Finite difference methods for the time fractional order differential equations

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In this talk, I'd like to present an overview of our recent works on the finite difference methods for the time fractional differential equation.

Outline

1.1 Definition of the Caputo fractional derivative

For a given positive real number γ, $n-1<\gamma\leqslant n$, the Caputo fractional derivative with the order of γ, is defined by

$$ {}_0^C\mathcal{D}_t^{\gamma} f(t)=\frac{1}{\Gamma(n-\gamma)}\int_0^t \frac{f^{(n)}(\xi)}{(t-\xi)^{\gamma-n+1}}d\xi. $$

- Case $\gamma\in(0,1)$:

$$ {}_0^C\mathcal{D}_t^{\gamma} f(t)=\frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{f'(\xi)}{(t-\xi)^{\gamma}}d\xi. $$

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$$ {}_0^C\mathcal{D}_t^{\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{f'(\xi)}{(t-\xi)^{\gamma}} d\xi. $$

- Case $\gamma \in (1,2)$:

$$ {}_0^C\mathcal{D}_t^{\gamma} f(t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{f''(\xi)}{(t-\xi)^{\gamma-1}} d\xi. $$

1.2 Approximation of the fractional derivative: $\gamma = \frac{1}{2}$

Theorem [Sun and Wu 2004 ANM]

Suppose $f(t) \in C^2[0, t_n]$ Let

$$R(f(t_n)) \equiv {}_0^C\mathcal{D}_t^{\frac{1}{2}} f(t_n) - \frac{\tau^{-\frac{1}{2}}}{\Gamma(2-\frac{1}{2})}\left[a_0 f(t_n) - \sum_{k=1}^{n-1}\left(a_{n-k-1} - a_{n-k}\right) f(t_k) - a_{n-1} f(t_0)\right],$$

then

$$|R(f(t_n))| \leqslant \frac{1}{6\sqrt{\pi}}\left(10\sqrt{2} - 11\right) \max_{0 \leqslant t \leqslant t_n} |f''(t)| \tau^{\frac{3}{2}},$$

where $a_l = \sqrt{l+1} - \sqrt{l}, \quad l \geqslant 0.$

1.3 Approximation of the fractional derivative: $\gamma \in (0,1)$

Theorem [Sun and Wu 2006 ANM]

Suppose $f(t) \in C^2[0, t_n]$ and $\gamma \in (0,1)$. Let

$$R(f(t_n)) \equiv {}_0^C\mathcal{D}_t^\gamma f(t_n) - \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\left[a_0 f(t_n) - \sum_{k=1}^{n-1}\left(a_{n-k-1} - a_{n-k}\right) f(t_k) - a_{n-1} f(t_0)\right],$$

then

$$|R(f(t_n))| \leqslant \frac{1}{\Gamma(2-\gamma)}\left[\frac{1-\gamma}{12} + \frac{2^{2-\gamma}}{2-\gamma} - (1+2^{-\gamma})\right] \max_{0\leqslant t\leqslant t_n} |f''(t)|\tau^{2-\gamma},$$

where $a_l = (l+1)^{1-\gamma} - l^{1-\gamma}, \quad l \geqslant 0.$

1.4 Approximation of the fractional derivative: $\gamma \in (1,2)$

Theorem [Sun and Wu 2006 ANM]

Suppose $f(t) \in C^3[0, t_n]$ and $\gamma \in (1,2)$. Let

$$\overline{R}(f(t_n)) \equiv \frac{1}{2}\Big[{}_0^C\mathcal{D}_t^{\gamma} f(t_n) + {}_0^C\mathcal{D}_t^{\gamma} f(t_{n-1})\Big] -$$

$$\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\Big[b_0 \delta_t f^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} \big(b_{n-k-1} - b_{n-k}\big)\delta_t f^{k-\frac{1}{2}} - \frac{1}{2} b_{n-1} f'(t_0)\Big],$$

then

$$|\overline{R}(f(t_n))| \leqslant \frac{1}{\Gamma(3-\gamma)}\left(\frac{2-\gamma}{12} + \frac{2^{3-\gamma}}{3-\gamma} - 2^{1-\gamma} - \frac{5}{6}\right)\max_{0\leqslant t\leqslant t_n}|f'''(t)|\tau^{3-\gamma},$$

where

$$b_l = (l+1)^{2-\gamma} - l^{2-\gamma}, \ \ l \geqslant 0, \ \delta_t f^{k-\frac{1}{2}} = \frac{f(t_k) - f(t_{k-1})}{\tau}, \ \ 1 \leqslant k \leqslant n.$$

2.1 Dirichlet boundary problem of the sub-diffusion equation

Consider the following one-dimensional problem

$$ {}_0^C\mathcal{D}_t^{\alpha} u(x,t) = \kappa_\alpha \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad a < x < b, \ 0 < t \leqslant T, \tag{1} $$

$$ u(x,0) = \psi(x), \quad a \leqslant x \leqslant b, \tag{2} $$

$$ u(a,t) = \varphi_1(t), \quad u(b,t) = \varphi_2(t), \quad 0 < t \leqslant T, \tag{3} $$

where $\alpha \in (0,1)$.

The fractional equation (1) is called the time fractional sub-diffusion equation.

2.1 Dirichlet boundary problem of the sub-diffusion equation

For finite difference approximation, discretize equally the interval $[a, b]$ with $x_i = a + ih$ $(0 \leqslant i \leqslant M)$, $[0, T]$ with $t_k = k\tau$ $(0 \leqslant k \leqslant N)$, where $h = 1/M$ and $\tau = T/N$ are the spatial and temporal step sizes, respectively. First the following notations are introduced.

$$\delta_x u_{i-\frac{1}{2}} = \frac{1}{h}(u_i - u_{i-1}), \quad \delta_x^2 u_i = \frac{1}{h}\Big(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}\Big),$$

$$\|u\|_\infty = \max_{0 \le i \le M} |u_i|, \ \mathcal{A}u_i = \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}), \quad 1 \leqslant i \leqslant M - 1$$

In addition, denote a discrete fractional derivative operator D_τ^α

$$D_\tau^\alpha u_i^k = \frac{1}{\mu}\Big[u_i^k - \sum_{j=1}^{k-1}(a_{k-j-1} - a_{k-j})u_i^j - a_{k-1}u_i^0\Big], \ 0 \leqslant i \leqslant M, \ 1 \leqslant k \leqslant N.$$

Define the grid function

$$U_i^k = u(x_i, t_k), \quad 0 \leqslant i \leqslant M, \quad 0 \leqslant k \leqslant N.$$

2.1 Dirichlet boundary problem of the sub-diffusion equation

In 2006, we constructed the following difference scheme

$$D_\tau^\alpha u_i^k = \kappa_\alpha \delta_x^2 u_i^k + f_i^k,\ 1 \leqslant i \leqslant M-1,\quad 1 \leqslant k \leqslant N, \tag{4}$$

$$u_i^0 = \psi(x_i),\quad 0 \leqslant i \leqslant M, \tag{5}$$

$$u_0^k = \varphi_1(t_k),\qquad u_M^k = \varphi_2(t_k),\qquad 1 \leqslant k \leqslant N. \tag{6}$$

We proved that

2.1 Dirichlet boundary problem of the sub-diffusion equation

Theorem (Stability) [Sun and Wu 2006 ANM]

The finite difference scheme (4)-(6) is unconditionally stable to the initial value ψ and the right hand term f.

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Theorem (Convergence) [Sun and Wu 2006 ANM]

Assume that $u(x,t) \in \mathcal{C}^{4,2}_{x,t}([a,b] \times [0,T])$ is the solution of (1)-(3) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is solution of the finite difference scheme (4)-(6), respectively. Then there exists a positive constant C such that

$$\|U^k - u^k\|_\infty \leqslant C(\tau^{2-\alpha} + h^2), \qquad 0 \leqslant k \leqslant N.$$

2.1 Dirichlet boundary problem of the sub-diffusion equation

In 2011, we established the following difference scheme

$$\mathcal{A}D_\tau^\alpha u_i^k = \kappa_\alpha \delta_x^2 u_i^k + \mathcal{A}f_i^k,\ 1 \leqslant i \leqslant M-1,\quad 1 \leqslant k \leqslant N, \tag{7}$$

$$u_i^0 = \psi(x_i),\quad 0 \leqslant i \leqslant M, \tag{8}$$

$$u_0^k = \varphi_1(t_k),\quad u_M^k = \varphi_2(t_k),\quad 1 \leqslant k \leqslant N. \tag{9}$$

We proved that

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Theorem (Stability) [Gao and Sun 2011 JCP]

The finite difference scheme (7)-(9) is unconditionally stable to the initial value ψ and the right hand term f.

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Assume that $u(x,t) \in \mathcal{C}^{6,2}_{x,t}([a,b]\times[0,T])$ is the solution of (1)-(3) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is solution of the finite difference scheme (7)-(9), respectively. Then there exists a positive constant C such that

$$\|U^k - u^k\|_\infty \leqslant C(\tau^{2-\alpha} + h^4), \qquad 0 \leqslant k \leqslant N.$$

2.1 Dirichlet boundary problem of the sub-diffusion equation

In (1)-(3), let $a = 0,\ b = 1,\ T = 1, \kappa_\gamma = 1,$

$$f(x,t) = \mathrm{e}^x\left[(1+\gamma)t^\gamma - \frac{\Gamma(2+\gamma)}{\Gamma(1+2\gamma)}t^{2\gamma}\right],$$

$\varphi_1(t) = t^{1+\gamma}$, $\varphi_2(t) = \mathrm{e}t^{1+\gamma}$, $u(x,0) = 0$.

Then the exact solution is

$$u(x,t) = \mathrm{e}^x t^{1+\gamma}.$$

2.1 Dirichlet boundary problem of the sub-diffusion equation

Table: Convergence orders of the difference scheme (7)-(9) in temporal direction with $h = \frac{1}{20000}$.

α	τ	$e_\infty(h,\tau)$	Order
$\alpha = 0.3$	1/20	1.341e-4	$*$
	1/40	4.341e-5	1.6276
	1/80	1.389e-5	1.6444
	1/160	4.404e-6	1.6567
$\alpha = 0.5$	1/20	4.544e-4	$*$
	1/40	1.656e-4	1.4558
	1/80	5.979e-5	1.4700
	1/160	2.144e-5	1.4794
$\alpha = 0.7$	1/20	1.237e-3	$*$
	1/40	5.111e-4	1.2753
	1/80	2.098e-4	1.2844
	1/160	8.579e-5	1.2903

2.1 Dirichlet boundary problem of the sub-diffusion equation

Table: Convergence orders of the difference scheme (7)-(9) in spatial direction ($\tau = \frac{1}{200000}, \alpha = 0.75$).

h	$e_\infty(h, \tau)$	Order
1/2	4.613e-5	$*$
1/4	2.881e-6	4.001
1/8	1.588e-7	4.181

2.2 Neumann boundary problem of the sub-diffusion equation

Consider the following one-dimensional time fractional sub-diffusion equation

$$ {}_0^C\mathcal{D}_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad a < x < b, \ 0 < t \leqslant T, \tag{10} $$

$$ u(x,0) = \varphi(x), \quad a \leqslant x \leqslant b, \tag{11} $$

$$ \frac{\partial u(a,t)}{\partial x} = 0, \quad \frac{\partial u(b,t)}{\partial x} = 0, \quad 0 < t \leqslant T. \tag{12} $$

where $\alpha \in (0,1)$.

2.2 Neumann boundary problem of the sub-diffusion equation

In 2011, We constructed the following spatial second order difference scheme

$$D_\tau^\alpha u_{\frac{1}{2}}^k = \frac{2}{h}\delta_x u_{\frac{1}{2}}^k + f_{\frac{1}{2}}^k,\ 1 \leqslant k \leqslant N, \tag{13}$$

$$\frac{1}{2}(D_\tau^\alpha u_{i-\frac{1}{2}}^k + D_\tau^\alpha u_{i+\frac{1}{2}}^k) = \delta_x^2 u_i^k + \frac{1}{2}(f_{i-\frac{1}{2}}^k + f_{i+\frac{1}{2}}^k),$$
$$1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N, \tag{14}$$

$$D_\tau^\alpha u_{M-\frac{1}{2}}^k = -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}^k + f_{M-\frac{1}{2}}^k,\ 1 \leqslant k \leqslant N, \tag{15}$$

$$u_i^0 = \psi(x_i),\ 0 \leqslant i \leqslant M. \tag{16}$$

We proved that

2.2 Neumann boundary problem of the sub-diffusion equation

Theorem (Stability) [Zhao and Sun 2011 JCP, Ren]

The finite difference scheme (13)-(16) and (17)-(20) is unconditionally stable to the initial value ψ and the right hand term f.

2.2 Neumann boundary problem of the sub-diffusion equation

Theorem (Stability) [Zhao and Sun 2011 JCP, Ren]

The finite difference scheme (13)-(16) and (17)-(20) is unconditionally stable to the initial value ψ and the right hand term f.

Theorem (Convergence) [Zhao and Sun 2011 JCP]

Assume that $u(x,t) \in \mathcal{C}_{x,t}^{4,2}([a,b] \times [0,T])$ is the solution of (10)-(12) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is solution of the finite difference scheme (13)-(16), respectively. Then there exists a positive constant C such that

$$\|U^k - u^k\|_\infty \leqslant C(\tau^{2-\alpha} + h^2), \qquad 0 \leqslant k \leqslant N.$$

2.2 Neumann boundary problem of the sub-diffusion equation

In 2013, we presented the following spatial fourth order finite difference scheme

$$\mathcal{B}D_\tau^\alpha u_0^k = \frac{2}{h}\delta_x u_{\frac{1}{2}}^k + \frac{h}{6}(f_x)_0^k + \mathcal{B}f_0^k, \quad 1 \leqslant k \leqslant N, \tag{17}$$

$$\mathcal{B}D_\tau^\alpha u_i^k = \delta_x^2 u_i^k + \mathcal{B}f_i^k, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant k \leqslant N, \tag{18}$$

$$\mathcal{B}D_\tau^\alpha u_M^k = -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}^k - \frac{h}{6}(f_x)_M^k + \mathcal{B}f_M^k, \quad 1 \leqslant k \leqslant N, \tag{19}$$

$$u_i^0 = \psi(x_i), \quad 0 \leqslant i \leqslant M, \tag{20}$$

where

$$\mathcal{B}u_i = \begin{cases} \frac{1}{6}(5u_0 + u_1), & i = 0, \\ \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}), & 1 \leqslant i \leqslant M-1, \\ \frac{1}{6}(u_{M-1} + 5u_M), & i = M. \end{cases}$$

2.2 Neumann boundary problem of the sub-diffusion equation

Theorem (Stability) [Sun and Zhao, 2013 JCP]

The finite difference schemes (17)-(20) is unconditionally stable to the initial value ψ and the right hand term f.

2.2 Neumann boundary problem of the sub-diffusion equation

Theorem (Stability) [Sun and Zhao, 2013 JCP]

The finite difference schemes (17)-(20) is unconditionally stable to the initial value ψ and the right hand term f.

Theorem (Convergence) [Ren, Sun and Zhao 2013 JCP]

Assume that $u(x,t) \in \mathcal{C}_{x,t}^{6,2}([a,b]\times[0,T])$ is the solution of (10)-(12) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is solution of the finite difference scheme (17)-(20), respectively. Then there exists a positive constant C such that

$$\|U^k - u^k\| \leqslant C(\tau^{2-\alpha} + h^4), \qquad 0 \leqslant k \leqslant N.$$

2.2 Neumann boundary problem of the sub-diffusion equation

In (10), let $T = 1$. In order to test the convergence rate of the proposed methods, we consider the exact solution of the problem (10)-(12) as follows

$$u(x,t) = e^x x^2 (1-x)^2 t^{\gamma+2}.$$

Then it can be checked that the corresponding forcing term $f(x,t)$ and initial condition $\varphi(x)$ are respectively

$$f(x,t) = \frac{\Gamma(\gamma+3)}{2} t^2 e^x x^2 (1-x)^2 - e^x t^{\gamma+2} (2 - 8x + x^2 + 6x^3 + x^4),$$

and

$$\varphi(x) = 0.$$

2.2 Neumann boundary problem of the sub-diffusion equation

Table: Convergence orders of both schemes in temporal direction with $h = \frac{1}{20000}$.

α	τ	$e_\infty(h, \tau)$	Order
$\alpha = 0.3$	$1/20$	1.341e-4	$*$
	$1/40$	4.337e-5	1.6469
	$1/80$	1.385e-5	1.6647
	$1/160$	4.368e-6	1.6915
$\alpha = 0.5$	$1/20$	4.543e-4	$*$
	$1/40$	1.656e-4	1.4704
	$1/80$	5.977e-5	1.4806
	$1/160$	2.142e-5	1.4892
$\alpha = 0.7$	$1/20$	1.237e-3	$*$
	$1/40$	5.111e-4	1.2845
	$1/80$	2.098e-4	1.2905
	$1/160$	8.577e-5	1.2946

2.2 Neumann boundary problem of the sub-diffusion equation

Table: Convergence order of difference scheme (13)-(16) in spatial direction ($\tau = \frac{1}{20000}, \alpha = 0.5$).

h	$e_\infty(h,\tau)$	Order
1/10	3.100e-2	*
1/20	7.763e-3	1.9977
1/40	1.943e-3	1.9981
1/80	4.859e-4	2.0000

Table: Convergence order of difference scheme (17)-(20) in spatial direction ($\tau = \frac{1}{150000}, \alpha = 0.5$).

h	$e_\infty(h,\tau)$	Order
1/5	2.338e-3	*
1/10	1.483e-4	3.9788
1/20	9.304e-6	3.9946
1/40	5.827e-7	3.9969
1/80	3.713e-8	3.9723

2.2 Neumann boundary problem of the sub-diffusion equation

Table: The maximum norm error and CPU time of two schemes.

α	N	scheme (17)-(20)			scheme (13)-(16)		
		M	$e_\infty(h,\tau)$	CPU time(s)	M	$e_\infty(h,\tau)$	CPU time(s)
0.3	585	15	3.335e-5	0.4907	225	7.645e-5	6.6628
	1151	20	1.056e-5	1.4478	400	2.419e-5	26.9830
	1947	25	4.330e-6	3.6058	625	9.906e-6	85.8940
	2989	30	2.089e-6	7.9330	900	4.777e-6	232.3370
0.5	1368	15	3.022e-5	1.3981	225	6.057e-5	19.1151
	2947	20	9.571e-6	5.3454	400	1.916e-5	101.8454
	5344	25	3.922e-6	16.7761	625	7.850e-6	406.5071
	8689	30	1.892e-6	44.7167	900	3.785e-6	1344.5115
0.7	4157	15	2.747e-5	6.8338	225	4.692e-5	98.2291
	10073	20	8.701e-6	40.6099	400	1.484e-5	794.8375
	20015	25	3.566e-6	172.7832	625	6.080e-6	4294.0118
	35074	30	1.720e-6	589.2156	900	2.932e-6	18178.9099

2.3 Space unbounded domain problem for the time fractional sub-diffusion equation

We are concerned with the fractional sub-diffusion equations on the whole-space

$$ {}_0^C\mathcal{D}_t^{\gamma} u(x,t) - K_{\gamma} u_{xx}(x,t) = f(x,t), \quad (x,t) \in \Omega = \mathcal{R} \times [0,T], \tag{21} $$

$$ u(x,0) = \psi(x), \quad x \in \mathcal{R}, \tag{22} $$

$$ u(x,t) \to 0, \quad \text{when} \quad x \to \pm\infty, \ t \in (0,T], \tag{23} $$

where $\mathrm{supp}\{f(x,t)\} \subseteq [X_L, X_R] \times [0,T]$, $\mathrm{supp}\{\psi(x)\} \subseteq [X_L, X_R]$.

2.3 Space unbounded domain problem for the time fractional sub-diffusion equation

Using the Laplace transform, the original problem on the space unbounded domain is reduced to the initial-boundary value problem on a space bounded domain, i.e.,

$$ {}_0^C\mathcal{D}_t^{\gamma} u(x,t) - K_\gamma u_{xx}(x,t) = f(x,t), \quad (x,t) \in [X_L, X_R] \times (0, T], \tag{24} $$

$$ u(x,0) = \psi(x), \quad x \in [X_L, X_R], \tag{25} $$

$$ \frac{\partial u(X_L,t)}{\partial x} = \frac{1}{\sqrt{K_\gamma}} {}_0^C\mathcal{D}_t^{\gamma/2} u(X_L,t), \;\; t \in (0, T], \tag{26} $$

$$ \frac{\partial u(X_R,t)}{\partial x} = -\frac{1}{\sqrt{K_\gamma}} {}_0^C\mathcal{D}_t^{\gamma/2} u(X_R,t), \;\; t \in (0, T]. \tag{27} $$

2.3 Space unbounded domain problem for the time fractional sub-diffusion equation

We constructed the following spatial second order difference scheme

$$D_\tau^\gamma u_i^k - K_\gamma \delta_x^2 u_i^k = f_i^k, \quad 1 \le i \le M-1, \quad 1 \le k \le N, \tag{28}$$

$$D_\tau^\gamma u_0^k - \frac{2K_\gamma}{h}\left[\delta_x u_{\frac{1}{2}}^k - \frac{1}{\sqrt{K_\gamma}} D_\tau^{\gamma/2} u_0^k\right] = f_0^k, \quad 1 \le k \le N, \tag{29}$$

$$D_\tau^\gamma u_M^k - \frac{2K_\gamma}{h}\left[-\frac{1}{\sqrt{K_\gamma}} D_\tau^{\gamma/2} u_M^k - \delta_x u_{M-\frac{1}{2}}^k\right] = f_M^k, \; 1 \le k \le N, \tag{30}$$

$$u_i^0 = \psi(x_i), \quad 0 \le i \le M, \tag{31}$$

and the following spatial fourth order finite difference scheme for the case of $\gamma \le 2/3$,

2.3 Space unbounded domain problem for the time fractional sub-diffusion equation

$$\mathcal{B}D_\tau^\gamma u_0^k - \frac{2K_\gamma}{h}\Big(\delta_x u_{\frac{1}{2}}^k - \frac{1}{\sqrt{K_\gamma}} D_\tau^{\gamma/2} u_0^k\Big)$$
$$= -\frac{h}{6}\frac{1}{K_\gamma}\Big[\frac{1}{\sqrt{K_\gamma}}\mathbb{D}_t^{3\gamma/2} u_0^k - f_x(x_0, t_k)\Big] + \mathcal{B}f_0^k, \quad 1 \leqslant k \leqslant N, \quad (32)$$

$$\mathcal{B}D_\tau^\alpha u_i^k - K_\gamma \delta_x^2 u_i^k = \mathcal{B}f_i^k, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant k \leqslant N, \quad (33)$$

$$\mathcal{B}D_\tau^\gamma u_M^k - \frac{2K_\gamma}{h}\Big(- \frac{1}{\sqrt{K_\gamma}} D_\tau^{\gamma/2} u_M^k - \delta_x u_{M-\frac{1}{2}}^k\Big)$$
$$= \frac{h}{6}\frac{1}{K_\gamma}\Big[- \frac{1}{\sqrt{K_\gamma}}\mathbb{D}_t^{3\gamma/2} u_M^k - f_x(x_M, t_k)\Big] + \mathcal{B}f_M^k, \quad 1 \leqslant k \leqslant N, \quad (34)$$

$$u_i^0 = \psi(x_i), \quad 0 \leqslant i \leqslant M, \quad (35)$$

2.3 Space unbounded domain problem for the time fractional sub-diffusion equation

where

$$\mathbb{D}_t^{3\gamma/2} U_i^k = \begin{cases} D_t^{3\gamma/2} U_i^k, & \gamma < 2/3, \\ (U_i^k - U_i^{k-1})/\tau, & \gamma = 2/3, \end{cases} \quad i = 0, \ M.$$

We proved that

2.3 Space unbounded domain problem for the time fractional sub-diffusion equation

Theorem (Stability) [Gao, Sun and Zhang 2013 JCP]

The finite difference schemes (28)-(31) and (32)-(35) are unconditionally stable to the initial value ψ and the right hand term f.

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Theorem (Convergence) [Gao, Sun and Zhang 2013 JCP]

Assume that $u(x,t) \in \mathcal{C}_{x,t}^{4,2}([a,b] \times [0,T])$ is the solution of (24)-(27) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is solution of the finite difference scheme (28)-(31), respectively. Then there exists a positive constant C such that

$$\sqrt{\tau \sum_{k=1}^{n} \|U^k - u^k\|_\infty^2} \leqslant C(\tau^{2-\gamma} + h^2), \qquad 0 \leqslant n \leqslant N.$$

2.3 Space unbounded domain problem for the time fractional sub-diffusion equation

Theorem (Convergence) [Gao, Sun and Zhang 2013 JCP]

Assume that $u(x,t) \in \mathcal{C}^{6,2}_{x,t}([a,b]\times[0,T])$ is the solution of (24)-(27) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is solution of the finite difference scheme (32)-(35), respectively. Then there exists a positive constant C such that

$$\sqrt{\tau \sum_{k=1}^{n} \|U^k - u^k\|_\infty^2} \leqslant C(\tau^{2-\alpha} + h^4), \qquad 0 \leqslant n \leqslant N.$$

2.3 Space unbounded domain problem for the time fractional sub-diffusion equation

Let see the following numerical results. For the problem, see (Convergence) [Gao, Sun and Zhang 2013 JCP].

Table: Convergence orders in temporal direction with $h = \frac{1}{20000}$.

γ	τ	scheme (28)-(31) $e_\infty(h,\tau)$	scheme (28)-(31) Order	scheme (32)-(35) $e_\infty(h,\tau)$	scheme (32)-(35) Order
2/3	1/ 10	1.615936e-1	*	1.615934e-1	*
	1/ 20	6.326587e-2	1.35	6.326573e-2	1.35
	1/ 40	2.508100e-2	1.33	2.508087e-2	1.33
	1/ 80	9.986011e-3	1.33	9.985876e-3	1.33
	1/160	3.978914e-3	1.33	3.978784e-3	1.33
1/2	1/ 10	8.150871e-2	*	8.150857e-2	*
	1/ 20	2.922531e-2	1.48	2.922517e-2	1.48
	1/ 40	1.052246e-2	1.47	1.052232e-2	1.47
	1/ 80	3.784390e-3	1.48	3.784252e-3	1.48
	1/160	1.357142e-3	1.48	1.357003e-3	1.48

2.3 Space unbounded domain problem for the time fractional sub-diffusion equation

Table: Convergence orders of scheme (28)-(31) and scheme (32)-(35) in spatial direction ($\tau = \frac{1}{20000}, \alpha = 0.5$).

γ	h	scheme (28)-(31) $e_\infty(h,\tau)$	Order	scheme (32)-(35) $e_\infty(h,\tau)$	Order
1/2	1/ 10	8.652041e-1	*	1.710801e-1	*
	1/ 20	2.086740e-1	2.05	1.132355e-2	3.92
	1/ 40	5.177437e-2	2.01	7.183828e-4	3.98
	1/ 80	1.292664e-2	2.00	4.533485e-5	3.99
2/3	1/ 10	8.580534e-1	*	1.619886e-1	*
	1/ 20	2.077051e-1	2.05	1.073701e-2	3.92
	1/ 40	5.156900e-2	2.01	6.826498e-4	3.98
	1/ 80	1.288174e-2	2.00	4.431081e-5	3.95

2.3 Space unbounded domain problem for the time fractional sub-diffusion equation

Table: The maximum norm error and CPU time of two schemes.

	scheme (32)-(35)			scheme (28)-(31)		
N	M	$e_\infty(h,\tau)$	CPU (s)	M	$e_\infty(h,\tau)$	CPU (s)
410	30	2.3608e-3	0.66	287	1.2825e-3	3.45
884	40	7.4933e-4	2.43	509	4.0789e-4	19.10
1603	50	3.0741e-4	7.89	796	1.6702e-4	79.31
2606	60	1.4839e-4	22.09	1146	8.0633e-5	269.93
3931	70	8.0141e-5	53.95	1560	4.3537e-5	788.90

3.1 Dirichlet boundary problem for time fractional diffusion-wave equation

Consider the following one-dimensional time fractional diffusion-wave equation

$$
\begin{aligned}
&{}_{0}^{C}\mathcal{D}_{t}^{\alpha}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad a < x < b,\ 0 < t \leqslant T, && (36)\\
&u(x,0) = \psi(x),\ u_t(x,0) = \phi(x), \quad a \leqslant x \leqslant b, && (37)\\
&u(a,t) = \varphi_1(t), \quad u(b,t) = \varphi_2(t), \quad 0 < t \leqslant T, && (38)
\end{aligned}
$$

where $\alpha \in (1,2)$.

The fractional equation in (36) is called the time fractional diffusion-wave equation.

3.1 Dirichlet boundary problem for time fractional diffusion-wave equatio

For the grid function $v = \{v^k \mid 0 \leqslant k \leqslant N\}$, define

$$v^{k+\frac{1}{2}} = \frac{1}{2}(v^{k+1}+v^k), \quad \delta_t v^{k+\frac{1}{2}} = \frac{1}{\tau}(v^{k+1}-v^k), \quad 0 \leqslant k \leqslant N-1.$$

In addition, denote a discrete fractional derivative operator D_τ^α as follows

$$D_\tau^\alpha g^{k-\frac{1}{2}} \equiv \frac{1}{\bar{\mu}}\Big[\delta_t g^{k-\frac{1}{2}} - \sum_{j=1}^{k-1}(b_{k-j-1} - b_{k-j})\delta_t g^{j-\frac{1}{2}} - b_{k-1}g'(0)\Big],$$

where $\bar{\mu} = \tau^{\alpha-1}\Gamma(3-\alpha)$ and $b_k = (k+1)^{2-\alpha} - k^{2-\alpha}$.

We showed that

3.1 Dirichlet boundary problem for time fractional diffusion-wave equation

In 2006, we constructed the following second order and compact finite difference schemes, respectively. i.e.,

$$D_\tau^\alpha u_i^{k-\frac{1}{2}} = \delta_x^2 u_i^{k-\frac{1}{2}} + f_i^{k-\frac{1}{2}},\ 1 \leqslant i \leqslant M-1,\quad 1 \leqslant k \leqslant N, \tag{39}$$

$$u_i^0 = \psi(x_i),\quad 0 \leqslant i \leqslant M, \tag{40}$$

$$u_0^k = \varphi_1(t_k),\quad u_M^k = \varphi_2(t_k),\quad 1 \leqslant k \leqslant N. \tag{41}$$

3.1 Dirichlet boundary problem for time fractional diffusion-wave equation

Theorem (Stability) [Sun and Wu 2006 ANM]

The finite difference schemes (39)-(41) is unconditionally stable to the initial values ψ, ϕ and the right hand term f.

3.1 Dirichlet boundary problem for time fractional diffusion-wave equation

Theorem (Stability) [Sun and Wu 2006 ANM]

The finite difference schemes (39)-(41) is unconditionally stable to the initial values ψ, ϕ and the right hand term f.

Theorem (Convergence) [Sun and Wu 2006 ANM]

Assume that $u(x,t) \in \mathcal{C}_{x,t}^{4,3}([a,b] \times [0,T])$ is the solution of (36)-(38) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is solution of the finite difference scheme (39)-(41), respectively. Then there exists a positive constant c_2 such that

$$\|U^k - u^k\|_\infty \leqslant c_2(\tau^{3-\alpha} + h^2), \qquad 0 \leqslant k \leqslant N.$$

3.1 Dirichlet boundary problem for time fractional diffusion-wave equation

In 2010, we established the following spatial fourth order difference scheme

$$\mathcal{A}D_\tau^\alpha u_i^{k-\frac{1}{2}} = \delta_x^2 u_i^{k-\frac{1}{2}} + \mathcal{A}f_i^{k-\frac{1}{2}}, \ 1 \leqslant i \leqslant M-1, \ \ 1 \leqslant k \leqslant N, \tag{42}$$

$$u_i^0 = \psi(x_i), \ \ 0 \leqslant i \leqslant M, \tag{43}$$

$$u_0^k = \varphi_1(t_k), \quad u_M^k = \varphi_2(t_k), \quad 1 \leqslant k \leqslant N. \tag{44}$$

We proved that

3.1 Dirichlet boundary problem for time fractional diffusion-wave equation

Theorem (Stability) [Du, Cao and Sun 2010 AMM]

The finite difference schemes (42)-(44) is unconditionally stable to the initial values ψ, ϕ and the right hand term f.

3.1 Dirichlet boundary problem for time fractional diffusion-wave equation

Theorem (Stability) [Du, Cao and Sun 2010 AMM]

The finite difference schemes (42)-(44) is unconditionally stable to the initial values ψ, ϕ and the right hand term f.

Theorem (Convergence) [Du, Cao and Sun 2010 AMM]

Assume that $u(x,t) \in \mathcal{C}_{x,t}^{6,3}([a,b] \times [0,T])$ is the solution of (36)-(38) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is solution of the finite difference scheme (42)-(44), respectively. Then there exists a positive constant c_4 such that

$$\|U^k - u^k\|_\infty \leqslant c_4(\tau^{3-\alpha} + h^4), \qquad 0 \leqslant k \leqslant N.$$

3.1 Dirichlet boundary problem for time fractional diffusion-wave equation

Table: Convergence order of difference scheme (42)-(44) in temporal direction with $h = \frac{1}{200}$.

α	τ	$e_\infty(h,\tau)$	Order
$\alpha = 1.3$	$1/20$	1.341e-4	$*$
	$1/40$	4.341e-5	1.6276
	$1/80$	1.389e-5	1.6444
	$1/160$	4.404e-6	1.6567
$\alpha = 1.5$	$1/20$	4.544e-4	$*$
	$1/40$	1.656e-4	1.4558
	$1/80$	5.979e-5	1.4700
	$1/160$	2.144e-5	1.4794
$\alpha = 1.7$	$1/20$	1.237e-3	$*$
	$1/40$	5.111e-4	1.2753
	$1/80$	2.098e-4	1.2844
	$1/160$	8.579e-5	1.2903

3.1 Dirichlet boundary problem for time fractional diffusion-wave equation

Table: Convergence order of difference scheme (42)-(44) in spatial direction ($\tau = \frac{1}{100000}, \alpha = 1.5$).

h	$e_\infty(h,\tau)$	Order
1/2	4.4035e-3	*
1/4	2.5365e-4	4.118
1/8	1.5560e-5	4.027
1/16	9.7010e-7	4.004
1/32	6.2794e-8	3.949

3.2 Neumann boundary problem for time fractional diffusion-wave equation

Considering the following one-dimensional time fractional diffusion-wave equation

$$ {}_0^C\mathcal{D}_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad a < x < b, \ 0 < t \leqslant T, \tag{45} $$

$$ u(x,0) = \varphi(x), \ \frac{\partial u(x,0)}{\partial t} = \psi(x), \quad a \leqslant x \leqslant b, \tag{46} $$

$$ \frac{\partial u(a,t)}{\partial x} = 0, \quad \frac{\partial u(b,t)}{\partial x} = 0, \quad 0 < t \leqslant T, \tag{47} $$

where $\alpha \in (1,2)$.

3.2 Neumann boundary problem for time fractional diffusion-wave equation

In 2013, we constructed the following spatial second order difference scheme

$$\mathcal{H}D_\tau^\alpha u_i^{k-\frac{1}{2}} = \delta_x^2 u_i^{k-\frac{1}{2}} + \mathcal{H}f_i^{k-\frac{1}{2}}, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N, \tag{48}$$

$$u_i^0 = \varphi_i, \quad 0 \leqslant i \leqslant M, \tag{49}$$

where

$$\mathcal{H}u_i = \begin{cases} \frac{1}{3}(2u_0 + u_1), & i = 0, \\ u_i, & 1 \leqslant i \leqslant M-1, \\ \frac{1}{3}(u_{M-1} + 2u_M), & i = M. \end{cases}$$

and the spatial fourth order finite difference scheme

3.2 Neumann boundary problem for time fractional diffusion-wave equation

$$\mathcal{B}D_\tau^\alpha u_0^{k-\frac{1}{2}} = \frac{2}{h}\delta_x u_{\frac{1}{2}}^{k-\frac{1}{2}} + \frac{h}{6}(f_x)_0^{k-\frac{1}{2}} - \frac{h^3}{90}\Big[(f_{xxx})_0^{k-\frac{1}{2}} + ({}_0^C\mathcal{D}_t^\alpha f_x)_0^{k-\frac{1}{2}}\Big] + \mathcal{B}f_0^{k-\frac{1}{2}}, \quad 1 \leqslant k \leqslant N, \tag{50}$$

$$\mathcal{B}D_\tau^\alpha u_i^{k-\frac{1}{2}} = \delta_x^2 u_i^{k-\frac{1}{2}} + \mathcal{B}f_i^{k-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant k \leqslant N, \tag{51}$$

$$\mathcal{B}D_\tau^\alpha u_M^{k-\frac{1}{2}} = -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}^{k-\frac{1}{2}} - \frac{h}{6}(f_x)_M^{k-\frac{1}{2}} + \frac{h^3}{90}\Big[(f_{xxx})_M^{k-\frac{1}{2}} + ({}_0^C\mathcal{D}_t^\alpha f_x)_M^{k-\frac{1}{2}}\Big] + \mathcal{B}f_M^{k-\frac{1}{2}}, \quad 1 \leqslant k \leqslant N, \tag{52}$$

$$u_i^0 = \varphi_i, \quad 0 \leqslant i \leqslant M. \tag{53}$$

3.2 Neumann boundary problem for time fractional diffusion-wave equation

Theorem (Stability) [Ren and Sun 2013 JSC]

The finite difference schemes (48)-(49) and (50)-(53) are both unconditionally stable to the initial values ψ, φ and the right hand term f.

3.2 Neumann boundary problem for time fractional diffusion-wave equation

Theorem (Stability) [Ren and Sun 2013 JSC]

The finite difference schemes (48)-(49) and (50)-(53) are both unconditionally stable to the initial values ψ, φ and the right hand term f.

Theorem (Convergence) [Ren and Sun 2013 JSC]

Assume that $u(x,t) \in \mathcal{C}_{x,t}^{4,3}([a,b] \times [0,T])$ is the solution of (45)-(47) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is solution of the finite difference scheme (48)-(49), respectively. Then there exists a positive constant C such that

$$\|U^k - u^k\|_\infty \leqslant C(\tau^{3-\alpha} + h^2), \qquad 0 \leqslant k \leqslant N.$$

3.2 Neumann boundary problem for time fractional diffusion-wave equation

Theorem (Convergence) [Ren and Sun 2013 JSC]

Assume that $u(x,t) \in \mathcal{C}^{6,3}_{x,t}([a,b]\times[0,T])$ is the solution of (45)-(47) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is solution of the finite difference scheme (50)-(53), respectively. Then there exists a positive constant C such that

$$\|U^k - u^k\|_\infty \leqslant C(\tau^{3-\alpha} + h^4), \qquad 0 \leqslant k \leqslant N.$$

3.2 Neumann boundary problem for time fractional diffusion-wave equation

In (45), let $T = 1$. In order to test the convergence rate of the proposed methods, we consider the exact solution of the problem (45)-(47) as follows

$$u(x,t) = e^x x^2 (1-x)^2 t^{\gamma+2}.$$

Then it can be checked that the corresponding forcing term $f(x,t)$ and initial conditions $\varphi(x), \psi(x)$ are respectively

$$f(x,t) = \frac{\Gamma(\gamma+3)}{2} t^2 e^x x^2 (1-x)^2 - e^x t^{\gamma+2}(2 - 8x + x^2 + 6x^3 + x^4),$$

and

$$\varphi(x) = 0, \quad \psi(x) = 0.$$

3.2 Neumann boundary problem for time fractional diffusion-wave equation

Table: Convergence orders of scheme (48)-(49) and scheme (50)-(53) in temporal direction with $h = \frac{1}{2000}$.

α	τ	scheme (48)-(49) $e_\infty(h,\tau)$	scheme (48)-(49) Order	scheme (50)-(53) $e_\infty(h,\tau)$	scheme (50)-(53) Order
1.3	1/10	1.148e-3	*	1.148e-3	*
	1/20	3.549e-4	1.694	3.547e-4	1.694
	1/40	1.096e-4	1.695	1.095e-4	1.696
	1/80	3.392e-5	1.693	3.374e-5	1.698
	1/160	1.058e-5	1.681	1.039e-5	1.699
1.7	1/10	5.201e-3	*	5.201e-3	*
	1/20	2.096e-3	1.311	2.096e-3	1.311
	1/40	8.499e-4	1.302	8.498e-4	1.302
	1/80	3.452e-4	1.300	3.452e-4	1.300
	1/160	1.403e-4	1.299	1.402e-4	1.299

3.2 Neumann boundary problem for time fractional diffusion-wave equation

Table: Convergence order of scheme (48)-(49) in spatial direction ($\tau = \frac{1}{2000}, \alpha = 1.1$).

h	$e_\infty(h,\tau)$	Order
1/20	2.929e-3	*
1/40	8.485e-4	1.788
1/80	2.281e-4	1.895
1/160	5.915e-5	1.947
1/320	1.507e-5	1.973

Table: Convergence order of scheme (50)-(53) in spatial direction ($\tau = \frac{1}{10000}, \alpha = 1.1$).

h	$e_\infty(h,\tau)$	Order
1/4	3.168e-3	*
1/8	1.960e-4	4.015
1/16	1.221e-5	4.004
1/32	7.620e-7	4.003
1/64	4.677e-8	4.026

3.2 Neumann boundary problem for time fractional diffusion-wave equation

Table: The maximum norm error and CPU time of two schemes.

α	N	scheme (50)-(53)			scheme (48)-(49)		
		M	$e_\infty(h,\tau)$	CPU time (s)	M	$e_\infty(h,\tau)$	CPU time (s)
1.3	346	12	2.605e-5	0.5648	144	5.878e-5	1.8133
	681	16	8.233e-6	1.4255	256	1.894e-5	6.9851
	1151	20	3.370e-6	3.1365	400	7.821e-6	21.0186
	1768	24	1.625e-6	6.3181	576	3.789e-6	53.7504
1.5	755	12	1.742e-5	1.3204	144	4.636e-5	4.5104
	1625	16	5.502e-6	4.0712	256	1.495e-5	21.4531
	2947	20	2.252e-6	10.8390	400	6.177e-6	79.2962
	4793	24	1.086e-6	26.0553	576	2.993e-6	247.9911
1.7	2092	12	1.061e-5	4.5861	144	3.627e-5	17.4946
	5070	16	3.349e-6	20.1830	256	1.170e-5	122.2289
	10073	20	1.370e-6	75.9918	400	4.838e-6	631.5692
	17652	24	6.606e-7	231.0118	576	2.345e-6	2390.2986

4. Multi-term time fractional diffusion-wave equation

Consider the following two-term time fractional mixed diffusion-wave equation

$$
\begin{aligned}
&{}_0^C\mathcal{D}_t^{\alpha_1}u(x,t) + {}_0^C\mathcal{D}_t^{\alpha}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),\\
&0 < x < L,\ 0 < t \leqslant T, && (54)\\
&u(x,0) = \varphi_1(x), \quad u_t(x,0) = \varphi_2(x), \quad 0 \leqslant x \leqslant L, && (55)\\
&u(0,t) = \psi_1(t), \quad u(L,t) = \psi_2(t), \ \ 0 < t \leqslant T, && (56)
\end{aligned}
$$

where $0 < \alpha_1 < 1 < \alpha < 2$, $\varphi_1(x), \varphi_2(x), \psi_1(t), \psi_2(t)$ and $f(x,t)$ are known smooth functions.

4. Multi-term time fractional diffusion-wave equation

We established the following spatial fourth order (compact) difference scheme

$$\frac{\tau}{2\mu_1}\mathcal{A}\Big(\sum_{j=1}^{k} a_{k-j}^{\alpha_1}\delta_t u_i^{j-\frac{1}{2}} + \sum_{j=1}^{k-1} a_{k-j-1}^{\alpha_1}\delta_t u_i^{j-\frac{1}{2}}\Big) + \mathcal{A}\bar{D}_\tau^\alpha u_i^{k-\frac{1}{2}}$$

$$= \delta_x^2 u_i^{k-\frac{1}{2}} + \mathcal{A} f_i^{k-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant k \leqslant N, \tag{57}$$

$$u_i^0 = \varphi_1(x_i), \quad 0 \leqslant i \leqslant M, \tag{58}$$

$$u_0^k = \psi_1(t_k), \quad u_M^k = \psi_2(t_k), \quad 1 \leqslant k \leqslant N. \tag{59}$$

In (57), $\mathcal{A}u_i^k = \frac{1}{12}(u_{i-1}^k + 10u_i^k + u_{i+1}^k)$,

$$\bar{D}_\tau^\alpha u_i^{k-\frac{1}{2}} = \frac{1}{\bar{\mu}_0}\Big[\delta_t u_i^{k-\frac{1}{2}} - \sum_{j=1}^{k-1}(b_{k-j-1}^\alpha - b_{k-j}^\alpha)\delta_t u_i^{j-\frac{1}{2}} - b_{k-1}^\alpha \varphi_2(x_i)\Big],$$

where $\bar{\mu}_0 = \tau^{\alpha-1}\Gamma(3-\alpha)$, $b_k^\alpha = (k+1)^{2-\alpha} - k^{2-\alpha}$.
We have proved that

4. Multi-term time fractional diffusion-wave equation

Theorem (Stability) [Zhang, Sun]

The difference scheme (57)-(59) is unconditionally stable to the initial values $\varphi_1(x)$ and $\varphi_2(x)$ and the right hand term f.

4. Multi-term time fractional diffusion-wave equation

Theorem (Stability) [Zhang, Sun]

The difference scheme (57)-(59) is unconditionally stable to the initial values $\varphi_1(x)$ and $\varphi_2(x)$ and the right hand term f.

Theorem (Convergence) [Zhang, Sun]

Assume that $u(x,t) \in \mathcal{C}^{6,3}_{x,t}([0,L]\times[0,T])$ is the solution of (54)-(56) and $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ is the solution of the difference scheme (57)-(59), respectively. Then, for $k\tau \leqslant T$, it holds that

$$\|e^k\|_\infty \leqslant \frac{c_1 L}{4}\sqrt{6\Gamma(2-\alpha)T^\alpha}(\tau^{\min\{2-\alpha_1,3-\alpha\}} + h^4), \qquad 0 \leqslant k \leqslant N.$$

4. Multi-term time fractional diffusion-wave equation

Let $L = \pi$, $T = 1$. We consider the exact solution of the problem (54)-(56) as follows

$$u(x,t) = t^{1+\alpha_1+\alpha} \sin x.$$

Then it can be checked that the corresponding source term $f(x,t)$, initial and boundary conditions are respectively

$$f(x,t) = \Big(\frac{\Gamma(2+\alpha_1+\alpha)}{\Gamma(2+\alpha)} t^{1+\alpha} + \frac{\Gamma(2+\alpha_1+\alpha)}{\Gamma(2+\alpha_1)} t^{1+\alpha_1} + t^{1+\alpha_1+\alpha}\Big) \sin x,$$

and

$$\varphi(x) = 0, \quad \psi_1(t) = 0, \quad \psi_2(t) = 0.$$

4. Multi-term time fractional diffusion-wave equation

Table: Numerical convergence of the difference scheme (57)-(59) in temporal direction with $h = \frac{\pi}{100}$.

$\alpha_1, \quad \alpha$	τ	$e_\infty(h,\tau)$	Order1
$\alpha_1 = 0.2, \ \alpha = 1.7$	$1/10$	4.145e-2	$*$
	$1/20$	1.707e-2	1.280
	$1/40$	6.969e-3	1.293
	$1/80$	2.833e-3	1.298
$\alpha_1 = 0.3, \ \alpha = 1.2$	$1/10$	5.329e-3	$*$
	$1/20$	1.667e-3	1.677
	$1/40$	5.164e-4	1.691
	$1/80$	1.589e-4	1.700

4. Multi-term time fractional diffusion-wave equation

Table: Numerical convergence of the difference scheme (57)-(59) in spatial direction with $\tau = \frac{1}{200000}$.

$\alpha_1, \quad \alpha$	h	$e_\infty(h, \tau)$	Order2
$\alpha_1 = 0.1, \ \alpha = 1.3$	$\pi/2$	4.074e-3	$*$
	$\pi/4$	2.413e-4	4.078
	$\pi/8$	1.482e-5	4.025
	$\pi/16$	9.223e-7	4.006

5.1 ADI methods for 2D time fractional sub-diffusion equation

Consider the following two-dimensional time fractional sub-diffusion equation

$$ {}_{0}^{C}\mathcal{D}_{t}^{\alpha} u(x,y,t) = \Delta u(x,y,t) + f(x,y,t), \quad (x,y) \in \Omega,\ 0 < t \leqslant T, \tag{60} $$

$$ u(x,y,0) = \psi(x,y), \quad (x,y) \in \bar{\Omega} = \Omega \cup \partial\Omega, \tag{61} $$

$$ u(x,y,t) = \varphi(x,y,t), (x,y) \in \partial\Omega, \quad 0 < t \leqslant T, \tag{62} $$

where $0 < \alpha < 1$, Δ is the two-dimensional Laplacian operator, the domain $\Omega = (0, L_1) \times (0, L_2)$, and $\partial\Omega$ is the boundary, $\varphi(x,y,t), \psi(x,y)$ and $f(x,y,t)$ are known smooth functions.

5.1 ADI methods for 2D time fractional sub-diffusion equation

Taking two positive integers M_1, M_2, let $x_i = ih_1$ and $y_j = jh_2$ with $h_1 = L_1/M_1$ and $h_2 = L_2/M_2$. Define $\Omega_{h_1} = \{x_i |\ 0 \leqslant i \leqslant M_1\}$ and $\Omega_{h_2} = \{y_j |\ 0 \leqslant j \leqslant M_2\}$, then the domain $\bar{\Omega}$ is covered by $\bar{\Omega}_h = \Omega_{h_1} \times \Omega_{h_2}$. For any mesh function $u = \{u_{ij} |\ 0 \leqslant i \leqslant M_1, 0 \leqslant j \leqslant M_2\}$ defined on $\Omega_{h_1} \times \Omega_{h_2}$, denote

$$\delta_x u_{i-\frac{1}{2},j} = \frac{1}{h_1}(u_{ij} - u_{i-1,j}), \quad \delta_y u_{i,j-\frac{1}{2}} = \frac{1}{h_2}(u_{ij} - u_{i,j-1}),$$

$$\delta_x^2 u_{ij} = \frac{1}{h_1}\left(\delta_x u_{i+\frac{1}{2},j} - \delta_x u_{i-\frac{1}{2},j}\right), \delta_y^2 u_{ij} = \frac{1}{h_2}\left(\delta_y u_{i,j+\frac{1}{2}} - \delta_y u_{i,j-\frac{1}{2}}\right).$$

$$\mathcal{A}_x u_{ij} = \begin{cases} \frac{1}{12}(u_{i-1,j} + 10u_{ij} + u_{i+1,j}), & 1 \leqslant i \leqslant M_1 - 1,\ 0 \leqslant j \leqslant M_2, \\ u_{ij}, & i = 0 \quad \text{or} \quad M_1, \ \ 0 \leqslant j \leqslant M_2, \end{cases}$$

$$\mathcal{A}_y u_{ij} = \begin{cases} \frac{1}{12}(u_{i,j-1} + 10u_{ij} + u_{i,j+1}), & 1 \leqslant j \leqslant M_2 - 1,\ 0 \leqslant i \leqslant M_1, \\ u_{ij}, & j = 0 \quad \text{or} \quad M_2, \ \ 0 \leqslant i \leqslant M_1. \end{cases}$$

5.1 ADI methods for 2D time fractional sub-diffusion equation

We construct the following L1-ADI scheme and BD-ADI scheme, respectively, i.e.,

$$D_\tau^\alpha\left(u_{ij}^n + \mu^2\delta_x^2\delta_y^2 u_{ij}^n\right) - \Delta_h u_{ij}^n = f_{ij}^n, (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{63}$$

$$u_{ij}^n = \phi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h,\ 1 \le n \le N, \tag{64}$$

$$u_{ij}^0 = \psi(x_i, y_j), \quad (x_i, y_j) \in \bar{\Omega}_h. \tag{65}$$

and

$$D_\tau^\alpha u_{ij}^n + \tau\mu\delta_x^2\delta_y^2\delta_t u_{ij}^{n-\frac{1}{2}} - \Delta_h u_{ij}^n = f_{ij}^n, (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{66}$$

$$u_{ij}^n = \phi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h,\ 1 < n \le N, \tag{67}$$

$$u_{ij}^0 = \psi(x_i, y_j), \quad (x_i, y_j) \in \bar{\Omega}_h. \tag{68}$$

5.1 ADI methods for 2D time fractional sub-diffusion equation

Theorem (Stability) [Zhang and Sun 2011 JCP]

The finite difference schemes (63)-(65) and (66)-(68) are unconditionally stable to the initial value ψ and the right hand term f.

5.1 ADI methods for 2D time fractional sub-diffusion equation

Theorem (Stability) [Zhang and Sun 2011 JCP]

The finite difference schemes (63)-(65) and (66)-(68) are unconditionally stable to the initial value ψ and the right hand term f.

Theorem (Convergence) [Zhang and Sun 2011 JCP]

Assume that the problem (60)-(62) has smooth solution $u(x, y, t)$ in the domain $\Omega \times [0, T]$ and $\{u_{ij}^n | (x_i, y_j) \in \Omega_h, 1 \leq n \leq N\}$ be the solution of the difference schemes (63)-(65) and (66)-(68). Then there exists a positive constant C such that

$$|U^n - u^n|_{H_1} \leqslant C(\tau^{\min\{2\alpha, 2-\alpha\}} + h_1^2 + h_2^2), \quad 1 \leq n \leq N.$$

$$\sqrt{\tau \sum_{k=1}^{n} |U^k - u^k|_{H_1}^2} \leqslant C(\tau^{\min\{1+\alpha, 2-\alpha\}} + h_1^2 + h_2^2), \quad 1 \leq n \leq N.$$

5.1 ADI methods for 2D time fractional sub-diffusion equation

Table: Convergence order of (63)-(65) in temporal direction with $h = \frac{\pi}{200}$.

α	N	$e_\infty(\tau, h)$	Order	$e_\infty(\tau, h)$	Order
1/2	10	3.4910e-3	*	4.9026e-4	*
	20	1.9907e-3	0.8104	1.7394e-4	1.4949
	40	1.0823e-3	0.8791	6.0359e-5	1.5270
	80	5.7133e-4	0.9217	*	*
2/3	10	1.4331e-3	*	4.0710e-5	*
	20	5.9327e-4	1.2724	1.1591e-5	1.8124
	40	2.4243e-4	1.2911	4.4132e-6	1.3931
	80	9.8870e-5	1.2940	*	*
3/4	10	3.8502e-3	*	2.3582e-4	*
	20	1.7555e-3	1.1331	7.6934e-5	1.6160
	40	7.8267e-4	1.1654	2.6593e-5	1.5326
	80	3.4449e-4	1.1840	*	*

5.1 ADI methods for 2D time fractional sub-diffusion equation

Table: Convergence orders of scheme (66)-(68) in temporal direction with $h = \frac{\pi}{200}$.

α	N	$e_\infty(\tau, h)$	Order	$e_\infty(\tau, h)$	Order
1/3	10	3.8684e-3	*	1.8001e-4	*
	20	1.6437e-3	1.2347	4.5085e-5	1.9973
	40	6.7951e-4	1.2744	1.1695e-5	1.9467
	80	2.7672e-4	1.2961	*	*
1/2	10	8.8067e-4	*	2.1851e-5	*
	20	3.2549e-4	1.4360	1.7096e-6	3.6760
	40	1.1618e-4	1.4862	6.1627e-7	1.4721
	80	4.0678e-5	1.5141	*	*
2/3	10	2.2326e-3	*	2.1368e-4	*
	20	1.0149e-3	1.1374	6.8734e-5	1.6364
	40	4.4422e-4	1.1920	2.1960e-5	1.6461
	80	1.8953e-4	1.2288	*	*

5.1 ADI methods for 2D time fractional sub-diffusion equation

Table: Convergence orders in spatial direction ($\tau = \frac{1}{2000}, \alpha = 0.5$).

	M	$e_\infty(\tau, h)$	Order
difference scheme(63)-(65)	4	5.1273e-3	*
	8	1.2844e-3	1.9972
	16	3.2219e-4	1.9951
	32	8.1568e-5	1.9818
difference scheme (66)-(68)	4	6.0514e-3	*
	8	1.5096e-3	2.0031
	16	3.7696e-4	2.0017
	32	9.3984e-5	2.0039

5.2 ADI methods for 2D time fractional diffusion-wave equation

Consider the following two-dimensional time fractional diffusion-wave equation

$$ {}_0^C\mathcal{D}_t^{\gamma} u(x,y,t) = \Delta u(x,y,t) + f(x,y,t), (x,y) \in \Omega,\ 0 < t \leqslant T, \tag{69} $$

$$ u(x,y,0) = \psi(x,y), u_t(x,y,0) = \phi(x,y), (x,y) \in \Omega = \Omega \cup \partial\Omega, \tag{70} $$

$$ u(x,y,t) = \varphi(x,y,t), (x,y) \in \partial\Omega, \quad 0 < t \leqslant T, \tag{71} $$

where $1 < \gamma < 2$, Δ is the two-dimensional Laplacian operator, the domain $\Omega = (0, L_1) \times (0, L_2)$, and $\partial\Omega$ is the boundary, $\varphi(x,y,t), \psi(x,y), \phi(x,y)$ and $f(x,y,t)$ are known smooth functions.

5.2 ADI methods for 2D time fractional diffusion-wave equation

We constructed the following Crank-Nicolson scheme

$$D_\tau^\gamma u_{ij}^{n-\frac{1}{2}} = \Delta_h u_{ij}^{n-\frac{1}{2}} - \frac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t u_{ij}^{n-\frac{1}{2}} + f_{ij}^{n-\frac{1}{2}},$$
$$(x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{72}$$
$$u_{ij}^n = \phi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h,\ 1 \le n \le N, \tag{73}$$
$$u_{ij}^0 = \psi(x_i, y_j), \quad (x_i, y_j) \in \Omega_h. \tag{74}$$

The difference scheme (72) can be decomposed into the ADI form.

5.2 ADI methods for 2D time fractional diffusion-wave equation

Theorem 10(Stability) [Zhang, Sun and Zhao 2012 SINUM]

The finite difference scheme (72)-(74) is unconditionally stable to the initial values ψ, ϕ and the right hand term f.

5.2 ADI methods for 2D time fractional diffusion-wave equation

Theorem 10(Stability) [Zhang, Sun and Zhao 2012 SINUM]

The finite difference scheme (72)-(74) is unconditionally stable to the initial values ψ, ϕ and the right hand term f.

Theorem 11(Convergence) [Zhang, Sun and Zhao 2012 SINUM]

Assume that the problem (69)-(71) has smooth solution $u(x, y, t)$ in the domain $\Omega \times [0, T]$ and $\{u_{ij}^n | (x_i, y_j) \in \Omega_h, 1 \le n \le N\}$ be the solution of the difference schemes (72)-(74). Then there exists a positive constant C such that

$$|U^n - u^n|_{H_1} \leqslant C(\tau^{3-\gamma} + h_1^2 + h_2^2), \quad 1 \le n \le N.$$

5.2 ADI methods for 2D time fractional diffusion-wave equation

We presented the following compact scheme

$$\mathcal{A}_x\mathcal{A}_y D_\tau^\gamma u_{ij}^{n-\frac{1}{2}} = (\mathcal{A}_y\delta_x^2 + \mathcal{A}_x\delta_y^2)u_{ij}^{n-\frac{1}{2}} - \frac{\Gamma(3-\gamma)}{4}\tau^{1+\gamma}\delta_x^2\delta_y^2\delta_t u_{ij}^{n-\frac{1}{2}} + \mathcal{A}_x\mathcal{A}_y f_{ij}^{n-\frac{1}{2}}, (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{75}$$

$$u_{ij}^n = \phi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h,\ 1 \le n \le N, \tag{76}$$

$$u_{ij}^0 = \psi(x_i, y_j), \quad (x_i, y_j) \in \Omega_h. \tag{77}$$

The difference scheme (75) can be decomposed into the ADI form.

5.2 ADI methods for 2D time fractional diffusion-wave equation

Theorem 12(Stability) [Zhang, Sun and Zhao 2012 SINUM]

The finite difference scheme (75)-(77) is unconditionally stable to the initial values ψ, ϕ and the right hand term f.

5.2 ADI methods for 2D time fractional diffusion-wave equation

Theorem 12(Stability) [Zhang, Sun and Zhao 2012 SINUM]

The finite difference scheme (75)-(77) is unconditionally stable to the initial values ψ, ϕ and the right hand term f.

Theorem 13(Convergence) [Zhang, Sun and Zhao 2012 SINUM]

Assume that the problem (69)-(71) has smooth solution $u(x, y, t)$ in the domain $\Omega \times [0, T]$ and $\{u_{ij}^n | (x_i, y_j) \in \Omega_h, 1 \leq n \leq N\}$ be the solution of the difference scheme (75)-(77). Then there exists a positive constant C such that

$$|U^n - u^n|_{H_1} \leqslant C(\tau^{3-\gamma} + h_1^4 + h_2^4), \quad 1 \leq n \leq N.$$

5.2 ADI methods for 2D time fractional diffusion-wave equation

In (36)-(38), let $\Omega = (0, \pi) \times (0, \pi)$,

$$f(x, y, t) = \sin x \ \sin y \ \left[\frac{\Gamma(3+\gamma)}{2} t^2 - 2t^{2-\gamma}\right],$$

$u(x, y, 0) = 0, u_t(x, y, 0) = 0, \ \varphi(x, y, t) = 0.$
Then the exact solution is

$$u(x, y, t) = \sin x \ \sin y \ t^{2-\gamma}.$$

5.2 ADI methods for 2D time fractional diffusion-wave equation

Table: Convergence order of difference scheme (72)-(74) in temporal direction with $h = \frac{\pi}{200}$.

γ	τ	scheme (72)-(74) $e_\infty(h,\tau)$	scheme (72)-(74) Order	scheme (75)-(77) $e_\infty(h,\tau)$	scheme (75)-(77) Order
1.25	1/5	2.7052e-2	*	2.7048e-2	*
	1/10	8.4520e-3	1.6784	8.4482e-3	1.6788
	1/20	2.5914e-3	1.7056	2.5877e-3	1.7070
	1/40	7.8867e-4	1.7162	7.8500e-4	1.7209
	1/80	2.4108e-4	1.7099	2.3742e-4	1.7253
1.75	1/5	1.9341e-1	*	1.9340e-1	*
	1/10	8.1579e-2	1.2454	8.1577e-2	1.2454
	1/20	3.4381e-2	1.2466	3.4379e-2	1.2466
	1/40	1.4485e-2	1.2470	1.4484e-2	1.2471
	1/80	6.1002e-3	1.2477	6.0986e-3	1.2479

5.2 ADI methods for 2D time fractional diffusion-wave equation

Table: Convergence order scheme (72)-(74) in spatial direction ($\gamma = 1.1$).

	h	$e_\infty(\tau, h)$	Order
scheme (72)-(74)	$\pi/4$	1.5865e-2	$*$
scheme ($\tau = 1/1000$)	$\pi/8$	3.9882e-3	1.9921
	$\pi/16$	9.9877e-4	1.9975
	$\pi/32$	2.5019e-4	1.9971
scheme (75)-(77)	$\pi/4$	5.0632e-4	$*$
scheme ($\tau = 1/10000$)	$\pi/8$	3.1104e-5	4.0249
	$\pi/16$	1.9421e-6	4.0014
	$\pi/32$	1.2814e-7	3.9218

5.2 ADI methods for 2D time fractional diffusion-wave equation

Table: The maximum norm error and CPU time of two schemes.

γ	N	scheme (75)-(77) M	$e_\infty(\tau, h)$	CPU time(s)	scheme (72)-(74) M	$e_\infty(\tau, h)$	CPU time(s)
1.25	24	4	2.1782e-3	0.0470	16	2.5190e-3	0.2650
	60	6	4.5846e-4	0.4220	36	5.3406e-4	3.4380
	116	8	1.4733e-4	1.7190	64	1.7218e-4	21.4370
	193	10	6.0965e-5	5.6250	100	7.1313e-5	91.0620
1.5	40	4	3.9333e-3	0.1100	16	4.1774e-3	0.4530
	118	6	8.0156e-4	0.9060	36	8.5395e-4	6.8910
	256	8	2.5323e-4	5.0780	64	2.7022e-4	50.7810
	464	10	1.0412e-4	20.7810	100	1.1115e-4	252.4060
1.75	84	4	5.7572e-3	0.2340	16	5.9206e-3	0.9370
	309	6	1.1535e-3	3.5320	36	1.1872e-3	19.5630
	776	8	3.6654e-4	30.2970	64	3.7736e-4	197.9530
	1585	10	1.5035e-4	178.5930	100	1.5480e-4	1680.1100

6. Conclusion

In this review, I report some works on the difference method for the time fractional differential equations. At first, two discrete fractional numerical differential formulae with their truncation errors are presented. Then some difference schemes are constructed for the Dirichlet problem, Neumann problem of the subdiffusion equation and diffusion-wave equation, respectively. For the 2d problem, we concentrate on the ADI schemes. At last the multi term problems are considered. Both spatial second order and fourth order difference schemes are established for each problem. The stability and convergence of the difference schemes are proved. The main tool for analyzing the difference schemes is the energy method. Some numerical examples are provided and the numerical results are accordance with the theoretical results.

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Thanks for your attention!