Correlation Structure of Fractional Pearson Diffusions

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Abstract

In a heterogeneous environment, the coefficients of the diffusion equation will naturally vary in space.

Pearson diffusions form a tractable class of variable coefficient diffusion models with polynomial coefficients.

Fractional Pearson diffusions are governed by the corresponding time-fractional diffusion equation.

We present the explicit formula for the covariance function of fractional Pearson diffusions in steady state.

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Diffusion and fractional diffusion

- Fractional differential equations are an important and useful tool in science and engineering (Mainardi (1997), Podlubny (1999), Magin (2006), Scalas (2006), Sabatier et al. (2007), Baleanu et al. (2012)).
- There are some interesting and fundamental connections between fractional calculus and probability (Meerschaert and Sikorskii (2012)).
- The diffusion equation with constant coefficients governs Brownian motion, the long-time scaling limit of a simple random walk (Einstein (1905)).
- If the first time derivative is replaced by a Caputo fractional derivative of order $0 < \alpha < 1$, the result is a fractional diffusion equation that governs the scaling limit of a continuous time random walk (Meerschaert and Scheffler (2004), Chen (2006)).
- The resulting sub-diffusive process spreads at a slower rate $t^{\alpha/2}$ than the usual rate $t^{1/2}$ for a traditional Brownian motion.

Diffusion equation with coefficients varying in space

- In a heterogeneous environment, the coefficients of the diffusion equation will naturally vary in space.
- Pearson diffusions form a tractable class of variable coefficient diffusion models with polynomial coefficients.
- They govern a class of Markov processes whose steady state distributions belong to the class of Pearson distributions (Pearson (1914)).
- In a fractional Pearson diffusion, the time variable is replaced by an inverse α-stable subordinator.

Pearson diffusions

Consider the stochastic differential equation

$$dX_1(t) = \mu(X_1(t))dt + \sigma(X_1(t))dW(t)$$

where $W(t)$ is a standard Brownian motion.

In the case

$$\mu(x) = a_0 + a_1 x \quad \text{and} \quad D(x) = \frac{\sigma^2(x)}{2} = d_0 + d_1 x + d_2 x^2$$

the process $X_1(t)$ is called a *Pearson diffusion*.

If $\sigma(x)$ is a positive constant, this is the Ornstein-Uhlenback process.

If $d_2 = 0$, this is the Cox-Ingersoll-Ross (CIR) process, which is used in finance.

The study of Pearson diffusions began with Kolmogorov (1931), and Wong (1964), and continued recently.

Forward and backward equations

Let $p_1(x, t; y)$ denote the conditional probability density of $x = X_1(t)$ given $y = X_1(0)$, i.e., the transition density of this time-homogeneous Markov process with the state space (l, L). This transition density solves the Kolmogorov forward equation (Fokker-Planck equation)

$$\frac{\partial p_1(x, t; y)}{\partial t} = -\frac{\partial}{\partial x}\left[\mu(x) p_1(x, t; y)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x) p_1(x, t; y))\right]$$

and the backward equation

$$\frac{\partial p_1(x, t; y)}{\partial t} = \mu(y)\frac{\partial p_1(x, t; y)}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2 p_1(x, t; y)}{\partial y^2}$$

with the same initial condition $p_1(x, 0; y) = \delta(x - y)$.

Then we say that $X_1(t)$ is the *stochastic solution* to the forward and the backward equations.

Caputo fractional derivative

The Caputo fractional derivative of order $0 < \alpha < 1$, defined by

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(\tau)(t-\tau)^{-\alpha} \, d\tau,$$

has Laplace transform $s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0)$, where
$\tilde{f}(s) = \int_0^\infty e^{-st} f(t) \, dt$.
The stochastic solution of the time-fractional forward equation

$$\frac{\partial^\alpha p_\alpha(x,t;y)}{\partial t^\alpha} = -\frac{\partial}{\partial x}\left[\mu(x) p_\alpha(x,t;y)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x) p_\alpha(x,t;y))\right]$$

and the time-fractional backward equation

$$\frac{\partial^\alpha p_\alpha(x,t;y)}{\partial t^\alpha} = \mu(y)\frac{\partial p_\alpha(x,t;y)}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2 p_\alpha(x,t;y)}{\partial y^2}$$

with point source initial condition $p_\alpha(x,0;y) = \delta(x-y)$ is called a fractional Pearson diffusion and denoted by $X_\alpha(t)$ (Leonenko, Meerschaert, Sikorskii (2013)).

Fractional time derivative models

- The fractional time derivative models particle sticking and trapping (Kochubei (1989), Meerschaert and Scheffler (2004)).
- Because particle resting times are distributed like a power law, $X_\alpha(t)$ is no longer a Markov process. Hence the conditional probability density $p_\alpha(x, t; y)$ of $x = X_1(t)$ given $y = X_1(0)$ is not enough to determine the process.
- We specify the process and derive an explicit formula for the correlation between $X_\alpha(t)$ and $X_\alpha(s)$ in terms of Mittag-Leffler functions.

Generator

Let $\mathbf{m}(x)$ be the steady state distribution of $X_1(t)$. The generator associated with the backward equation

$$\mathcal{G}g(y) = \left[\mu(y)\,\frac{\partial}{\partial y} + \frac{\sigma^2(y)}{2}\,\frac{\partial^2}{\partial y^2}\right] g(y)$$

has a set of eigenfunctions $\mathcal{G}Q_n(y) = -\lambda_n Q_n(y)$ with eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ that form an orthonormal basis for $L_2(\mathbf{m}(y)\,dy)$.

Eigenfunctions of the generator

If $d_1 = d_2 = 0$ and $d_0 > 0$ ($\sigma^2(y)$ is constant), then $\mathbf{m}(y)$ is a normal density and Q_n are Hermite polynomials.

In the case $d_2 = 0$ ($\sigma^2(y)$ is a first degree polynomial), $\mathbf{m}(y)$ is a gamma density and Q_n are Laguerre polynomials.

For $D''(y) < 0$ with two positive real roots, $\mathbf{m}(y)$ is a beta density and Q_n are Jacobi polynomials.

In the remaining cases, the spectrum of $\mathcal{G}$ has a continuous part, and some moments of $X_1(t)$ do not exist.

In every case, $\mathbf{m}(y)$ is one of the Pearson distributions. We will assume one of the three cases (Hermite, Laguerre, Jacobi) so that all moments exist.

Solving backward equation

Assume a solution $p_1(x, t; y) = f(t)\phi(y)$ to the backward equation and separate variables to see that

$$\frac{df(t)}{dt}\varphi(y) = f(t)\mathcal{G}\varphi(y) \quad \text{or} \quad \frac{1}{f(t)}\frac{df(t)}{dt} = \frac{\mathcal{G}\varphi(y)}{\varphi(y)} := -\lambda$$

so that $f(t)\phi(y) = e^{-\lambda_n t}Q_n(y)$ solves the backward equation for any $n \geq 0$.

Then any linear combination $\sum_n b_n e^{-\lambda_n t}Q_n(y)$ is also a solution, with initial condition $g(x) = \sum_n b_n Q_n(x)$, where

$$b_n = \langle g, Q_n \rangle_{L^2(\mathbf{m}(x)\,dx)} := \int g(x)Q_n(x)\mathbf{m}(x)\,dx,$$

Solving backward equation

It follows that

$$\sum_{n=0}^{\infty} b_n e^{-\lambda_n t} Q_n(y) = \sum_{n=0}^{\infty} \left(\int g(x) Q_n(x) \mathbf{m}(x)\, dx \right) e^{-\lambda_n t} Q_n(y)$$
$$= \int \left(\mathbf{m}(x) \sum_{n=0}^{\infty} e^{-\lambda_n t} Q_n(x) Q_n(y) \right) g(x)\, dx,$$

and hence

$$p_1(x, t; y) = \mathbf{m}(x) \sum_{n=0}^{\infty} e^{-\lambda_n t} Q_n(x) Q_n(y)$$

is the transition density of $X_1(t)$, i.e., the point source solution to the backward equation and also to the forward equation.

Solving fractional backward equation

Since the time-fractional backward equation is a *fractional Cauchy problem* of the form

$$\frac{\partial^\alpha p_\alpha(x,t;y)}{\partial t^\alpha} = \mathcal{G}_y p_\alpha(x,t;y),$$

a general semigroup result (Baeumer and Meerschaert (2001)) implies that

$$p_\alpha(x,t;y) = \int_0^\infty p_1(x,u;y) f_t(u)\,du$$

where

$$f_t(x) = \frac{t}{\alpha} x^{-1-\frac{1}{\alpha}} g_\alpha(t x^{-\frac{1}{\alpha}}),$$

and $g_\alpha(t)$ is the probability density of a stable subordinator with Laplace transform $\tilde{g}_\alpha(s) = \exp(-s^\alpha)$.

Defining fractional Person diffusion process

If $D(u)$ is the standard stable subordinator, a strictly increasing stochastic process with stationary independent increments such that $D(1)$ has probability density g_α, then the *inverse stable subordinator*

$$E_t = \inf\{u > 0 : D(u) > t\}$$

has density f_t.

Then it follows that

$$p_\alpha(x, t; y) = \int_0^\infty p_1(x, u; y) f_t(u)\, du$$

is also the conditional probability density of $x = X_\alpha(t)$ given $y = X_\alpha(0)$, where $X_\alpha(t) := X_1(E_t)$ and the time change E_t is independent of the outer process $X_1(t)$.

Sub-diffusion

Since E_t has the same distribution as $t^{\alpha} E_1$, the fractional Pearson diffusion $X_{\alpha}(t)$ is a kind of sub-diffusion, where particles move along the same trajectories, but more slowly than the Pearson diffusion $X_1(t)$.

Bingham (1971) shows that

$$\int_0^{\infty} e^{-su} f_t(u)\, du = E_{\alpha}(-st^{\alpha}) := \sum_{j=0}^{\infty} \frac{(-st^{\alpha})^j}{\Gamma(1+\alpha j)}$$

using the Mittag-Leffler function.

Transition density representation

The transition density has the following representation:

$$
\begin{aligned}
p_\alpha(x,t;y) &= \sum_{n=0}^{\infty} \mathbf{m}(x) Q_n(x) Q_n(y) \int_0^\infty e^{-\lambda_n u} f_t(u)\, du \\
&= \mathbf{m}(x) \sum_{n=0}^{\infty} E_\alpha(-\lambda_n t^\alpha) Q_n(x) Q_n(y).
\end{aligned}
$$

An alternative proof in Leonenko, Meerschaert, Sikorskii (2013) uses separation of variables, and the fact that $E_\alpha(-\lambda t^\alpha)$ is an eigenfunction of the Caputo derivative with eigenvalue $-\lambda$ (Mainardi and Gorenflo (2000)).

Lemma

For the three classes of fractional Pearson diffusions (OU, CIR, Jacobi) with invariant density $\mathbf{m}$ and system of orthonormal polynomials $\{Q_n,\ n \in \mathbb{N}\}$, for any $0 < \alpha < 1$, the series

$$p_\alpha(x, t; y) = \mathbf{m}(x) \sum_{n=0}^{\infty} E_\alpha\left(-\lambda_n t^\alpha\right) Q_n(y) Q_n(x)$$

where E_α is Mittag-Leffler function, converges for fixed $t > 0$, $x,\ y \in (l, L)$.

Strong solution of fractional backward equation

Suppose that the function $g \in L_2(\mathbf{m}(x)dx)$ is such that $\sum_n g_n Q_n$ with $g_n = \int_l^L g(x)Q_n(x)\mathbf{m}(x)dx$ converges to g uniformly on finite intervals $[y_1, y_2] \subset (l, L)$. Then the fractional Cauchy problem

$$\frac{\partial^\alpha u(t;y)}{\partial t^\alpha} = \mathcal{G}u(t;u) = \mu(y)\frac{\partial u(t;y)}{\partial y} + \frac{1}{2}\sigma^2(y)\frac{\partial^2 u(t;y)}{\partial y^2} \tag{1}$$

with initial condition $u(0;y) = g(y)$ has a strong solution $u = u(t;y)$ given by

$$u(t;y) = u_\alpha(t;y) = \int_l^L p_\alpha(x,t;y)g(x)dx = \sum_{n=0}^{\infty} E_\alpha\left(-\lambda_n t^\alpha\right) Q_n(y) g_n. \tag{2}$$

The series in (2) converges absolutely for each fixed $t > 0$, $y \in (l, L)$, and (1) holds pointwise.

Strong solution of the fractional forward equation

Suppose that the function $f/\mathbf{m} \in L_2(\mathbf{m}(\mathbf{x})dx)$, and $\sum_n f_n Q_n$ with $f_n = \int_l^L f(y)Q_n(y)dy$ converges to $f/\mathbf{m}$ uniformly on finite intervals $[y_1, y_2] \subset (l, L)$. Then the fractional Cauchy problem

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mathcal{L}u(x,t) = -\frac{\partial}{\partial x}\left[\mu(x)u(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)u(x,t)\right] \tag{3}$$

with the initial condition $u(x,0) = f(x)$ has a strong solution $u = u(x,t)$ given by

$$\begin{aligned} u(x,t) = u_\alpha(x,t) = \int_l^L p_\alpha(x,t;y)f(y)dy \\ = \mathbf{m}(x)\sum_{n=0}^{\infty} E_\alpha\left(-\lambda_n t^\alpha\right) Q_n(x) f_n. \end{aligned} \tag{4}$$

The series in (4) converges absolutely for each $t > 0$, $x \in (l, L)$, and equation (3) holds pointwise (u is a strong solution).

Correlation function

If the time-homogeneous Markov process $X_1(t)$ is in steady state, then its probability density $\mathbf{m}(x)$ stays the same over all time.

We will say that fractional Pearson diffusion is in steady state if it starts with the distribution $\mathbf{m}(x)$. The fractional Pearson diffusion in steady state is first order stationary, i.e., $X_\alpha(t)$ has the same probability density $p_\alpha(x,t) = \mathbf{m}(x)$ for all $t > 0$. Indeed

$$\begin{aligned} p_\alpha(x,t) &= \int p_\alpha(x,t;y)\mathbf{m}(y)dy \\ &= \int \int_0^\infty p_1(x,u;y) f_t(u) du \, m(y) dy \\ &= \int_0^\infty \mathbf{m}(x) f_t(u) \, du = \mathbf{m}(x). \end{aligned}$$

Correlation function continued

Thus the fractional Pearson diffusion in steady state has mean $\mathbb{E}[X_\alpha](t) = \mathbb{E}[X_1(t)] = m_1$ and variance $\mathrm{Var}[X_\alpha(t)] = \mathrm{Var}[X_1(t)] = m_2^2$ which do not vary over time.

The stationary Pearson diffusion has correlation function

$$\mathrm{corr}[X_1(s), X_1(t)] = \exp(-\theta|t-s|)$$

where the correlation parameter $\theta = \lambda_1$ is the smallest positive eigenvalue of the generator.

Thus the Pearson diffusion exhibits *short range dependence*, with a correlation function that falls off exponentially.

Theorem 1

Suppose that $X_1(t)$ is a Pearson diffusion in steady state. Then the correlation function of the corresponding fractional Pearson diffusion $X_\alpha(t) = X_1(E_t)$, where E_t is the standard inverse α-stable subordinator independent of $X_1(t)$, is given by

$$\begin{aligned} &\mathrm{corr}[X_\alpha(t), X_\alpha(s)] \\ &= E_\alpha(-\theta t^\alpha) + \frac{\theta \alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{E_\alpha(-\theta t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz \end{aligned}$$

for $t \geq s > 0$, where $E_\alpha(\cdot)$ is the Mittag-Leffler function.

Idea of proof

Write

$$\begin{aligned}\operatorname{corr}[X_\alpha(t), X_\alpha(s)] &= \operatorname{corr}[X_1(E_t), X_1(E_s)] \\ &= \int_0^\infty \int_0^\infty e^{-\theta|u-v|} H(du, dv),\end{aligned}$$

a Lebesgue-Stieltjes integral with respect to the bivariate distribution function $H(u, v) := \mathbb{P}[E_t \le u, E_s \le v]$ of the process E_t.

Idea of proof continued

To compute the integral, we use the bivariate integration by parts formula (Gill et al. (1995))

$$\begin{aligned}\int_0^a \int_0^b F(u,v)H(du,dv) = & \int_0^a \int_0^b H([u,a]\times[v,b])F(du,dv) + \\ & + \int_0^a H([u,a]\times(0,b])F(du,0) \\ & + \int_0^b H((0,a]\times[v,b])F(0,dv) \\ & + F(0,0)H((0,a]\times(0,b]).\end{aligned}$$

with $F(u,v) = e^{-\theta|u-v|}$ and infinite limits of integration.

Idea of proof continued

The application of bivariate integration by parts gives

$$
\begin{aligned}
\int_0^\infty \int_0^\infty F(u,v)H(du,dv) &= \int_0^\infty \int_0^\infty \mathbb{P}[E_t \geq u, E_s \geq v] F(du,dv) \\
&+ \int_0^\infty \mathbb{P}[E_t \geq u] F(du,0) \\
&+ \int_0^\infty \mathbb{P}[E_s \geq v] F(0,dv) + 1.
\end{aligned}
$$

Analysis of the three integrals leads to the formula stated in the Theorem.

Asymptotics of the correlation function

To determine the asymptotics of the correlation function, fix $s > 0$ and recall that

$$E_\alpha(-\theta t^\alpha) \sim \frac{1}{\Gamma(1-\alpha)\theta t^\alpha} \quad \text{as } t \to \infty.$$

Then

$$E_\alpha(-\theta t^\alpha(1-sy/t)^\alpha) \sim \frac{1}{\Gamma(1-\alpha)\theta t^\alpha(1-sy/t)^\alpha}$$

as $t \to \infty$ for any $y \in [0,1]$. Using the dominated convergence theorem we get

$$\begin{aligned}&\frac{\theta\alpha t^\alpha}{\Gamma(1+\alpha)}\int_0^{s/t}\frac{E_\alpha(-\theta t^\alpha(1-z)^\alpha)}{z^{1-\alpha}}dz\\ &\sim \left(\frac{s}{t}\right)^\alpha \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\end{aligned}$$

as $t \to \infty$.

Asymptotics of the correlation function continued

Combining the two terms, for any fixed $s > 0$ we have

$$\mathrm{corr}(X_\alpha(t), X_\alpha(s)) \sim \frac{1}{t^\alpha \Gamma(1-\alpha)} \left(\frac{1}{\theta} + \frac{s^\alpha}{\Gamma(\alpha+1)} \right) \quad \text{as } t \to \infty.$$

Recall that stationary Pearson diffusions exhibit short range dependence, since their correlation function falls off exponentially fast.

However, the correlation function of a fractional Pearson diffusion falls off like a power law with exponent $\alpha \in (0, 1)$ equal to the order of the fractional derivative in time, and so this process exhibits *long range dependence*.

Remarks

In contrast to a Pearson diffusion in steady state, a fractional Pearson diffusion is not stationary.

Since fractional Pearson diffusions are not Markovian, neither the governing fractional backward equation nor the corresponding fractional forward equation uniquely determine the process.

The joint distribution of the inverse stable subordinator E_t at multiple times has recently been computed in Meerschaert and Straka (2012), and in principle, this can be used to give a different proof of the expression of the correlation. However, the resulting integrals do not seem tractable.

For diffusions with constant coefficients, there has been some work on identifying and solving the governing equations of the joint density for multiple times (Baulle and Friedrich (2007), Meerschaert and Straka (2012b). It would be interesting to extend this work, to obtain the governing equations for fractional Pearson diffusions at multiple times.

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