Treatment of Nonlinear Stochastic PDE with Fractional Derivative Terms: Iterative Analytical Solution and Monte Carlo Simulations

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Outline of the content

- Preliminaries on fractional derivative representations
- Nonlinear fractional PDE governing the large deflection of a beam
- Approximate solution via a statistical linearization based procedure
- Monte Carlo simulation: an efficient algorithm for computing the dynamical response in the time domain
- Reliability of the analytical solution: statistical linearization vis-à-vis Monte Carlo data
- References

Fractional derivative

- Definition of fractional integral of a certain function w(t) (Podlubny, 1998):

$$ {}_0D_t^{-\alpha} w(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{w(\tau)}{(t-\tau)^{-\alpha+1}} d\tau; \quad for\ \alpha > 0 $$

- $(t\text{-}\tau)^{-1+\alpha}$: power law decay kernel
- $\Gamma(\alpha)$: gamma function
- $-\alpha$: order of the fractional integral

Fractional derivative

- Fractional derivative representation
  - Riemann-Liouville fractional derivative

$$ {}_0^{RL}D_t^{\alpha} w(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{w(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau; \quad for\ \ m-1 \le \alpha < m $$

  - Caputo fractional derivative (Caputo, 1967)

$$ {}_0^{C}D_t^{\alpha} w(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{w^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau; \quad for\ m-1 \le \alpha < m $$

Remark: the Caputo representation accommodates vibration analyses, as the initial conditions involve integer order derivatives.

Fractional derivative

Integral transforms of a Caputo fractional derivative

- Fourier transform: $\widehat{{}_0D_t^{\alpha} w(t)} = (i\omega)^{\alpha} \, \widehat{w(t)}$
- Laplace transform: $\widetilde{{}_0D_t^{\alpha} w(t)} = s^{\alpha}\widetilde{w}(t) - \sum_{k=0}^{m-1} s^{\alpha-k-1} w^{(k)}(0)$

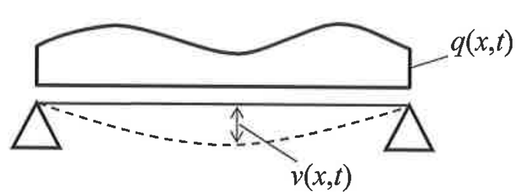
$w^{(k)}(0)$: derivative of order k □ N

Governing equation

Nonlinear fractional PDE with stochastic forcing function

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} + c_0 \partial_t^{\alpha} v(x,t) - N\frac{\partial^2 v(x,t)}{\partial x^2} = q(x,t)$$

$v(x,t)$ = displacement
E = elastic modulus
I = moment of inertia
ρ = mass density
A = cross-sectional area
c = damping
α = order of the fractional derivative
$q(x,t)$ = load



Remark: in this context the fractional term can describe the effect of an external damping or the material viscoelastic properties (Bagley and Torvik, 1983).

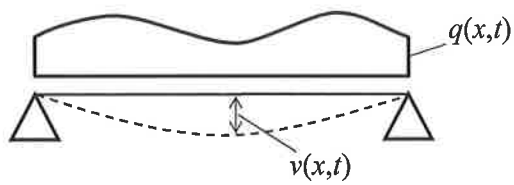
Governing equation

Nonlinear fractional PDE with stochastic forcing function

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} + c_0 \partial_t^\alpha v(x,t) - N\frac{\partial^2 v(x,t)}{\partial x^2} = q(x,t)$$

Nonlinear term N arising from the moderately large deflection of the beam

$$N = \frac{EA}{2L}\int_0^L \left(\frac{\partial v(x,t)}{\partial x}\right)^2 dx$$



Governing equation

Assumptions on the load $q(x,t)$:

•Separable, that is: $q(x,t) = p(x) f(t)$

•Space-wise deterministic

•Time-wise random: $f(t)$ is a stationary Gaussian process with power spectral density function $S(\omega)$ and autocorrelation function

$$< f(t-\tau_1) f(t-\tau_2) > = \int_{-\infty}^{\infty} S(\omega)\exp[i\omega(\tau_2 - \tau_1)]d\omega$$

Statistical Linearization

The solution of the nonlinear fractional PDE is sought via an optimal equivalent linear system. The equivalence is posed in the time domain according to a certain error criterion. Then, the solution of the linear system is readily estimated.

Statistical Linearization

Representation of the solution by the linear modes of vibration of the beam:

$$v(x,t) = \sum_{m=1}^{\infty} w_m(t)\Phi_m(x)$$

Properties of the modes:

•Compatible with the eq. $EI\Phi_m^{iv} = \rho A \omega_m^2 \Phi_m$

•Orthogonality: $\int_0^L \Phi_m \Phi_n dx = L\delta_{mn}$

•Define: $K_{mn} = K_{nm} = \int_0^L \Phi'_m \Phi'_n dx$ then: $\int_0^L \Phi_m \Phi''_n dx = -K_{mn}$

Statistical Linearization

Nonlinear equations associated to the amplitudes $w_m(t)$:

$$\ddot{w}_m + \frac{c}{\rho A}{}_0D_t^{\alpha} w_m + \omega_m^2 w_m + \frac{E}{2\rho L^2}\sum_n\sum_l\sum_j w_n w_i w_j K_{ij} K_{mn} = \frac{P_m}{\rho AL} f(t); \quad for \quad m = 1,2,...$$

Equivalent linear equations:

$$\ddot{w}_m + \frac{c}{\rho A}{}_0D_t^{\alpha} w_m + \omega_{eq,m}^2 w_m = \frac{P_m}{\rho AL} f(t); \quad for \quad m = 1,2,...$$

where $P_m = \int_0^L p(x)\Phi_m(x)dx$

Statistical Linearization

The equivalent stiffness is estimated by solving the minimization problem:

$$\frac{\partial}{\partial \omega_{eq,m}^2} < \varepsilon_m^2 >= 0; \quad for \quad m = 1,2,...$$

where ε_m are errors given by the equations:

$$\varepsilon_m = \omega_{eq,m}^2 w_m - \frac{E}{2\rho L^2}\sum_n\sum_j\sum_i w_n w_i w_j K_{ij} K_{mn} - \omega_m^2 w_m; \quad for \quad m = 1,2,...$$

Statistical Linearization

After a few algebraic manipulations, it is proven that

$$\omega_{eq,m}^2 = \omega_m^2 + \frac{E}{2\rho L^2}\frac{1}{<w_m^2>}\sum_n\sum_i\sum_j K_{ij}K_{mn} <w_m w_n w_i w_j>; \; for \;\; m=1,2,...$$

The average values are estimated by invoking the linear input-output relationships

$$w_m = \frac{P_m}{\rho AL}\int_{-\infty}^{\infty} h_m(\tau) f(t-\tau)d\tau; \; for \;\; m=1,2,...$$

$$h_m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_m(\omega)e^{i\omega t}d\omega \qquad H_m(\omega) = \int_{-\infty}^{\infty} h_m(t)e^{-i\omega t}dt$$

and the transfer function of a fractional linear system

$$H_m(\omega) = \frac{1}{-\omega^2 + \beta(i\omega)^\alpha + \omega_{eq,m}^2}$$

Statistical Linearization

Further, by recalling that

$$\begin{aligned} &< f(t-\tau_1)f(t-\tau_2)f(t-\tau_3)f(t-\tau_4) >= \\ &\quad =< f(t-\tau_1)f(t-\tau_2)>< f(t-\tau_3)f(t-\tau_4)>+ \\ &\quad + < f(t-\tau_1)f(t-\tau_3)>< f(t-\tau_2)f(t-\tau_4)>+ \\ &\quad + < f(t-\tau_1)f(t-\tau_4)>< f(t-\tau_2)f(t-\tau_3)> \end{aligned}$$

The average values are obtained:

$$< w_m^2 >= \left(\frac{P_m}{\rho AL}\right)^2 \int_{-\infty}^{\infty} H_m(-\omega)S(\omega)H_m(\omega)d\omega$$

$$< w_m w_n w_j w_i >= \frac{P_m P_n P_j P_i}{(\rho AL)^4}(S_{mn}S_{ij} + S_{mi}S_{nj} + S_{mj}S_{ni})$$

where $S_{mn} = \int_{-\infty}^{\infty} H_m(\omega)S(\omega)H_n(-\omega)d\omega$

Statistical Linearization

Thus,

$$\omega_{eq,m}^2 = \omega_m^2 + \frac{E}{2\rho^3 A^2 L^4} \frac{1}{P_m S_{mm}} \sum_n \sum_i \sum_j K_{ij} K_{mn} P_n P_i P_j (S_{mn} S_{ij} + S_{mi} S_{nj} + S_{mj} S_{ni}); \quad for \quad m = 1,2,...$$

The equivalent stiffness are calculated iteratively, as S_{mn} depends on $\omega_{eq,m}$. At the first iteration it is posed $\omega_{eq,m} = \omega_m; \; for \;\; m = 1,2,...$

The iteration is performed until convergence is reached.

Statistical Linearization

Standard deviation of the vertical displacement

$$\sigma^2(x) = <v^2(x,t)> = \frac{1}{(\rho A L)^2} \int_{-\infty}^{\infty} S(\omega) \sum_m \sum_n \Phi_m(x) \Phi_n(x) P_m P_n H_m(\omega) H_n(-\omega) d\omega$$

Power spectral density of the vertical displacement

$$S_v(x,\omega) = \frac{1}{(\rho A L)^2} S(\omega) \sum_m \sum_n \Phi_m(x) \Phi_n(x) P_m P_n H_m(\omega) H_n(-\omega)$$

Monte Carlo simulation

The PDE is integrated numerically by the Analog Equation Method (Babouskos and Katsikadelis, 2010; Katsikadelis, G.C. Tsiatas, 2003).

The method is based on the replacement of the original equation by an "analog" equation with certain favorable characteristics. The logic of the AEM is exploited twice: first, the nonlinear fractional PDE is converted to a set of nonlinear fractional ODESs. Then, they are replaced again by a single term fractional differential equations

Given the PDE: $EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} + c_0\partial_t^{\alpha} v(x,t) - N\frac{\partial^2 v(x,t)}{\partial x^2} = q(x,t)$

As it is of the fourth order with respect to x, it replaced by

$$v_{xxxx} = b(x,t)$$

In this context, the time variable is a parameter.

Monte Carlo simulation

The equation

$$v_{xxxx} = b(x,t)$$

is solved by a Boundary Integral Equation Method (BIEM).

Integral representation of the solution:

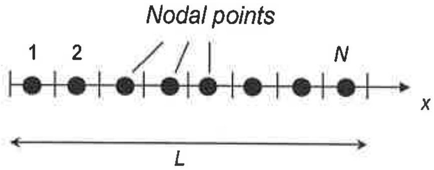
$$v = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \int_0^L G(x,\xi) b(\xi,t) d\xi$$

where c_i are time-dependent coefficients, and $G(x,\xi)$ is the source function

$$G = \frac{1}{12} | x - \xi | (x - \xi)^2$$

Monte Carlo simulation

Discretization of the beam



Resulting discretized solution

$$v = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \sum_{j=1}^{N} b(\xi_j, t) \int_j G(x, \xi_j) d\xi$$

By collocating each displacement to the equation of motion we have

$$EIb(x_j, t) + \rho A \sum_{k=1}^{N} \int_k G(x_j, \xi_k) d\xi \cdot \ddot{b}(\xi_k, t) +$$

$$+ c \sum_{k=1}^{N} \int_k G(x_j, \xi_k) d\xi \cdot {}_0 D_t^{\alpha} b(\xi_k, t) - \frac{EA}{2L} F_j(b, G) = q(x_j, t)$$

for $j=1,\ldots,N$

Monte Carlo simulation

In a matrix notation

$$\rho A \underline{\underline{G}} \cdot \ddot{\underline{b}}(t) + c \underline{\underline{G}} \cdot {}_0 D_t^{\alpha} \underline{b}(t) + EI \underline{b}(t) - \frac{EA}{2L} \underline{F}(\underline{b}(t), \underline{\underline{G}}) = \underline{q}(t)$$

$\underline{F}(\underline{b}(t),\underline{G})$ = nonlinear vector function encapsulating the nonlinearities.

Monte Carlo simulation

$$\rho A \underline{\underline{G}} \cdot \ddot{\underline{b}}(t) + c \underline{\underline{G}} \cdot {}_0D_t^{\alpha} \underline{b}(t) + EI \underline{b}(t) - \frac{EA}{2L} \underline{F}(\underline{b}(t), \underline{\underline{G}}) = \underline{q}(t)$$

The AEM is used for the second time by replacing this set of nonlinear fractional ODEs by the set of equations (Katsikadelis, 2009)

$$\rho A \underline{\underline{G}} \cdot \underline{p}(t) + c \underline{\underline{G}} \cdot \underline{\overline{p}}(t) + EI \underline{b}(t) - \frac{EA}{2L} \underline{F}(\underline{b}(t), \underline{\underline{G}}) = \underline{q}(t)$$

$$\ddot{\underline{b}}(t) = \underline{p}(t)$$

$${}_0D_t^{\alpha} \underline{b}(t) = \underline{\overline{p}}(t)$$

This set can be converted to a system of nonlinear algebraic equations!

Monte Carlo simulation

By taking the Laplace transform of the last equations

$$\ddot{\underline{b}}(t) = \underline{p}(t) \quad \Rightarrow \quad \underline{\tilde{b}} = \frac{1}{s^2} \underline{\tilde{p}} + \frac{1}{s} \underline{b}(0) + \frac{1}{s^2} \dot{\underline{b}}(0)$$

$${}_0D_t^{\alpha} \underline{b}(t) = \underline{\overline{p}}(t) \quad \Rightarrow \quad \underline{\tilde{b}} = \frac{1}{s^{\alpha}} \underline{\tilde{\overline{p}}} + \sum_{k=0}^{m-1} s^{-k-1} \frac{d^k}{dt^k} \underline{b}(0)$$

By taking the inverse Laplace transform of the first one and of the one obtained by equating the right hand sides of both equations:

$$\underline{b}(t) = \underline{b}(0) + t \dot{\underline{b}}(0) + \int_0^t \underline{p}(\tau)(t-\tau) d\tau$$

$$\underline{\overline{p}}(t) = [2 - ceil(\alpha)] \dot{\underline{b}}(0) \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)} \int_0^t \underline{p}(\tau)(t-\tau)^{2-\alpha-1} d\tau$$

Monte Carlo simulation

The following system of nonlinear algebraic equations is obtained in the time domain:

$$\begin{cases} \rho A \underline{\underline{G}} \cdot \underline{p}_n + c \underline{\underline{G}} \cdot \underline{\bar{p}}_n + EI \underline{b}_n - \dfrac{EA}{2L} \underline{F}(\underline{b}_n, \underline{\underline{G}}) = \underline{q}_n \\ -\dfrac{1}{4}\Delta t^2 \underline{p}_n + \underline{b}_n = \underline{b}_0 + t\dot{\underline{b}}_0 + \sum_{i=1}^{n-1} \dfrac{\underline{p}_{i-1} + \underline{p}_i}{2} \left(n - i + \dfrac{1}{2} \right) \Delta t^2 + \dfrac{1}{4} \Delta t^2 \underline{p}_{n-1} \\ -\dfrac{1}{2} \dfrac{\Delta t^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)} \underline{p}_n + \underline{\bar{p}}_n = [2 - ceil(\alpha)] \dot{\underline{b}}_0 \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \\ \quad + \dfrac{\Delta t^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)} \sum_{i=1}^{n-1} \dfrac{\underline{p}_{i-1} + \underline{p}_i}{2} \left[(n-i+1)^{2-\alpha} - (n-i)^{2-\alpha} \right] + \dfrac{1}{2} \dfrac{\Delta t^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)} \underline{p}_{n-1} \end{cases}$$

The unknonwn $\underline{p}_n, \underline{\bar{p}}_n, \underline{b}_n$ can be found at each time step by standard Newton-Raphson iterations

Reliability of the analytical solution

Comparison between the response obtained by statistical linearization and Monte Carlo simulations.

Assumptions:

- uniform load $p(x)=p$
- spectrum of the excitation: coloured white noise with expression

$$\hat{S}(w) = \frac{Cw^4}{[(w^2 - k_1)^2 + (c_1 w)^2][(w^2 - k_2)^2 + (c_2 w)^2]}$$

- simply supported beam, in which:

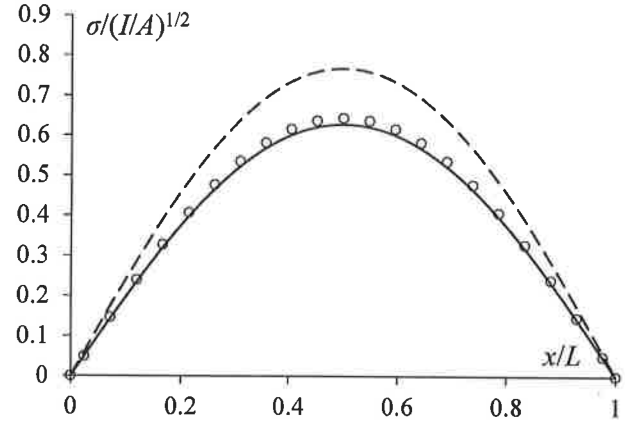
$$\Phi_m = \sqrt{2} \sin(\pi m x / L) \qquad K_{mn} = \frac{\pi^2 m^2}{L} \delta_{mn} \qquad P_m = p \frac{\sqrt{2} L}{m\pi} [1 - (-1)^m]$$

- order of the derivative:

$$\alpha = 0.5$$

Reliability of the analytical solution

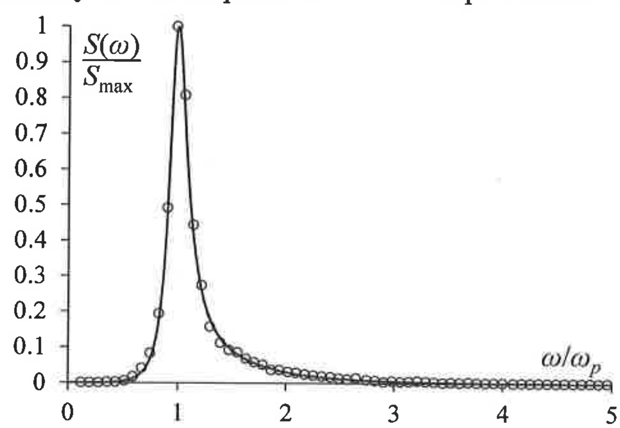
Standard deviation of the response



Statistical linearization (continuous line), Monte Carlo data (circles), linear solution obtained by neglecting the nonlinear term (dotted line)

Reliability of the analytical solution

Power spectral density of the response at the mid-span of the beam



Statistical linearization (continuous line), Monte Carlo data (circles), linear solution obtained by neglecting the nonlinear term (dotted line)

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Closure

- An approximate method has been developed for determining the solution of nonlinear partial differential equations and endowed with fractional derivative terms and stochastic forcing terms
- The method has been based on the concept of "an equivalent linear" PDE the parameters of which have been determined iteratively
- The lateral vibrations of a nonlinear beam have been considered as a paradigm problem
- A reliability of the proposed method has been assessed by extensive Monte Carlo simulations
- A challenging mathematical problem can be the derivation of a priori error estimates of the proposed procedure