{d}{\ell}$ we use $\frac{d}{\psi(\ell)}$ where

$$\psi(\ell) = \frac{L(t)}{\pi} \sin\left[\frac{\ell \pi}{L(t)}\right]$$

where $L(t)$ is the total length at time t.

The computation is similar and we leave it to the interested reader