Lecture II

In the last lecture we computed

$$\frac{d}{dt} L(\gamma + t\varphi\nu)\Big|_{t=0} = -\int k\varphi\, ds = -\int k^2 ds \leq 0$$

If $\varphi = k$ then this is decreasing (and it is the fastest decreasing direction)

Def: The curve shortening flow is the deformation of a curve in the normal direction with a speed equal to its curvature.

More precisely, we consider

$\gamma : I \times [0, \infty) \longrightarrow \mathbb{R}^2$

$$\frac{d\gamma}{dt} = k\nu = \gamma_{ss}$$

k is the signed curvature

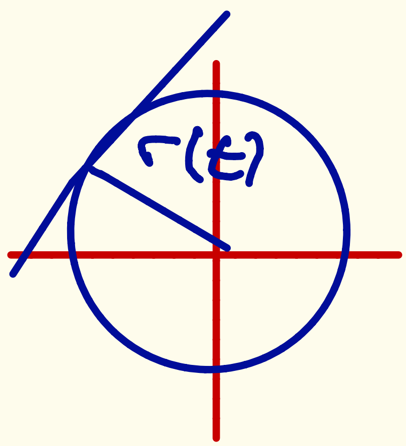
Remark: For each t $\gamma(\cdot, t)$ is a curve.

Example 1 :

$\gamma(\theta, t)$
$\parallel$

Assume that $r(t)(\cos\theta, \sin\theta) = \gamma(\theta,t)$ is a solution to curve shortening flow. Find $r(t)$.

$$\frac{d\gamma}{dt} = r'(t)\,(\cos\theta, \operatorname{sen}\theta)$$



$$\to K = \frac{1}{r(t)}$$

$$\nu = -(\cos\theta, \operatorname{sen}\theta)$$

$$\Rightarrow r'(t)\,(\cos\theta, \operatorname{sen}\theta) = -\frac{1}{r}\,(\cos\theta, \operatorname{sen}\theta)$$

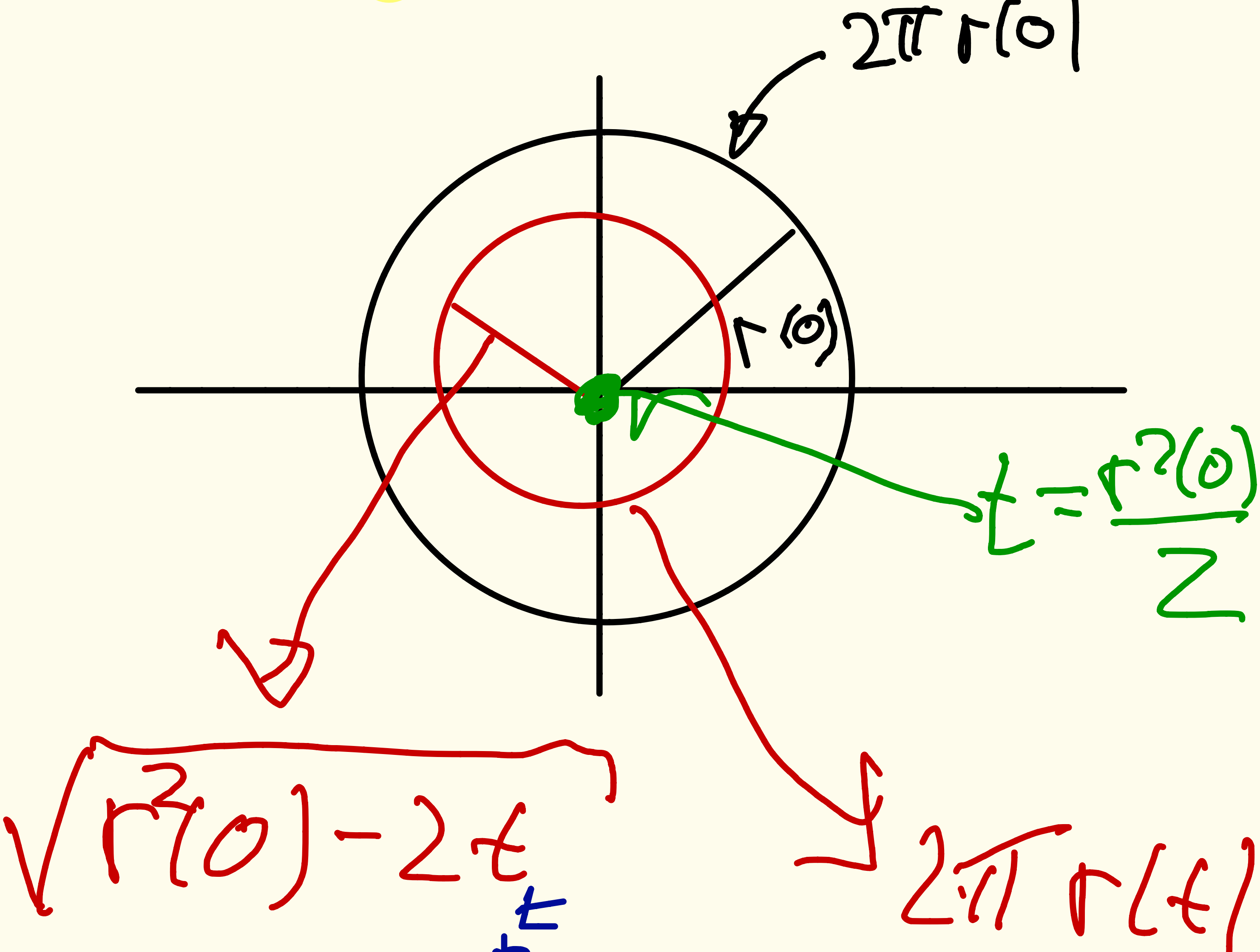
$$\Rightarrow (r^2)' = 2rr' = -2 \;\Big/ \int_0^t d\sigma$$

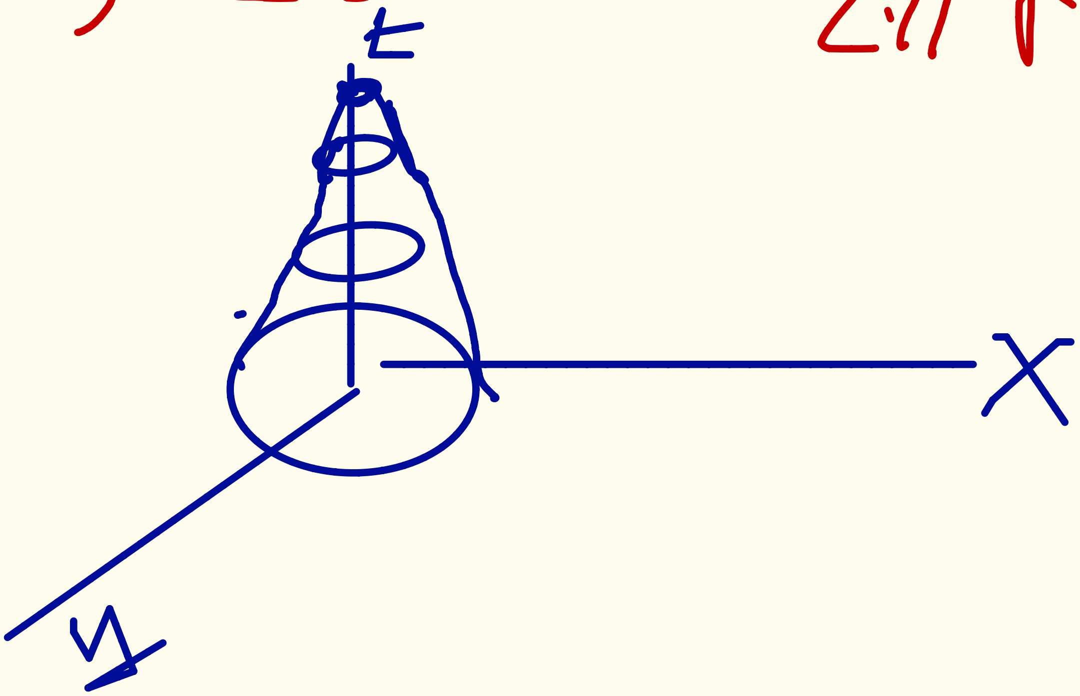
$$\Rightarrow r^2(t) - r^2(0) = -2t$$

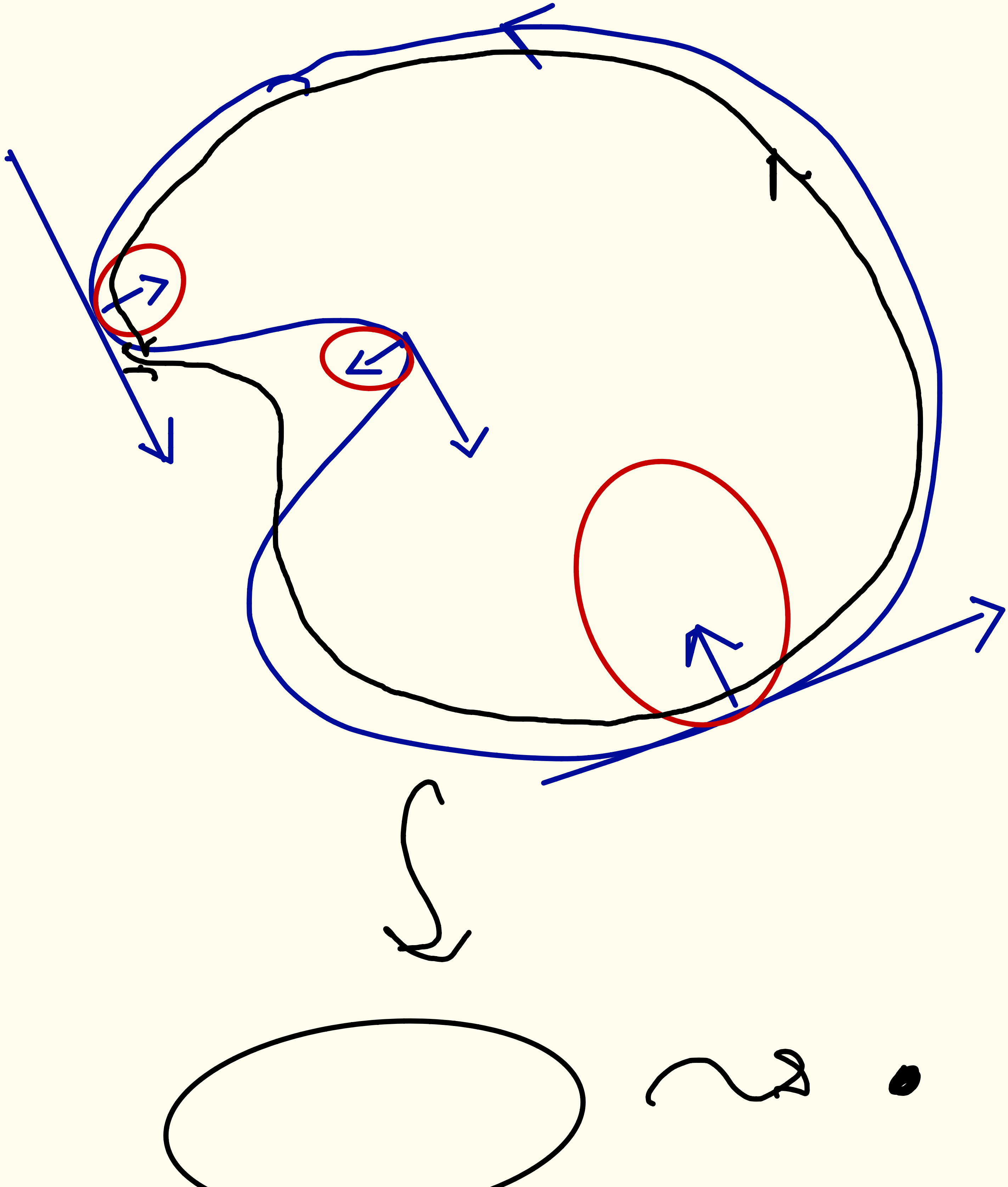
$$r(t) = \sqrt{r^2(0) - 2t}$$

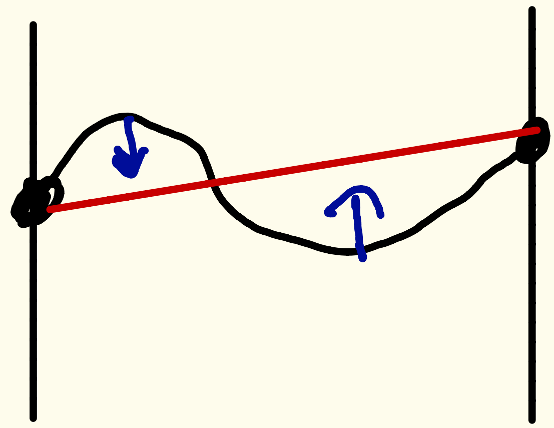
$$\to \gamma(\theta, t) = \sqrt{r^2(0) - 2t}\,(\cos\theta, \operatorname{sen}\theta)$$

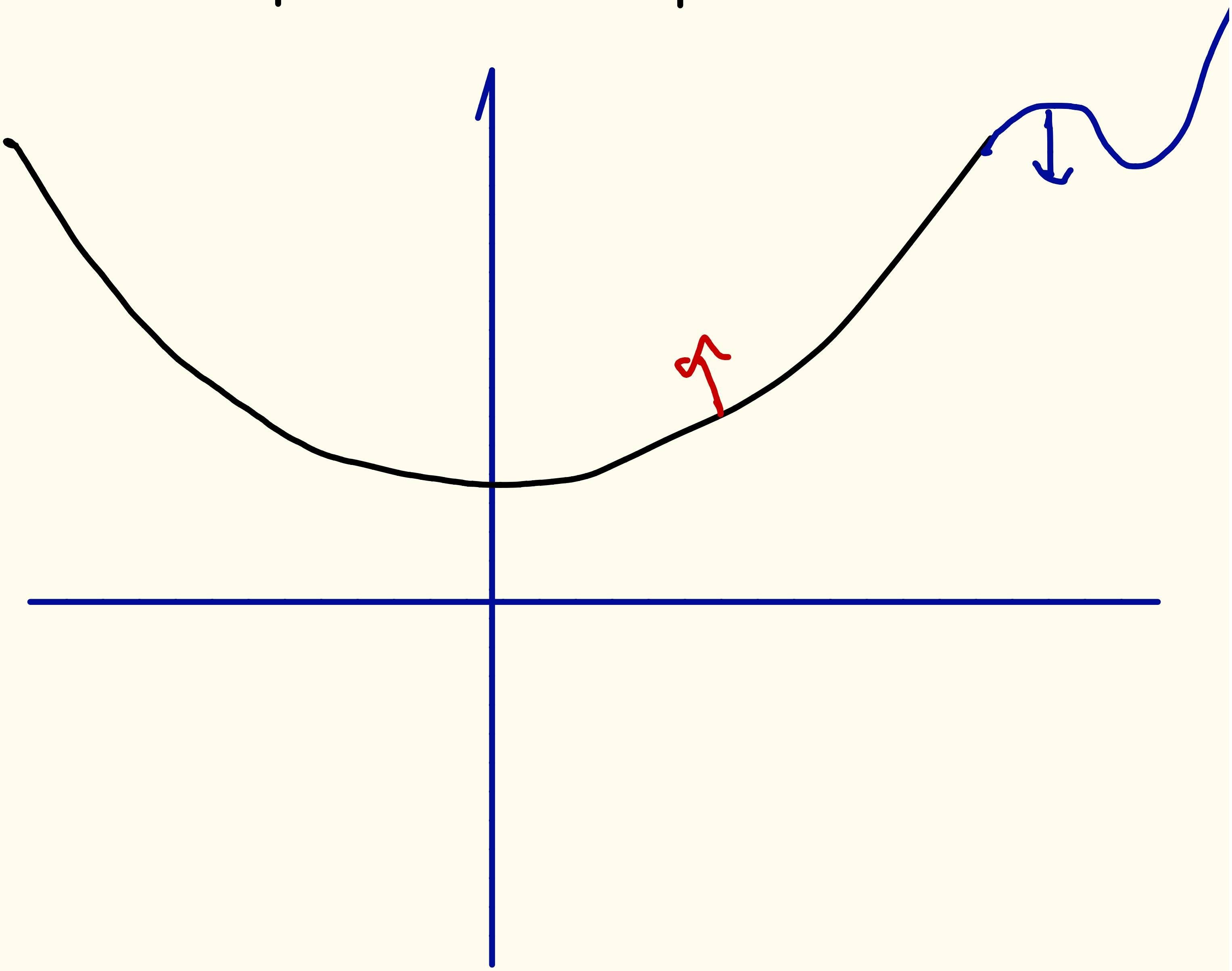
What does the previous example represent?











Another example:

We look for a solution of the form $(x, u(x)+t) = \gamma(x,t)$

$$\frac{d\gamma}{dt} = (0,1) \overset{!}{=} k\cdot\nu \quad /\cdot\nu$$

$$\Leftrightarrow \overset{\overset{!}{=}\frac{1}{\sqrt{1+u_x^2}}}{(0,1)\cdot\nu} = k = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$s(x) = \int_0^x |\gamma_x(\lambda,t)|\,d\lambda$$

$$= \int_0^x |(1, u_x)|\,d\lambda = \int_0^x \sqrt{1+u_x^2}(\lambda,t)\,d\lambda$$

$$\underbrace{\frac{ds}{ds}}_{=1} = \underbrace{\sqrt{1+u_x^2}}_{\left(\frac{d}{dx}\int_0^x\right)}\cdot\frac{dx}{ds}$$

$$\gamma_s(x,t) = \gamma_x(x,t)\cdot\frac{dx}{ds}$$ ← chain rule

$$= \frac{(1, u_x)}{\sqrt{1+u_x^2}} = \tau \rightsquigarrow \nu = \frac{(-u_x, 1)}{\sqrt{1+u_x^2}}$$

$$\gamma_{ss} = \left[\frac{(0, u_{xx})}{\sqrt{1+u_x^2}} - \tau\cdot\frac{u_{xx}u_x}{(1+u_x^2)}\right]\frac{dx}{ds} \rightsquigarrow k = \gamma_{ss}\cdot\nu = \frac{u_{xx}}{(1+u_x)^{3/2}}$$

$$\frac{1}{\sqrt{1+u_x^2}} = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$\rightsquigarrow \frac{u_{xx}}{1+u_x^2} = 1$$

$$\rightsquigarrow (\arctan u_x)_x = 1 \quad \Big/ \int dx$$

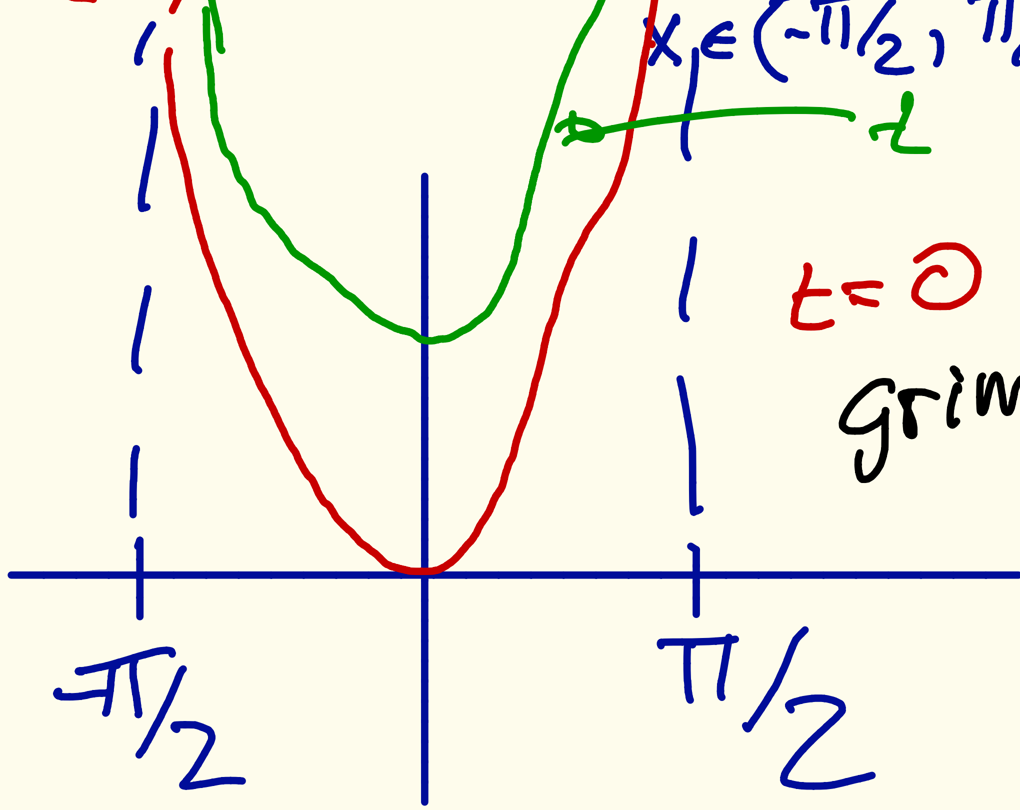
$$\arctan u_x = x$$

$$\Rightarrow \frac{du}{dx} = \tan x = \frac{\operatorname{sen} x}{\cos x}$$

$$u(x) = -\ln(\cos x)$$

$$\rightsquigarrow (x, -\ln(\cos x) + t)$$

$$x \in (-\pi/2, \pi/2)$$



grim reaper

The maximum principle

- We change now the topic for a bit.

Assume that we have a solution to an ODE of the form

$$-af'' + bf' + c(x) = 0 \quad \text{for } x\in(0,1)$$

Assume $a>0$ and $c>0$.
<0

Claim: f does not have a maximum (minimum) in the interior

Proof: If f has a maximum at x_0

$f'(x_0)=0, \quad f''(x_0)\le 0$

$$0 < c(x_0) = +af''(x_0) + b\underbrace{\cancel{f'}(x_0)}_{0} \le 0$$

This contradicts that $c>0$

→← //

Remark: The same statement is true for the minimum if $c<0$

Claim: If f satisfies the same as before but $c \geq 0$, the same statements hold.

Idea of the proof:
Check the equation satified by $f_\varepsilon = f(x) + \varepsilon e^{Lx}$

$$f_\varepsilon'(x) = f' + \varepsilon L e^{Lx}$$
$$f_\varepsilon''(x) = f'' + \varepsilon L^2 e^{Lx}$$
$$-a f_\varepsilon'' + b f_\varepsilon' = -a f'' + b f' + (\varepsilon L b - \varepsilon L^2 a) e^{Lx}$$
$$= \underbrace{-c + (\varepsilon L b - \varepsilon L^2 a) e^{Lx}}_{-c_\varepsilon}$$

Eligiendo L adecuado $c_\varepsilon > 0$

$\Rightarrow f_\varepsilon$ cumple las hipótesis del caso anterior $f_\varepsilon(x) \leq \max\{f_\varepsilon(0), f_\varepsilon(1)\}$

$\rightsquigarrow$ we conclude by taking $\varepsilon \to 0$ //

An application of the maximum principle

Assume that f_1 and f_2 satisfy:

$$-a f_i'' + b f_i' + c = 0 \quad \text{and} \quad \begin{matrix} f_1(0) = f_2(0) \\ f_1(1) = f_2(1) \end{matrix}$$

Then: $f_1 \equiv f_2$

<u>Proof</u>:

Check the equation satisfied

$$g = f_1 - f_2$$

$$g' = f_1' - f_2' \quad ; \quad g'' = f_1'' - f_2''$$

$$\rightsquigarrow -a g'' + b g' = (-a f_1'' + b f_1') - (-a f_2'' + b f_2')$$

$$= -c - (-c) = 0$$

$$g(0) = f_1(0) - f_2(0) = 0$$

$$g(1) = f_1(1) - f_2(1) = 0$$

$$\max g = 0 = \min g$$

$$\Rightarrow g \equiv 0 \Rightarrow f_1 = f_2 \quad //$$

We will be interested in equations of the form

$$-a f_{xx} + b f_x + c = f_t$$

Following the ideas before, we can define

$$\tilde{f}(t) = \max_{x\in[0,1]} f(x,t)$$

If $\tilde{f}$ is regular enough we would have

$$c \leq \tilde{f}_t \quad \left(\text{or } \tilde{f} = f(1,t) \text{ or } \tilde{f} = (0,t)\right)$$

In particular, if $c \geq 0$, $\tilde{f}$ is increasing

Similarly we can define

$$\underset{\sim}{f} = \min_{x\in[0,1]} f(x,t)$$

and

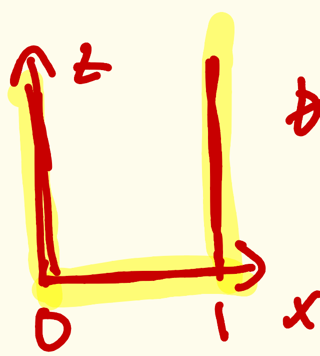
$$\tilde{f}_t \leq c \quad \left(\text{or } \underset{\sim}{f} = f(0,t) \text{ or } \underset{\sim}{f} = f(1,t)\right)$$

If $c \leq 0$, $\underset{\sim}{f}$ is decreasing.

Then

$$\max_{x,t} f(x,t) = \max\left\{ \max_{x} f(x,0),\ \max_{t}\{f(0,t), f(1,t)\} \right\}$$

Rem



this is usually called the parabolic boundary

Now we return to CSF:

Assume that $(x, u_1(x,t))$ and $(x, u_2(x,t))$ are solutions to CSF.

Assume in addition that

1) $u_1(x,0) > u_2(x,0)$, $x \in [-1,1]$
2) $u_1(\pm 1, t) > u_2(\pm 1, t)$

Then $u_1(x,t) > u_2(x,t)$ for every $x \in [-1,1]$ and $t \in [0,T)$

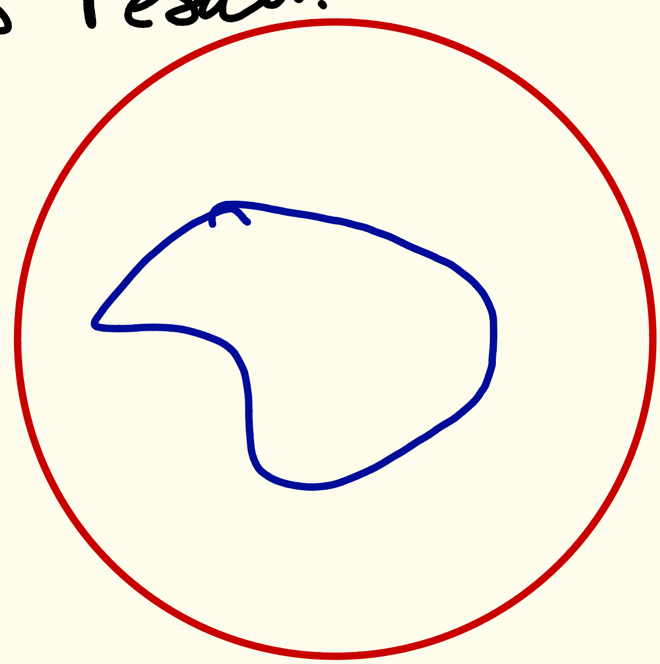
Exercise: Show that

$$(u_i)_t = \frac{(u_i)_{xx}}{1+(u_i)_x^2}$$

Proof:

Theorem If $\gamma_1(x,t)$ and $\gamma_2(x,t)$ are two bounded closed curves such that $\gamma_1(x,0) \cap \gamma_2(x,0) = \phi$ Then $\gamma_1(x,t) \cap \gamma_2(x,t) = \phi$ while the solutions are defined.

Proof: If they touch at some point, we can locally (after rotation and translation) write them as graphs and apply the previous result.



Corollary: If $\gamma(x,t)$ is a compact (bounded) curve the solution can exist at most for a finite point.