Deforming geometry in time

Lecture 1

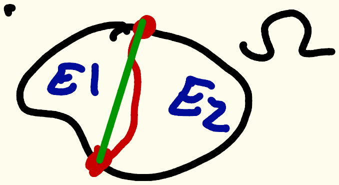
Contents:

- What is a curve?
- How do we compute the length of a curve
- Geometric quantities: arc-length parameter, tangent vector
- What is curvature?
- Why is curvature a natural quantity?

Goals of these lectures

Geometric flows: Deform continuously in time a geometric object to achieve an "optimal" shape

Ex:



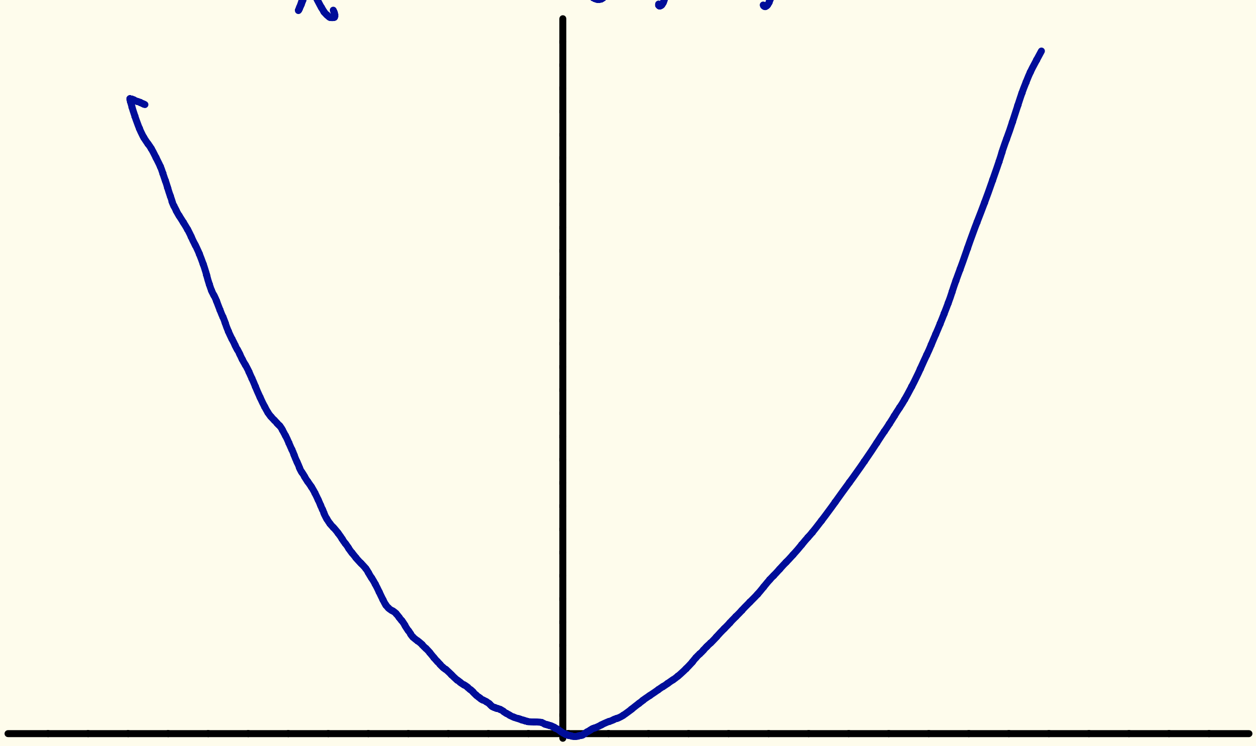
What is a curve?

<u>Def</u>: A curve (on M) is a function $\gamma : I \to M$, with $I \subseteq \mathbb{R}$.

In general, "I" will be an interval on $\mathbb{R}$.

<u>Example 1</u>

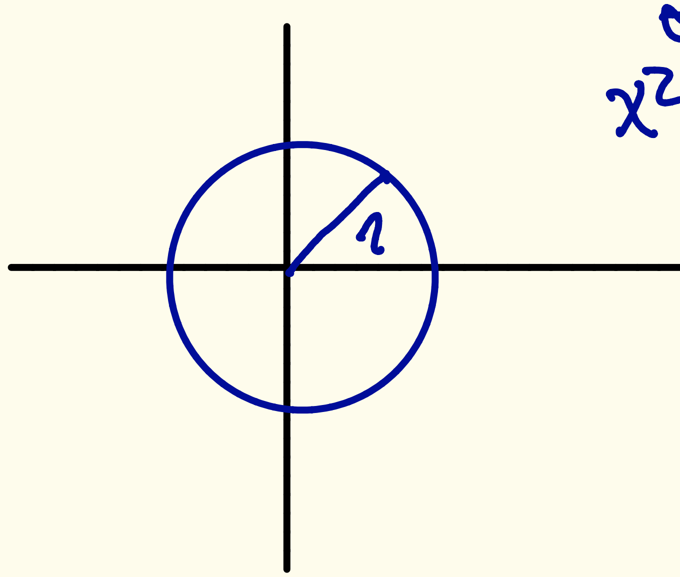
$$\gamma : \mathbb{R} \to \mathbb{R}^2$$
$$x \to (x, x^2)$$



Example 2:

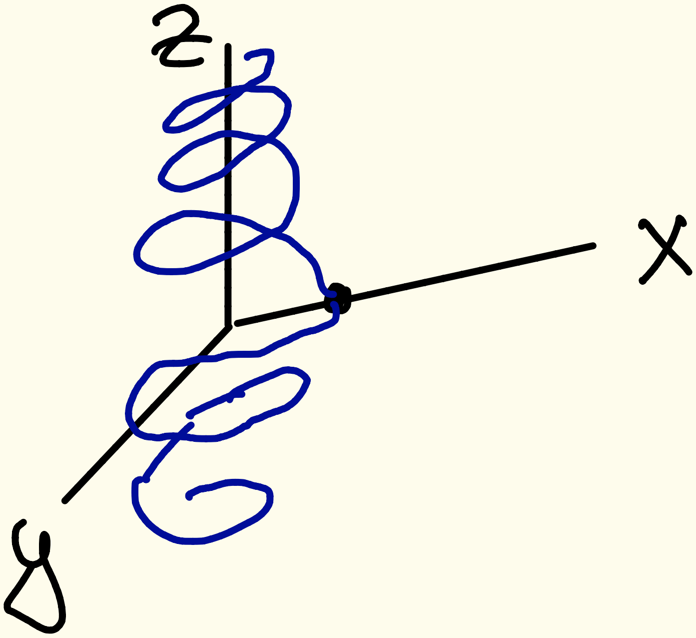
$$\gamma : [0, 2\pi) \longrightarrow \mathbb{R}^2$$
$$\theta \longrightarrow (\cos\theta, \sin\theta)$$

$$x^2 + y^2 = 1$$



Example 3:

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$t \longrightarrow (\cos t, \sin t, t)$$

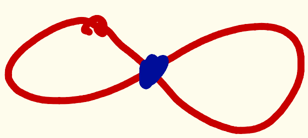


Def: A curve $\gamma : [a,b] \to M$ is closed if $\gamma(a) = \gamma(b)$

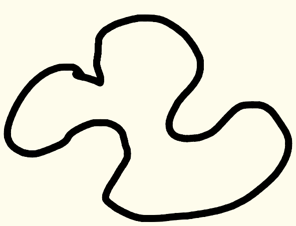
Def: A curve is embedded if it doesn't self intesect. That is, the function is injective.

Otherwise, we say is ireersed

Examples



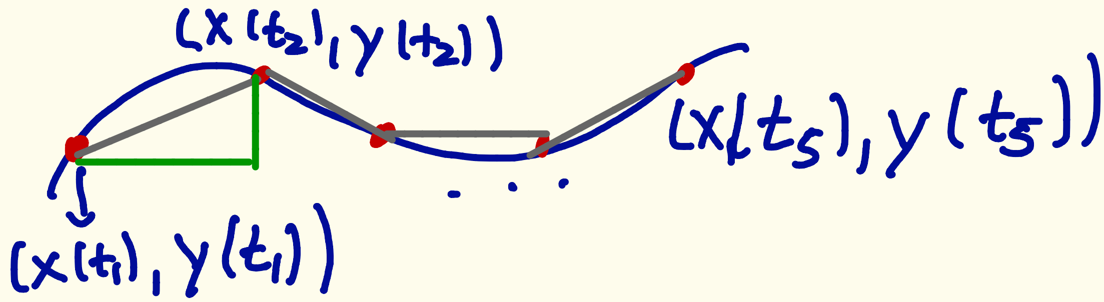
inmersed but not embedded



embedded

How do we compute the length?

Intuition:



$$\Delta x_i = x(t_{i+1}) - x(t_i)$$

$$\Delta y_i = y(t_{i+1}) - y(t_i)$$

$$L \sim \sum_{i=1}^{4} \underbrace{\frac{\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}}{\Delta t}}_{\sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2}} \quad \underbrace{\Delta t}_{\text{"}dt\text{"}}$$

$$\frac{\Delta x_i}{\Delta t} = \frac{x(t_i + \Delta t) - x(t_i)}{\Delta t} \xrightarrow[\Delta t \to 0]{} x'(t)$$

$$\frac{\Delta y}{\Delta t} \xrightarrow[\Delta t \to 0]{} y'(t)$$

$$\longrightarrow \int_{t_0}^{T} \sqrt{(x')^2 + (y')^2}\, dt$$

Def: Consider $\gamma : [a,b] \to \mathbb{R}^2$

The length of γ is defined by

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt$$

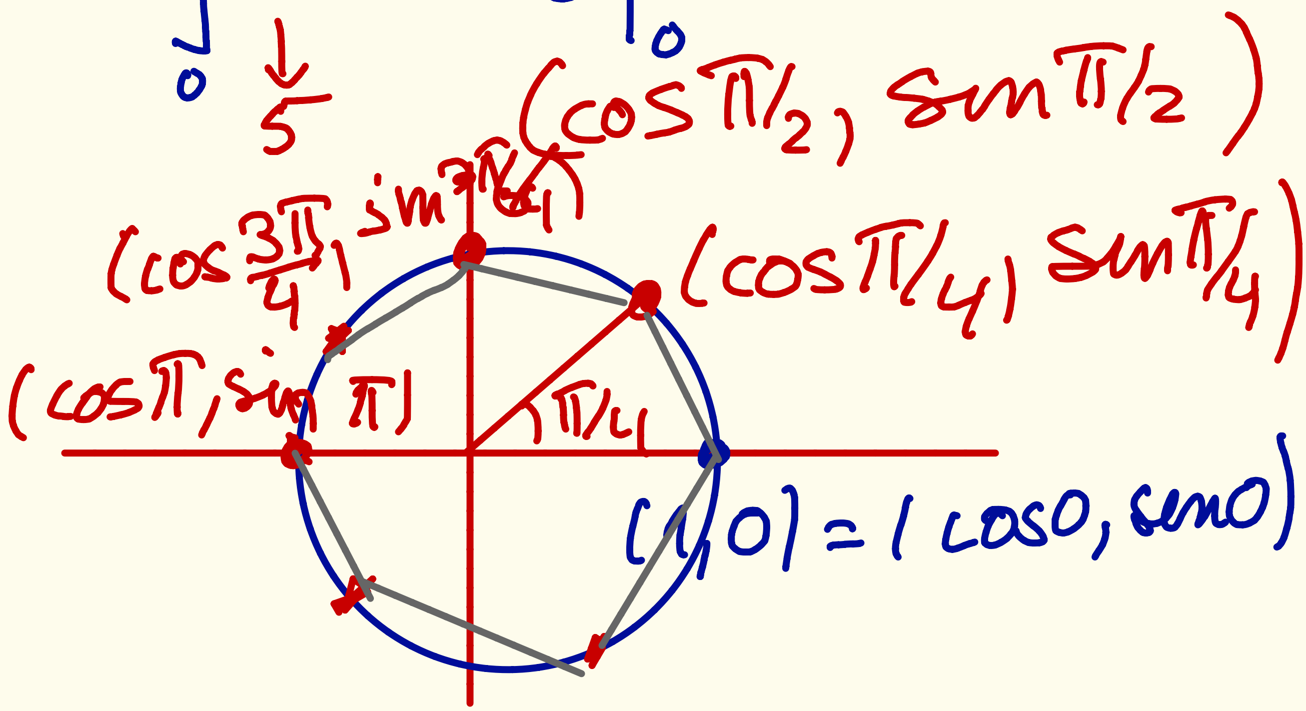
Examples:

① $\gamma(\theta) = 5(\cos\theta, \operatorname{sen}\theta) \quad \theta \in [0, 2\pi)$

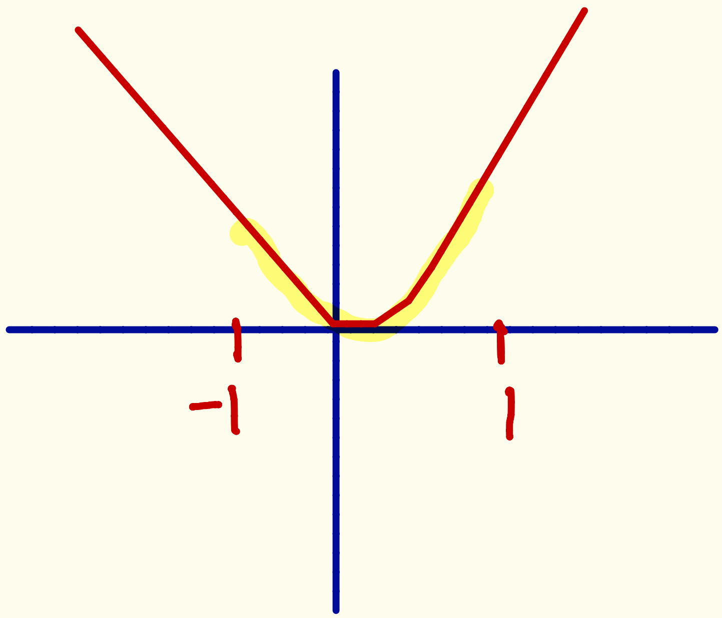
$\gamma'(\theta) = 5(-\operatorname{sen}\theta, \cos\theta)$

$|\gamma'(\theta)| = 5\sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = 1$

$$\int_0^{2\pi} \underset{\downarrow\, 5}{1}\, d\theta = \theta\Big|_0^{2\pi} = 2\pi \cdot 5$$



Ex 2: $\gamma(x) = (x, x^2)$, $x \in [-1, 1]$



$$\gamma'(x) = (1, 2x)$$

$$|\gamma'(x)| = \sqrt{1 + (2x)^2}$$

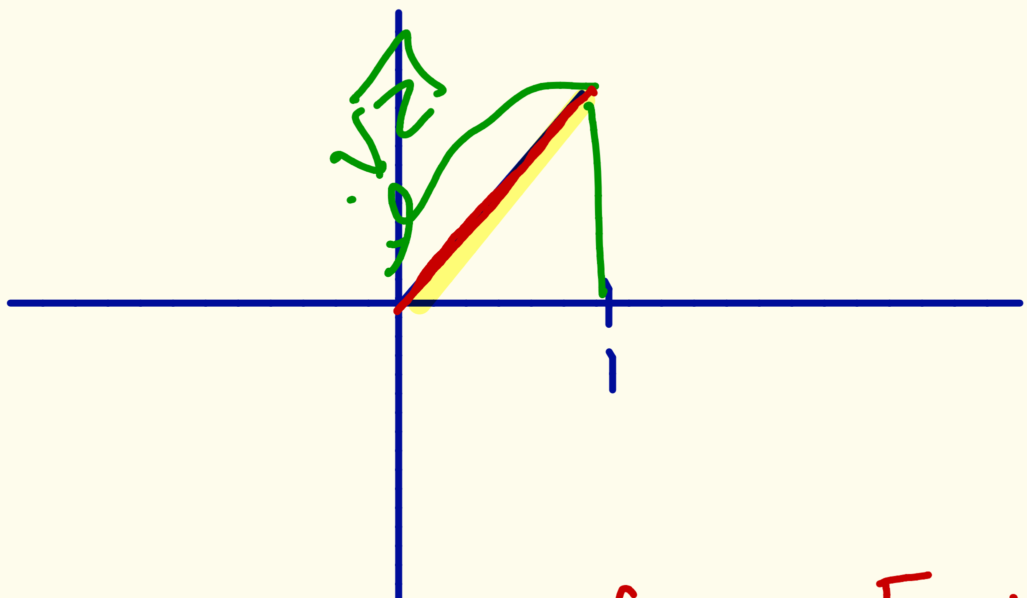
$$\text{Length} = \int_{-1}^{1} \sqrt{1 + (2x)^2}\, dx$$

Rmk: $(x, f(x))$

$$\text{Length} = \int_a^b \sqrt{1 + (f')^2(x)}\, dx$$

$\gamma(x) = (|x|, |x|) \qquad x \in [-1, 1]$

$$= \begin{cases} (x, x) & \text{if } x \in [0,1] \\ (-x, -x) & \text{if } x \in [-1, 0] \end{cases}$$



$$\gamma' = \begin{cases} (1, 1) & \text{if } x \in [0, 1] \\ (-1, -1), & \text{if } x \in [-1, 0] \end{cases}$$

$$\text{length} = \int_{0} \sqrt{2}\, dx + \int_{-1}^{0} \sqrt{2}\, dx$$

$$= 2\sqrt{2}$$

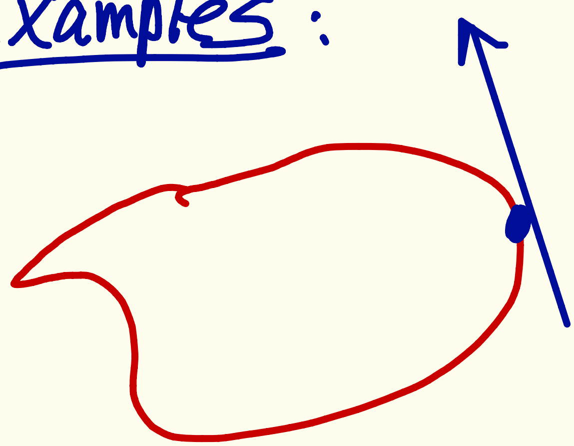
Some geometric quantities

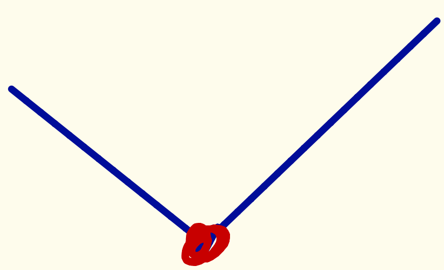
<u>Def</u>: Let $\gamma : [a,b] \to \mathbb{R}^2$ be a curve. The unit tangent vector at $t_0 \in (a,b)$ is defined by

$$\boxed{\frac{\gamma'(t_0)}{|\gamma'(t_0)|} = \tau(t_0)}$$

<u>Rmk</u>: This is only well defined if $\gamma'(t_0) \neq 0$. Points at which this holds are called regular

<u>Examples</u>:





What is the parametrization of a curve?

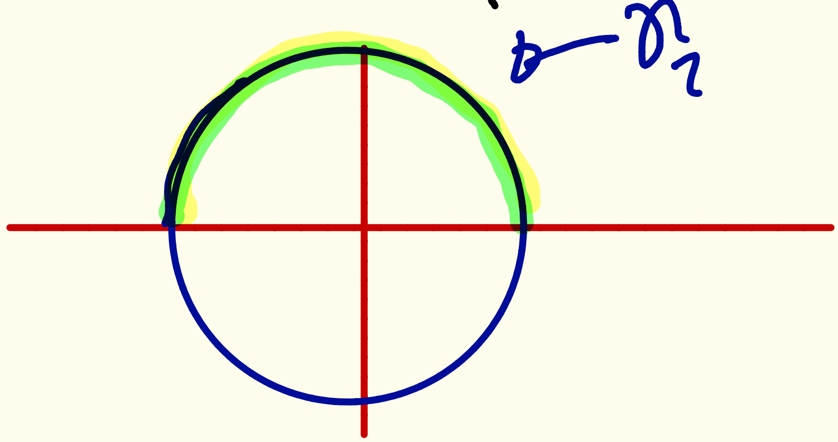
The same geometric object can be described in different ways.

Example

$\gamma_1 : [0,\pi] \to \mathbb{R}^2$
$\theta \to (\cos\theta, \sin\theta)$

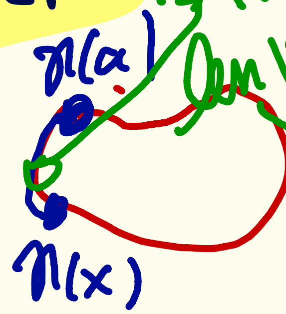
$\gamma_2 : [-1,1] \to \mathbb{R}^2$
$x \to (x, \sqrt{1-x^2})$

A choice of description is called a "parametrization" and the curve will be from now on the geometric object, that is the set $\{\gamma(t) : t \in [a,b]\}$



The arc-length parameter

Let $\gamma : [a,b] \longrightarrow \mathbb{R}^2$
$\qquad\qquad x \longrightarrow \gamma(x)$



We define the function

$$s(x) = \int_a^x |\gamma'(\lambda)|\, d\lambda \rightsquigarrow \frac{ds}{dx} = |\gamma'(x)| \geq 0$$

$s : [a, b] \longrightarrow [0, L(\gamma)]$ is increasing. This implies that s is invertible and we can use it to parametrize γ.

$\gamma : [0, L(\gamma)] \longrightarrow \mathbb{R}^2$
$\qquad\qquad s \longrightarrow \gamma(x(s))$

this is the inverse of s

This called the arc-length parametrization.

<u>Rmk</u>: $\dfrac{d\gamma}{ds} = \dfrac{d\gamma}{dx} \cdot \dfrac{dx}{ds} = \dfrac{\gamma'(x)}{|\gamma'(x)|} = \tau(s)$

Examples:

① $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$\theta \longrightarrow 2(\cos\theta, \sin\theta)$

$\gamma'(\theta) = 2(-\sin\theta, \cos\theta) \rightsquigarrow |\gamma'(\theta)| = 2$

$s(\theta) = \int_0^\theta 2\,d\phi = 2\theta \rightsquigarrow \theta = s/2$

$\gamma(s) = 2(\cos s/2, \sin \frac{s}{2}),\ s \in [0, 4\pi]$

$\gamma'(s) = \cancel{2}\left(\frac{1}{\cancel{2}}(-\sin s/2), \frac{1}{\cancel{2}}(\cos s/2)\right)$

② $\gamma : [-1, 1] \longrightarrow \mathbb{R}^2$

$x \longrightarrow (x, 2x+1)$

$\gamma'(x) = (1, 2) \rightsquigarrow |\gamma'(x)| = \sqrt{5}$

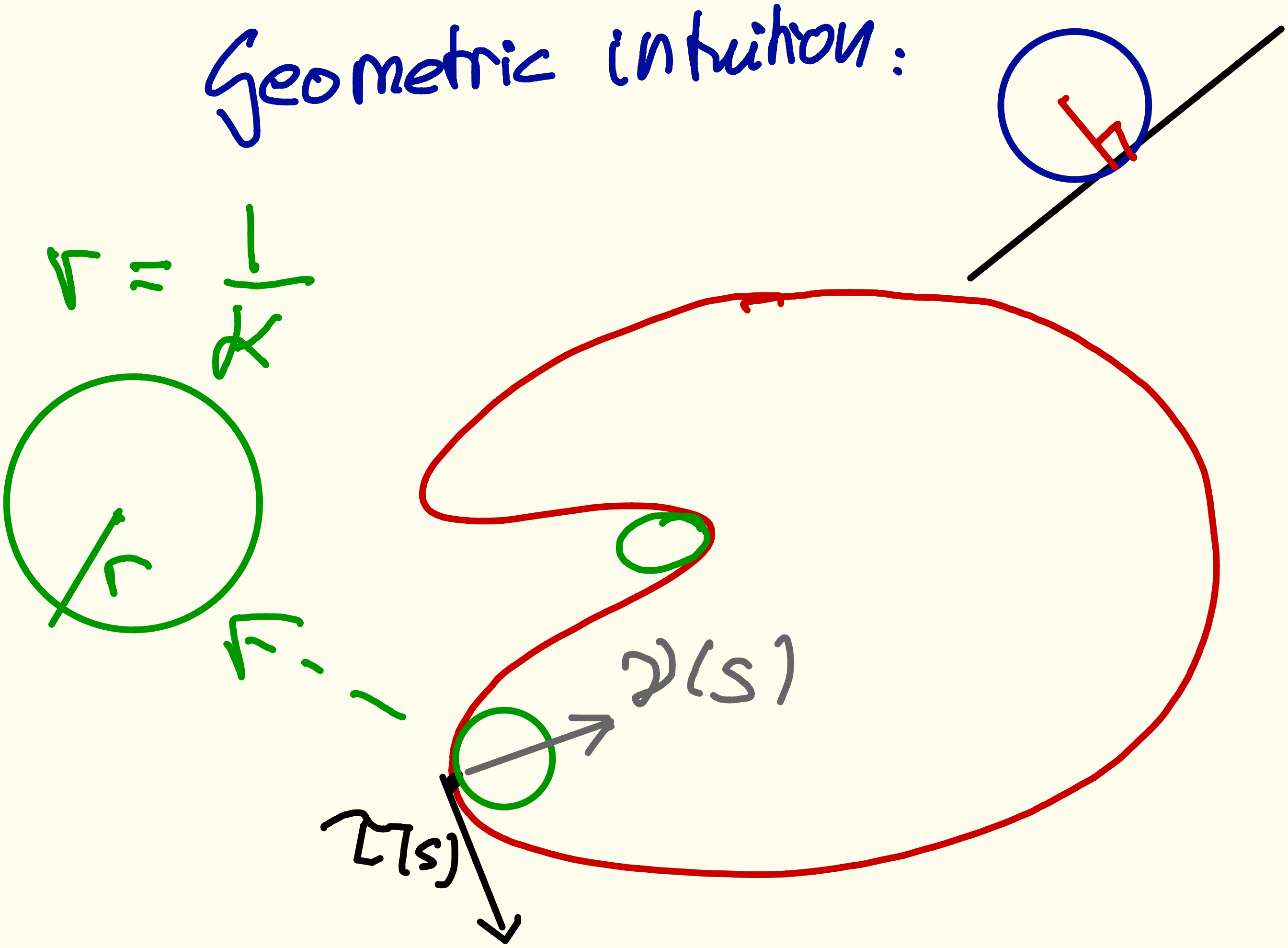
$s(x) = \int_0^x \sqrt{5}\,dt = \sqrt{5}\,x$

$\rightsquigarrow x = s/\sqrt{5} \rightsquigarrow \gamma(s) = \left(\frac{s}{\sqrt{5}}, \frac{2s}{\sqrt{5}} + 1\right)$

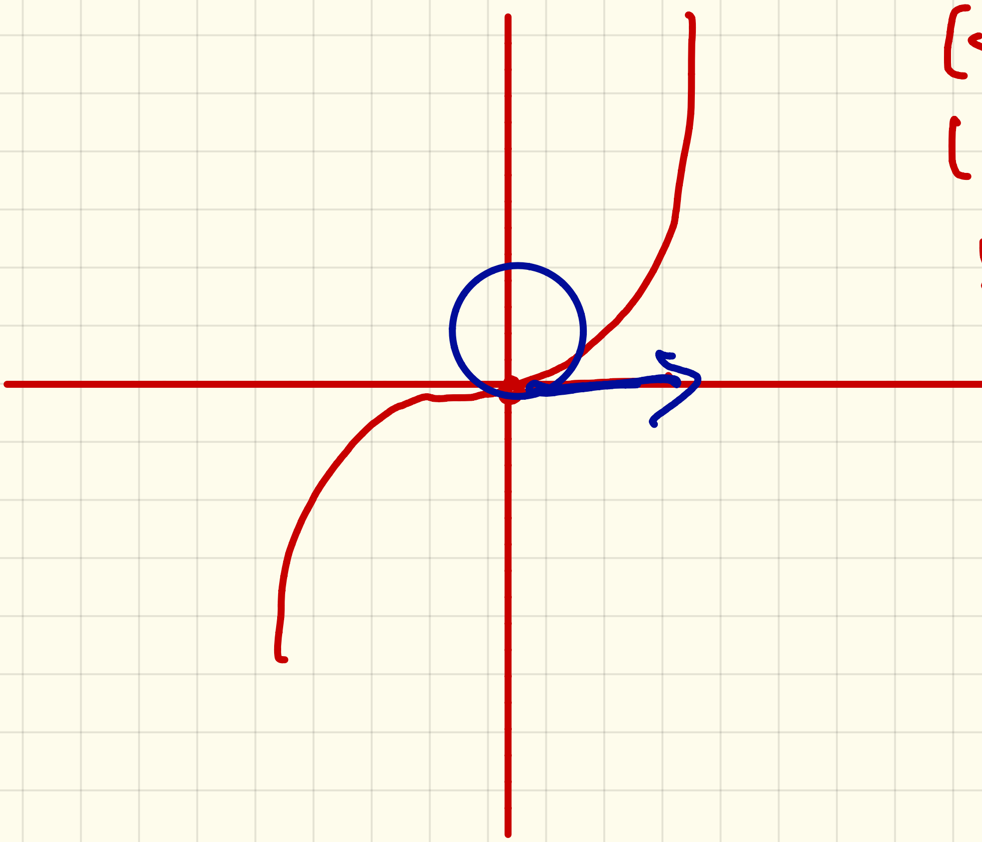
What is the curvature?

<u>Def</u>: Consider $\gamma : [0, L(s)] \to \mathbb{R}^2$ parametrized in arc-length parameter. We define the curvature of γ as $\kappa(s) := |\gamma''(s)|$

Geometric intuition:



ν is called "the normal vector"

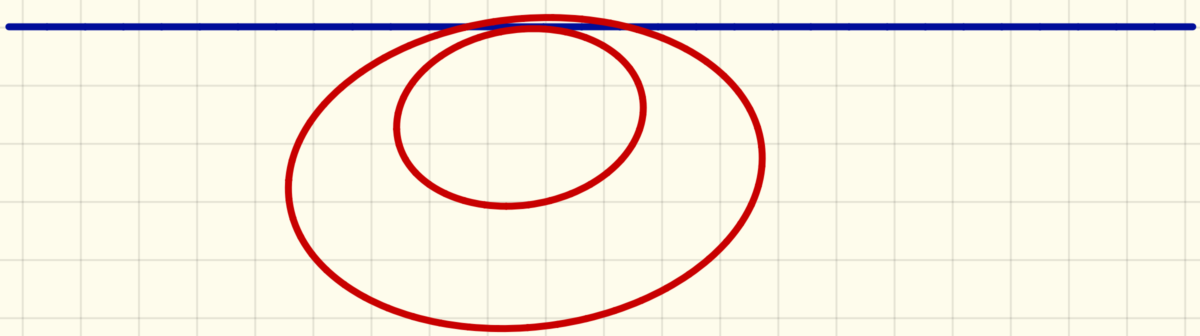


(x, x^3)

$(1, 2x^3)$

$x = 0$

$(1, 0)$



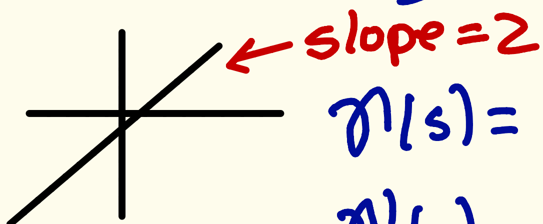
The curvature is the inverse of the radius of the "best approximating" circle. This the geometric equivalent of a Taylor approximation of order 2.

Example

$$\gamma : [0, 2\pi] \to \mathbb{R}^2$$
$$\theta \to 2(\cos\theta, \sin\theta)$$

$$\gamma(s) = 2(\cos s/2, \sin s/2) \; ; \; s \in [0, 4\pi)$$
$$\gamma'(s) = (-\sin s/2, \cos s/2)$$
$$\gamma'' = (-\cos s/2 \cdot \tfrac{1}{2}, -\sin s/2 \cdot \tfrac{1}{2})$$
$$|\gamma''(s)| = 1/2 = \frac{1}{\text{radio}}$$

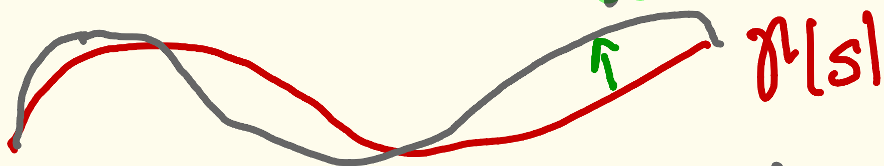
$$\gamma : [-1, 1] \to \mathbb{R}^2$$
$$t \to (t, 2t+1)$$



$$\gamma(s) = \left(\frac{s}{\sqrt{5}}, \frac{2 \cdot s}{\sqrt{5}} + 1\right)$$
$$\gamma'(s) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$
$$\gamma''(s) = (0, 0)$$
$$\kappa(s) = |\gamma''| = 0$$

Why is the curvature a natural quantity?

$\gamma(s) + t\varphi(s)\nu(s) = \tilde{\gamma}(s,t)$



$$\left|\frac{d\tilde{\gamma}}{ds}\right| = |\gamma'(s) + t\varphi'\nu + t\varphi\nu'|$$

$$L(\tilde{\gamma})(t) = \int_0^{L(\gamma)} |\tau(s) + t\varphi'\nu + t\varphi\nu'|\, ds$$

Rmk: $\{\tau(s), \nu(s)\}$ are a base of $\mathbb{R}^2$.

$\Rightarrow \nu'(s) = a\,\tau(s) + b\,\nu(s)$

We also know $\langle \nu, \nu \rangle = 1$ and $\langle \nu, \tau \rangle = 0$

$\Rightarrow 2\langle \nu', \nu \rangle = 0$ and $\langle \nu', \tau \rangle = -\langle \nu, \tau' \rangle$

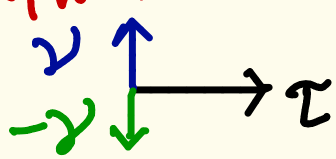
$\Rightarrow a = -\langle \tau', \nu \rangle$ and $b = 0$

$\tau = \gamma'(s) \Rightarrow \tau' = \gamma''(s)$

$\gamma'' = c\,\tau + d\,\nu$

As before, $c = 0$ and $|d| = \kappa(s)$

Remark: In principle there is an ambiguity in the choice of ν, since there are two possible orientations



We will choose ν with positive orientation and the signed curvature such that

$$\gamma''(s) = \kappa \nu$$

Then $\boxed{a = -\kappa}$ and $\boxed{\nu' = -\kappa \tau}$

We are now ready to compute

$$\frac{d}{dt} \mathcal{L} = \int_0^{L(\gamma)} \frac{d}{dt} \left| (1 - \kappa \varphi t)\tau + t\varphi' \nu \right| dx$$

Note:

$$\frac{d}{dt}\left|(1-t\kappa\varphi)\tau+t\varphi'\nu\right|^2$$

$$=\frac{d}{dt}\left[(1-t\kappa\varphi)^2+t^2(\varphi')^2\right]$$

$$=-2(1-t\kappa\varphi)\kappa\varphi+2t|\varphi'|^2$$

Then at $t=0$

$$\Rightarrow \frac{d}{dt}\mathcal{L}(0)=-\int_0^{L(0)}\kappa\varphi\,ds$$