# THE FUNDAMENTAL SOLUTION OF THE DISTRIBUTED ORDER FRACTIONAL WAVE EQUATION IN ONE SPACE DIMENSION IS A PROBABILITY DENSITY

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**June 2013**

## Operators of the Fractional Calculus

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-t')^{\alpha-1} f(t')\, dt' \,, \quad \alpha > 0 \,, \tag{1}$$

$$_{RL}D_t^\alpha = D_t^n J_t^{n-\alpha} \,, \; _C D_t^\alpha = J_t^{n-\alpha} D_t^n \,, \; n-1 < \alpha < n \,, \tag{2}$$

$$_{RL}D_t^n =_C D_t^n = D_t^n \,, \; n = 1, 2, , \cdots \tag{3}$$

For $\alpha \neq n$

$$_C D_t^\alpha \, f(t) =_{RL} D_t^\alpha \left( f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)(t)}}{k!} \right) . \tag{4}$$

Laplace transform of the Caputo derivative of $f(t)$ with $t > 0$

$$L\,[\,_C D_t^\alpha \, f(t),\, t \to s] = s^\alpha \, \widetilde{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} \, f^{(k)}(0) \,. \tag{5}$$

- **Time-Fractional Wave Equation** $-\infty < x < \infty, \ t > 0$

$$D_t{}^{\beta} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 1 < \beta \le 2, \tag{6}$$

$$u(x,0) = \delta(x), \quad \frac{\partial u(x,t=0)}{\partial t} = 0, \tag{7}$$

where $D_t^{\beta}$ denotes Caputo fractional differentiation. Then,

$$D_t{}^{\beta} u(x,t) = \frac{1}{\Gamma(2-\beta)} \int_0^t \frac{\partial^2 u(x,t')/\partial t'^2}{(t-t')^{\beta-1}} \, dt' \, . \tag{8}$$

## The fundamental solution and the M-Wright function

$$u(x,t) = g(x,t) = \frac{1}{2} t^{-\frac{\beta}{2}} M_{\frac{\beta}{2}}(|x| t^{-\frac{\beta}{2}}), \qquad (9)$$

with the M-Wright Function defined for $\nu \in (0,1)$ as

$$M_\nu(z) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\nu n) \sin(\pi \nu n), \qquad (10)$$

see Mainardi [5].

- The process with density $t^{-\frac{\beta}{2}} M_{\frac{\beta}{2}}(x, t^{-\frac{\beta}{2}})$ is a fractional drift process, called "Mittag-Leffler process" by some people. It is distinct from Pillai's ML-process.
  Hence $g(x,t)$ is the sojourn density of a randomly wandering particle, monotonically rightwards with probability $\frac{1}{2}$, monotonically leftwards with probability $\frac{1}{2}$.
- **Remark**: Let us recall that $t^{-\nu} M_\nu(t_* t^{-\nu})$ is the density (in $t_*$, evolving with $t$) of the inverse $\nu$-stable subordinator, see Gorenflo-Mainardi [3]-[4].

- **Distributed Order Fractional Wave Equation**

  $p(\beta)$ generalized function, $p(\beta) \geq 0$

$$0 < \int_{(1,2]} p(\beta) \mathrm{d}\beta < \infty \,. \tag{11}$$

Distributed order fractional Caputo

$$D_t{}^{p(.)} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = \delta(x), \; u_t(x,t=0) = 0 \tag{12}$$

where

$$D_t^{p(.)} f(t) = \int_1^2 p(\beta) \left( D_t^{\beta} f \right)(t) \, \mathrm{d}\beta \,. \tag{13}$$

Special Case:

$$p(\beta) = \delta(\beta - \beta_0).$$

## Fourier-Laplace Solution

$$\widehat{v}(\kappa) = \int_{-\infty}^{\infty} \mathrm{e}^{i\kappa x} v(x)\,\mathrm{d}x$$

$$\widetilde{w}(s) = \int_{0}^{\infty} \mathrm{e}^{-st} w(t)\,\mathrm{d}t$$

$$B(s) = \int_{(1,2]} p(\beta) s^{\beta}\,\mathrm{d}\beta$$

$$\widetilde{\widehat{g}}(\kappa, s) = \frac{B(s)/s}{B(s) + \kappa^2}$$

**Question:** Is $g(x, t)$ a density in $x$, evolving in $t > 0$? Yes

$$\widetilde{\widehat{g}}(\kappa = 0, s) = \frac{1}{s}, \quad 1 = \widehat{g}(\kappa = 0, t) = \int_{-\infty}^{\infty} g(x, t)\,\mathrm{d}x$$

Remains to show that always and everywhere $g(x,t) \geq 0$.

- **Remark**: It is known (see[1]) that the fundamental solution of the distributed order fractional diffusion equation,

$$\int_{(0,1]} p(\beta) D_t{}^{\beta} u(x,t) \mathrm{d}\beta = \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{14}$$

is a probability density evolving in time and that there exists a corresponding stochastic process.
But for the fractional wave equation a different method of proof is required.

- **Special types of functions** See the excellent book by Schilling et al. [8]
  (a) **Completely monotone**
  (b) **Stieltjes**
  (c) **Bernstein**
  (d) **Complete Bernstein**
  Let us consider a function $\varphi(\lambda) : (0, \infty) \to \mathbb{R}$.
- (a): $\varphi$ is **Completely monotone** $\Leftrightarrow$

$$(-1)^n \varphi^{(n)}(\lambda) \geq 0, \quad \text{for } n = 0, 1, 2, \cdots$$

$\Leftrightarrow$ It is the Laplace transform of a non-negative measure.
- **Properties**: Products and positive linear combinations of completely monotone functions are completely monotone
- **Examples**: $\lambda^{-\alpha}$, for $\alpha \geq 0$, $\mathrm{e}^{-\lambda^\alpha}$ for $0 < \alpha \leq 1$
  $E_{\alpha,\beta}(-\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\alpha k+\beta)}$ for $0 < \alpha \leq 1$ , $\beta \geq \alpha$
  (Mittag-Leffler)

- (b): $\varphi$ is **Stieltjes function** $\Leftrightarrow$
  $\varphi$ is Laplace transform of a completely monotone function.
  The functions $\lambda^{-\alpha}$, for $0 < \alpha \leq 1$ are Stieltjes.
- (c) $\varphi$ is **Bernstein function** $\Leftrightarrow$
  $\varphi \in C^\infty$ and $\varphi'$ is completely monotone.
- **Examples**: $\lambda^\alpha$, for $0 < \alpha \leq 1$, $1 - \mathrm{e}^{-\lambda}$
- **Properties**: Positive linear combinations and compositions are again Bernstein, likewise pointwise limits of sequences.
  $\varphi$ completely monotone and $\psi$ Bernstein $\Rightarrow \varphi(\psi)$ is completely monotone.
  $\varphi$ Bernstein $\Rightarrow \frac{\varphi(\lambda)}{\lambda}$ is completely monotone.

- (d): $\varphi$ is **Completely Bernstein** $\Leftrightarrow \frac{\varphi(\lambda)}{\lambda}$ is Stieltjes.
- **Properties**: If $\varphi$ is completely Bernstein and $\psi$ is Stieltjes, then $\varphi(\psi)$ is Stieltjes.
- $\varphi \neq 0$ is complete Bernstein $\Leftrightarrow \frac{1}{\varphi}$ is Stieltjes.
  The set of complete Bernstein functions is a convex cone.
- **Examples**: $\lambda^\alpha$, for $0 < \alpha \leq 1$

## Distributed order time-fractional diffusion equation

$$\int_{(0,1]} p(\beta) {D_t}^{\beta} u(x,t)\mathrm{d}\beta = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{15}$$

$$B(s) = \int_{(0,1]} p(\beta) s^{\beta} \mathrm{d}\beta, \quad \hat{\tilde{g}}(\kappa, s) = \frac{\frac{B(s)}{s}}{B(s) + \kappa^2} \tag{16}$$

$$g(x,t) = \int_0^{\infty} \frac{1}{2\sqrt{\pi r}} \mathrm{e}^{-\frac{x^2}{4r}} R(r,t)\,\mathrm{d}r \tag{17}$$

$$\widetilde{R}(r,s) = \frac{B(s)}{s} \mathrm{e}^{-rB(s)} \tag{18}$$

- $B(s)$ is Bernstein, $B(s)/s$ completely monotone, $\mathrm{e}^{-rB(s)}$ is completely monotone in $s$ for all $x \geq 0$. Hence $\widetilde{R}$ completely monotone in $s$, hence $R \geq 0$ and $g \geq 0$. That $g$ is normalized, shown as previously.

## Distributed order fractional wave equation

$$\widehat{\widetilde{g}}(\kappa, s) = \frac{\frac{B(s)}{s}}{B(s) + \kappa^2}, \quad B(s) = \int_{(1,2]} p(\beta) s^\beta \mathrm{d}\beta \tag{19}$$

By Fourier inversion

$$\widetilde{g}(x, s) = \frac{1}{2} \widetilde{f}_1(s)\, \widetilde{f}_2(s) \tag{20}$$

$$\widetilde{f}_2(s) = \frac{\sqrt{(B(s))}}{s}, \quad \widetilde{f}_1(s) = \exp(-|x| \sqrt{(B(s))}) \tag{21}$$

- Plan: Show $\widetilde{f}_1$ and $\widetilde{f}_2$ completely monotone, hence $\widetilde{g}(x, s)$, is completely monotone in $s$ for all $x \geq 0$, and consequently $g(x, t) \geq 0$.

- **Steps of proof**

$$\left(\widetilde{f_2}(s)\right)^2 = \int_{(1,2]} p(\beta) s^{\beta-2} \mathrm{d}\beta$$

  $s^{\beta-2}$ is Stieltjes, hence also $(\widetilde{f_2}(s))^2$;
  $s\widetilde{f_2}(s) = \sqrt{(B(s))}$ is completely Bernstein,
  $\widetilde{f_2}(s) = \sqrt{(B(s))}/s$ is completely monotone,
  $\widetilde{f_1}(s) = \exp(-|x|\sqrt{(B(s))})$ is a completely monotone function of a Bernstein function, hence completely monotone.

- Result: $\widetilde{g}(x,s)$ is completely monotone in $s$ for all $x \geq 0$, hence $g(x,t) \geq 0$.

- **Distributed order diffusion wave equation**

$$D_t^{p(\cdot)} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$$

$p(\beta) \geq 0, \quad \int_{(0,2]} p(\beta) \mathrm{d}\beta > 0$

Cases (a), (b), (c)

- (a): $p(\beta) \equiv 0$ in (1,2] distributed order fractional diffusion, we know that $g(x,t)$ is a probability density in $x$.
- (b): $p(\beta) \equiv 0$ in [0,1] distributed order wave, we have proved this.
- (c): $p(\beta) \neq 0$ in [0,1], $\neq 0$ in (1,2] distributed order diffusion wave equation.

- Case (c) is open .
  A few very special cases have been discussed affirmatively in e.g. Orsingher-Beghin [7].

$$p(\beta) = \delta(\beta - 2\alpha) + 2\lambda\delta(\beta - \alpha), \;\; 0 < \alpha \leq 1, \;\; \lambda > 0$$

- Of course $0 < \alpha \leq \frac{1}{2}$ is d.o. fractional diffusion.

- **Stochastic Process?**
  Does there exist a stochastic process having $g(x,t)$ as density in $x$ evolving in $t$?
- Clearly yes in the special case

$$p(\beta) = \delta(\beta - \beta_0), \ \ 0 < \beta_0 \leq 2.$$

Also yes for distributed order fractional diffusion

$$p(\beta) = 0\,, \ \text{for } 1 < \beta \leq 2\,.$$

- If $\beta_0 = 2$ the process is degenerated: the particle moves with constant velocity 1 monotonically rightwards with probability $\frac{1}{2}$, monotonically leftwards with probability $\frac{1}{2}$

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- **This lecture was presented to the "International Symposium on Fractional PDEs: Theory, Numerics and Applications", June 3-5, 2013, Salve Regina University, New Port, RI, USA**
- **The speaker appreciates collaboration with Yuri Luchko and Mira Stojanovic and work by Seema S. Nair in producing the slides. He is grateful to Francesco Mainardi for his assistance in producing (after the conference) a revised presentation.**


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