Correlation structure of fractional Pearson diffusions

N. N. Leonenko[1], M. M. Meerschaert[2], A. Sikorskii[3]

[1] *School of Mathematics, Cardiff University,Cardiff, UK, LeonenkoN @cardiff.ac.uk*
[2] *Department of Statistics and Probability, Michigan State University, East Lansing, USA, mcubed@stt.msu.edu*
[3] *Department of Statistics and Probability, Michigan State University, East Lansing, USA, sikorska@stt.msu.edu*

Fractional differential equations are an important and useful tool in many areas of science and engineering [1, 2, 3, 4, 5]. In a heterogeneous environment, the coefficients of the diffusion equation will naturally vary in space. Pearson diffusions form a tractable class of variable coefficient diffusion models with polynomial coefficients. The process $X_1(t)$ is called a Pearson diffusion if it solves the stochastic differential equation $dX_1(t) = \mu(X_1(t))dt + \sigma(X_1(t))dW(t)$ with $\mu(x) = a_0 + a_1 x$ and $D(x) = \frac{\sigma^2(x)}{2} = d_0 + d_1 x + d_2 x^2$.

These processes include the Ornstein-Uhlenback process and the Cox-Ingersoll-Ross (CIR) process, which are used in finance. Their steady state distributions belong to the class of Pearson distrbutions. In a fractional Pearson diffusion, the time variable is replaced by an inverse α-stable subordinator independent of the process X_1 [6]. The resulting stochastic process is non-Markovian, but its density $p_\alpha(x,t)$ of the one dimensional distribution of $X_\alpha(t)$ is governed by the fractional Pearson diffusion equation, obtained by replacing the first time derivative in the Pearson diffusion equation with a Caputo fractional derivative of the same order $0 < \alpha < 1$ [7]:

$$\frac{\partial^\alpha p_\alpha(x,t)}{\partial t^\alpha} = -\frac{\partial}{\partial x}\left[\mu(x) p_\alpha(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x) p_\alpha(x,t))\right].$$

The purpose of this paper is study the correlation structure of fractional Pearson diffusions in steady state [8]. We show that if $X_1(t)$ is a Pearson diffusion in steady state, so that its correlation function is given by $\mathrm{corr}[X_1(s), X_1(t)] = \exp(-\theta|t-s|)$, $\theta > 0$, $t, s > 0$, then the correlation function of the corresponding fractional Pearson diffusion $X_\alpha(t) = X_1(E_t)$, where E_t is the standard inverse α-stable subordinator independent of X_1, is given by

$$\mathrm{corr}[X_\alpha(t), X_\alpha(s)] = E_\alpha(-\theta t^\alpha) + \frac{\theta \alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{E_\alpha(-\theta t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz \tag{1}$$

for $t \geq s > 0$, where $E_\alpha(z)$ is the Mittag-Leffler function.

It follows from the expression for the correlation function that fractional Pearson diffusions exhibit long-range dependence in the following sense: if $s > 0$ is fixed and $t \to \infty$, then the correlation of $X_\alpha(t)$ and $X_\alpha(s)$ falls off like a power law $t^{-\alpha}$, with exponent equal to the order of the fractional derivative in time.

References

[1] F. Mainardi (1997) *Fractals and fractional calculus in continuum mechanics*. Number 378. Springer Verlag, Berlin.
[2] J. Sabatier, O. P. Agrawal, and J. A. T. Machado (2007) *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*. Springer, New York.
[3] R. Metzler and J. Klafter (2004) The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics. *J. Physics A: Math. Gen.*. **37**(31), R161–R208.
[4] D. Benson, S. Wheatcraft and M. Meerschaert (2000), Application of a fractional advection-dispersion equation. *Water Resour. Res.* **36**, 1403–1412.
[5] E. Scalas (2006) Five years of continuous-time random walks in econophysics. *The Complex Networks of Economic Interactions* **567**, 3–16.
[6] M. M. Meerschaert and A. Sikorskii (2012) *Stochastic Models for Fractional Calculus.* De Gruyter, Berlin.
[7] N. N. Leonenko, M. M. Meerschaert, and A. Sikorskii (2013) Fractional Pearson diffusion. To appear in *Journal of Mathematical Analysis and Applications*.
[8] N. N. Leonenko, M. M. Meerschaert, and A. Sikorskii (2013) Correlation stucure of fractional Pearson diffusion. To appear in *CAMWA*.