Nonlinear Fractional Filtering Problems and Associated Zakai Equations

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Contents

- Triangle: Brownian Motion (BM), Fokker-Planck equation (FP), and Îto stochastic calculus (IC);
- Fractional generalizations of the triangle BM-FP-IC;
- Beyond the triangle: Filtering Problem and Zakai Equation;
- Generalized Filtering Problems;
- Generalized Zakai Equations;
- Filtering Problems with Time-Changed Processes;
- Fractional Zakai Equation.

Triple relation: BM, Îto SDE, FPE

- Brownian motion B_t :
  - $B_0 = 0$;
  - For all t and s, $t > s$, the increments $B_t - B_s$ has the Gaussian distribution: $B_t - B_s \sim \mathcal{N}(0, t-s)$;
  - For any non-overlapping time intervals (s, r) and (r, t) $B_r - B_s$ and $B_t - B_r$ are independent;
  - B_t has a continuous path.
- Two key properties (for SDE):
  1. B_t is nowhere differentiable;
  2. B_t has the infinite total variation over any interval.
- Îto SDE:

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \;\; X_{t=0} = X_0.$$

Meaning:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s$$

Triple relation: BM, Îto SDE, FPE

- Fokker-Planck Equation (forward Kolmogorov equation):

$$\frac{\partial p}{\partial t} = A^* p, \;\; p(0,x) = f_{X_0}(x). \tag{1}$$

- A^* is the conjugate of the operator

$$A\varphi(x) = \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial^2 \varphi(x)}{\partial x_i \partial x_k} + \sum_{k=1}^{n} b_k(x)\frac{\partial \varphi(x)}{\partial x_k},$$

with the matrix $a(x) = \sigma(x) \times \sigma(x)^t$.

Fractional generalization: CTRW, semimartingale SDEs, FFPE

- CTRW is a random walk subordinated to a renewal process;
- Scaling limit process of CTRW:
  - Time-change process (B_{T_t}, L_{T_t}); (definition comes soon)
  - Semimartingale;
  - Driving process in stochastic calculus
- Associated Stochastic Differential Equation:

$$dX_t = b(X_t)dT_t + \sigma(X_t)dB_{T_t}, \ X_{t=0} = X_0.$$

- Fractional Fokker-Planck Equation:

$$\mathcal{D}^{\beta} p = A\, p, \ p(0, x) = f_{X_0}(x). \tag{2}$$

Fractional generalization: CTRW, semimartingale SDEs, FFPE - publications

- Hahn, Kobayashi, Umarov. J. Theoretical Probability, 25, 262-279. (2010)
- Hahn, Umarov. FCAA, 2011, 14 (1), 56-79. (2011) (survey)
- Kobayashi. J. Theoretical Probability, 24, 789-820, (2011)

Filtering Problem and Zakai Equation

- State process X_t:

$$dX_t = f(t, X_t)dt + g(t, X_t)dB_t, X_{t=0} = X_0$$

- Observation Process Z_t : $\quad (= \int h(s, X_s)ds + W_t)$:

$$dZ_t = h(t, X_t)dt + dW_t,\ Z_0 = 0.$$

- B_t and W_t are independent Brownian motions; X_0 is an independent r.v.
- f, g, and h satisfy the Lipschitz and growth conditions

Filtering Problem and Zakai Equation

- **The filtering problem:** *Find the best estimation X_t^* of X_t at time t, given Z_t* :
$$E[\|X_t - X_t^*\|^2] = \inf E[\|X_t - Y_t\|^2],$$
where inf is taken over all $\mathcal{Z}_t$-measurable stochastic processes $Y_t \in L_2(P)$.
- It follows from the abstract theory that X_t^* is the projection of X_t onto the space of stochastic processes $\mathcal{L}(Z_t) = \{Y \in L_2(P) : Y_t \text{ is } \mathcal{Z}_t - \text{measurable}\}$.
- The latter can be written in the form
$$X_t^* = E[f(X_t)|\mathcal{Z}_t].$$
- Compare with $E[f(X_t)|X_0 = x] = p(t,x)$ in the FP case

Filtering Problem and Zakai Equation

- Kalman and Bucy (1960-61) solved the linear filtering problem reducing it to:
  - Riccati type differential equation for coefficients in the SDE, and
  - SDE driven by Brownian motion.
- Linear filtering problem:
  - State process: $dX_t = F(t)X_t dt + C(t)dB_t$
  - Observation process: $dZ_t = G_(t)X_t dt + D(t)dW_t$

Filtering Problem and Zakai Equation

- Theorem
  *(**Kalman-Bucy filter**) The solution X_t^* satisfies the stochastic differential equation*
  $$dX_t^* = F^*(t)X_t^* dt + G^*(t)dZ_t, \quad X_0^* = E[X_0],$$
  where $F^*(t) = F(t) - \frac{G^2(t)S(t)}{D^2(t)}$, $G^*(t) = \frac{G(t)S(t)}{D^2(t)}$, and the function $S(t)$ is the solution to the initial value problem for the deterministic Riccati equation
  $$\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)S^2(t)} + C^2(t), \tag{3}$$
  $$S(0) = Var(X_0). \tag{4}$$

Filtering Problem and Zakai Equation

- Consider the measure P_0 defined as $dP_0 = \rho(t)dP$, where

$$\rho(t) = \exp\{-\sum_{k=1}^{m} \int_0^t h_k(X_s)dW_s - \frac{1}{2}\int_0^t |X_s|^2 ds\}.$$

  and the process

$$\Lambda_t = \hat{E}\Big(\frac{dP}{dP_0}\Big|\mathcal{Z}_t\Big),$$

  where the expectation $\hat{E}$ is under the measure P_0.

- Kallianpur-Striebel's formula:

$$X_t^* = E[f(X_t)|\mathcal{Z}_t] = \frac{\hat{E}[f(X_t)\Lambda_t|\mathcal{Z}_t]}{\hat{E}[\Lambda_t|\mathcal{Z}_t]}.$$

Filtering Problem and Zakai Equation

- Introduce the unnormalized filtering measure

$$p_t(f) = \hat{E}[f(X_t)\Lambda_t | \mathcal{Z}_t].$$

- **Zakai equation:**

$$p_t(f) = p_0(f) + \int_0^t p_s(Af)ds + \sum_{k=1}^{m} \int_0^t p_s(h_k f)dZ_s^k.$$

- Here A is the infinitesimal generator of X_t, and of the form

$$A = \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial^2}{\partial x_i \partial x_k} + \sum_{k=1}^{n} b_k(x)\frac{\partial}{\partial x_k},$$

where $a_{ik}(x)$ is (i,k)-th entry of the matrix $a(x) = \sigma\sigma^t$

Filtering Problem and Zakai Equation

- Introduce the unnormalized filtering density $U(t,x)$:

$$p_t(f) = \int f(x)U(t,x)dx.$$

- **Zakai Equation:**

$$dU(t,x) = A^* U(t,x)dt + \sum_{k=1}^{m} h_k(x)U(t,x)dZ_k(t),$$

with the initial condition $U(0,x) = p_0(x)$. Here A^* is the congugate of the operator A.

- compare with FP.

Generalized Filtering Problem (driven by a Lévy process)

- The state process

$$X_t = X_0 + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dB_s +$$

$$\int_0^t \int_{|w|<1} H(X_{s-}, w)\tilde{N}(ds, dw) + \int_0^t \int_{|w|\geq 1} K(X_{s-}, w)N(ds, dw),$$

- The obsrvation process is

$$Z_t = \int_0^t \mu(s, X_{s-})ds + \int_0^t \nu(s, X_{s-})dW_s +$$

$$\int_0^t \int_{|w|<1} g(X_{s-}, w)\tilde{M}(ds, dw) + \int_0^t \int_{|w|\geq 1} f(X_{s-}, w)M(ds, dw);$$

- Both processes are driven by **Lévy processes**.

Lévy process

- **Definition:** A stochastic process $X_t, t \geq 0$ is a Lévy process, if:
  - $X_0 = 0$ a.s.,
  - X_t has independent and stationary increments,
  - $P(|X_t - X_s| > \varepsilon) \to 0$ as $t \to s$ for all $\varepsilon > 0$.
- A Lévy measure μ is defined on $R^n \setminus 0$ and satisfies the condition

$$\int_{R^n - 0} \min(|x|^2, 1)\mu(dx) < \infty.$$

- $\mu(R^n \setminus \{|x| < \varepsilon\}) < \infty, \quad \forall \varepsilon > 0$

Characterization of LP: Lévy-Khintchine formula

- The characteristic function $\Phi_{L_t}(\xi) = E[e^{i\xi L_t}]$ of a Lévy process L_t has representation:

$$\Phi_{L_t}(\xi) = e^{t\Psi(\xi)},$$

where

$$\Psi(\xi) = i(b,\xi) - \frac{1}{2}(\Sigma\xi,\xi) + \int_{R^n-0}[e^{i(\xi,x)} - 1 - i(\xi,x)I_{|x|\leq 1}(x)]\mu(dx),$$

with $b \in R^n$; Σ, the covariance matrix of B_t, and μ, the Lévy measure.

Characterization of LP: Lévy-Khintchine formula

- Interpretation:
  - (i) $\mu = 0$: Brownian motion with drift bt and covariance matrix Σ;
  - (ii) $\mu = \lambda\delta_h$: $L_t = bt + B_t + N_t$, where N_t is a Poisson process with intensity parameter λ and with values kh.
  - (iii) $\mu = \sum \lambda_k \delta_{h_k}$
  - (iv) General case: Lévy-Itô decomposition.

Characterization of LP: Lévy-Itô decomposition

- If L_t is a Lévy process, then

$$L_t = b_1 t + \sigma B_t + \int_{0<|x|<1} x[N(t,dx) - t\mu(dx)] + \int_{|x|\geq 1} xN(t,dx),$$

  where
  - $N(t,dx)$ is a Poisson measure, and
  - $M(t,dx) = N(t,dx) - t\mu(dx)$ is a compensated Poisson measure (martingale measure).
- Both fundamental statements - Lévy-Khintchine formula or Lévy-Itô decomposition imply that any Lévy process L_t is completely defined by a triple (b, Σ, μ)!

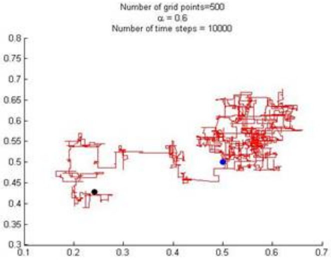
Examples of Lévy processes

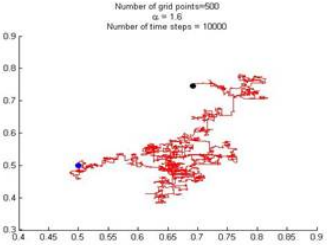
- 1. Brownian motion with (or w/o) drift;
- 2. Compound Poisson process;
- 3. α-stable Lévy processes ($0 < \alpha < 2$):

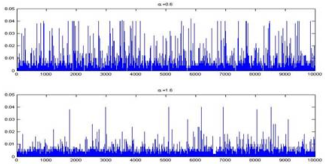
$$\Psi_\alpha(\xi) = i(b, \xi) - \int_{S^{n-1}} |(\xi, s)|^\alpha \left(1 - i \tan \frac{\pi\alpha}{2}\right) sgn(\xi, s)\nu(ds),$$

  where ν is a finite measure on S^{n-1} and $\alpha \neq 1, 2$.
- 4. $S\alpha S$: Take $b = 0$, $\Sigma = 0$ and $\mu(dx) = \frac{c}{|x|^{n+\alpha}} dx$.

Examples of Lévy processes: 2-D $S\alpha S$







Examples of Lévy processes

- 5. Subordinator: D_t is a 1-D Lévy process s.t.
  - $D_t \geq 0$ a.s. for all $t > 0$,
  - $D_{t_1} \leq D_{t_2}$, if $t_1 \leq t_2$.
- Lévy subordinator of index β $(0 < \beta < 1)$ is a subordinator D_t, which is also called a β-stable Lévy process.
- The Laplace symbol of β-stable subordinator is

$$E[e^{-sD_t}](s) = e^{-ts^\beta}$$

Two cases of generalized filtering problems

- Two particular cases:
  (1) The state process is driven by a Lévy process and the observation process is driven by a Brownian motion;
  (2) The state process is driven by a Brownian motion and the observation process is driven by a Lévy process.

Two cases of generalized filtering problems

- Zakai equations associated with the above two particular cases:
- Case 1: For $p_f(t,x) = \hat{E}[f(X_t)\Lambda_t|\mathcal{Z}_t]$:

$$p_f(t,x) = p_0(x) + \int_0^t p_{Af}(s,x)ds + \sum_{k=1}^{m} \int_0^t p_{fh_k}(s,x)dZ_{ks},$$

where A is the infinitezimal generator of X_t.

Two cases of generalized filtering problems

- A is a pseudo-DO with the symbol

$$\Psi(x,\xi) = i(b(x),\xi) - \frac{1}{2}(\Sigma(x)\xi,\xi)$$

$$+\int_{\mathbb{R}^n\setminus\{0\}}(e^{i(G(x,w),\xi)} - 1 - i(G(x,w),\xi)\chi_{(|w|<1)}(w))\nu(dw)$$

where $G(x,w) = H(x,w)$ if $|w| < 1$, and $G(x,w) = K(x,w)$ if $|w| \geq 1$

Two cases of generalized filtering problems

- Case 2:
  Observation process:

$$Z_t = Z_0 + \int_0^t h(s, X_s)ds + W_t + \int_R wN_\lambda(dt, dw)$$

- Zakai Equation:

$$\Phi(t,x) = p_0(x) + \int_0^t A^*\Phi(s,x)ds$$

$$+ \int_0^t h(s,x)\Phi(s,x)dB_s + \int_0^t \int_R (\lambda(s,x,w) - 1)\Phi(s,x)\tilde{N}(ds,dw)$$

where $\tilde{N}(ds,dw) = N(ds,dw) - dsd\nu$ and A^* is the dual operator to A.

Time-changed Lévy process

- A stochastic process T_t is a stopping time if:
  - (i) $T_t \geq 0$ for all $t \geq 0$;
  - (ii) $\{T \leq t\}$ is $\mathcal{F}_t$ measurable
- Example ("First hitting time"). Let X_t càdlàg and $B \subset R^n$. Then $T^B = \inf\{t : X_t \in B\}$ is a stopping time;
- If X_t is a Lévy process and D_t is a Lévy subordinator, then the time-changed process X_{D_t} is again a Lévy process.
- However, if the stopping time is defined as $E_t = \inf\{\tau : D_\tau > t\}$ then time-changed Lévy process X_{E_t} is not a Lévy process! It is a semimartingale.

Fractional Generalization of Filtering Problem

- The state process

$$X_t = X_0 + \int_0^t b(X_{s-})dT_s + \int_0^t \sigma(X_{s-})dB_{T_s} +$$

$$\int_0^t \int_{|w|<1} H(X_{s-}, w)\tilde{N}(dT_s, dw) + \int_0^t \int_{|w|\geq 1} K(X_{s-}, w)N(dT_s, dw),$$

- The obsrvation process is

$$Z_t = \int_0^t \mu(s, X_{s-})dE_s + \int_0^t \nu(s, X_{s-})dW_{E_s} +$$

$$\int_0^t \int_{|w|<1} g(X_{s-}, w)\tilde{M}(dE_s, dw) + \int_0^t \int_{|w|\geq 1} f(X_{s-}, w)M(dE_s, dw),$$

where T_t and E_t are inverse processes to Lévy subordinators.

Simplified version: Time changed Brownian Motion

- The state process with time-changed Brownian motion

$$dX_t = b(X_t)dE_t + \sigma(X_t)dB_{E_t},\ X_{t=0} = X_0$$

where E_t is the inverse processes to the Lévy subordinator with the stability index β.

- The obsrvation process:

$$dZ_t = h(t, X_t)dt + W_{E_t},\ Z_0 = 0$$

Fractional Zakai Equation

- Associated Fractional Zakai Equation:

$$\Phi(t,x) = p_0(x) + J^{\beta} A^{*}\Phi(t,x) + \sum_{k=1}^{m} \int_0^t h_k(x)\Phi(s,x)dZ^k_{E_s}$$

for $\Phi(t,x) = \hat{E}(f(X_t)\Lambda_{E_t}|(\mathcal{Z}\circ\mathcal{E})_t)$.

- Here $J^{\beta} f = \frac{1}{\Gamma(\beta)} \int_0^t \frac{f(s)ds}{(t-s)^{1-\beta}}$ is the fractional integral operator of order β.
- If $\beta = 1$ then the fractional Zakai equation recoveres the classical Zakai equation.

Fractional Zakai Equation

- Associated Fractional Zakai Equation in the differential form:

$$d\Phi(t,x) = {D_{RL}^{1-\beta}} A^{*}\Phi(t,x)dt + \sum_{k=1}^{m} h_k(x)\Phi(t,x)dZ_{E_t}^{k}$$

$$\Phi(0,x) = p_0(x).$$

- Here ${D_{RL}^{1-\beta}}$ is the fractional derivative in the sense of Riemann-Liouville: ${D_{RL}^{1-\beta}} = \frac{d}{dt}J^{\beta}$.

How to prove Fractional Zakai Equation

- Some facts about stable subordinators:

- Lemma

Let $f_t(\tau)$ be the density function of E_t. Then

(a) $\lim_{t\to+0} f_t(\tau) = \delta_0(\tau)$ *in the sense of the topology of the space of tempered distributions* $\mathcal{D}'(\mathbb{R})$;

(b) $\lim_{\tau\to+0} f_t(\tau) = \frac{t^{-\beta}}{\Gamma(1-\beta)},\ t>0$;

(c) $\lim_{\tau\to\infty} f_t(\tau) = 0,\ t>0$;

(d) $\mathcal{L}_{t\to s}[f_t(\tau)](s) = s^{\beta-1}e^{-\tau s^{\beta}},\ s>0,\ \tau\geq 0,$

where $\mathcal{L}_{t\to s}$ *denotes the Laplace transform with respect to the variable* t.

How to prove Fractional Zakai Equation

- Lemma

  The function $f_t(\tau)$ satisfies the following equation

$$D_{*,t}^{\beta} f_t(\tau) = -\frac{\partial}{\partial \tau} f_t(\tau) - \frac{t^{-\beta}}{\Gamma(1-\beta)} \delta_0(\tau),$$

  in the sense of distributions.

How to prove Fractional Zakai Equation

- Lemma
  (Time-change formula for stochastic integrals)

$$\int_0^{E_t} H_s dZ_s = \int_0^t H_{E_s} dZ_{E_s}.$$

- The above facts are used to derive the Zakai equation for

$$\Phi(t,x) = \hat{E}(f(X_t)\Lambda_{E_t} | \mathcal{Z} \circ \mathcal{E}_t) = \hat{E}(f(Y_{E_t})\Lambda_{E_t} | \mathcal{Z} \circ \mathcal{E}_t)$$

Fractional Filtering Problem: Time changed Lévy process

- The state process with time-changed Lévy process:
- The state process

$$X_t = X_0 + \int_0^t b(X_{s-})dE_s + \int_0^t \sigma(X_{s-})dB_{E_s} +$$

$$\int_0^t \int_{|w|<1} H(X_{s-}, w)\tilde{N}(dE_s, dw) + \int_0^t \int_{|w|\geq 1} K(X_{s-}, w)N(dE_s, dw),$$

- The obsrvation process:

$$dZ_t = h(t, X_t)dt + W_{E_t}, \;\; Z_0 = 0$$

Fractional Zakai Equation

- Associated Fractional Zakai Equation:

$$\Phi(t,x) = p_0(x) + J^{\beta} A^{*}\Phi(t,x) + \sum_{k=1}^{m} \int_0^t h_k(x)\Phi(s,x)dZ^k_{E_s}$$

for $\Phi(t,x) = \hat{E}(f(X_t)\Lambda_{E_t} | (\mathcal{Z} \circ \mathcal{E})_t)$.

- Here A^* is the dual to the pseudo-DO A with the symbol

$$\Psi(x,\xi) = i(b(x),\xi) - \frac{1}{2}(\Sigma(x)\xi,\xi)$$

$$+ \int_{\mathbb{R}^n \setminus \{0\}} (e^{i(G(x,w),\xi)} - 1 - i(G(x,w),\xi)\chi_{(|w|<1)}(w))\nu(dw)$$

Fractional Filtering Problem: Time changed Lévy process

- The observation process with time-changed Lévy process:

$$dZ_t = h(X_t)dE_t + dW_{E_t} + \int_R wN_\lambda(dE_t, dw), Z_0 = 0,$$

  where N_λ is a Poisson measure with a (predictable) compensator $\lambda(t, X_t, w)dtd\nu(w)$.
- The state process is driven by B_{E_t}, time-changed Brownian motion

Fractional Zakai Equation

- Associated Fractional Zakai Equation (in terms of filtering measure):

$$\phi_t(f) = p_0(f) + J^\beta \phi_t(Af) + \sum_{k=1}^{m} \int_0^t \phi(H_k f) dZ_{E_s}^k$$

$$+ \int_0^t \int_{R\backslash 0} \phi_s((\lambda(s, \cdot, w) - 1)f) dN_E(ds, dw).$$

Conclusion and Perspectives

- Fractional generalizations of the Zakai equation are obtained; the obtained equations represent a new type of Zakai equations;
- Solution to the fractional Zakai equation through the associated not time-changed driving process is obtained. This can be used to prove existence and uniqueness of a solution to the fractional Zakai equation (work ongoing; will be a separate paper);
- Direct solution of fractional Zakai equation as a stochastic partial differential equation is of interest (work ongoing). We expect a solution via Mittag-Leffler functions, which would lead to a new type of special stochastic processes;
- Numerical solution and possible computer simulation of the fractional Zakai equation is of interest (work ongoing);
- e.t.c.

THANK YOU!

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