# A Spectral Element Based Parareal Method for the Long Time Integration of Time-Fractional Differential Equations

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June 5, 2013

# Outline

# Introduction

We consider the long time integration of the time fractional differential equation (TFDE),

$$\begin{cases} {}_0D_t^{\alpha} u(t) = \lambda u(t) + f(t), \quad t \in \Omega, \\ u(0) = u_0, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{C}$. ${}_0D_t^{\alpha} u(t)$ is the Caputo derivative of order $\alpha$ of function $u$,

$${}_0D_t^{\alpha} u(t) \triangleq {}_0D_t^{\alpha-n} \frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-1-\alpha} \frac{\mathrm{d}^n u(s)}{\mathrm{d}s^n} \mathrm{d}s,$$

where $n$ is an integer and $\alpha \in (n-1, n]$.

**Challenges:**

1. insure high accuracy
2. reduce computational cost

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**Solutions:**

1. high order spectral element method
2. parallel-in-time (parareal) method

# Spectral collocation method based on Jacobian polynomial

Denote $J_j^{a,b}(t)$ the $j$-th order Jacobi polynomial with index $(a,b)$ defined on $[-1,1]$.
$\{J_j^{a,b}(t)\}_{j=0}^N$ satisfies the three terms recurrence relation:

$$\begin{aligned}
&J_0^{a,b}(t) = 1,\\
&J_1^{a,b}(t) = (a+b+2)t + (a-b),\\
&J_{j+1}^{a,b}(t) = (A_j^{a,b}t - B_j^{a,b})J_j^{a,b}(t) - C_j^{a,b}J_{j-1}^{a,b}(t), \quad 1 \leqslant j \leqslant N-1,
\end{aligned}$$

where $A_j^{a,b}, B_j^{a,b}, C_j^{a,b}$ are known coefficients.

# Fractional derivative of Jacobian polynomial

Consider the fractional integral of Jacobi polynomials,

$$\hat{J}_j^{a,b,\alpha}(t) \triangleq {}_{-1}D_t^{-\alpha} J_j^{a,b}(t) = \frac{1}{\Gamma(\alpha)} \int_{-1}^{t} (t-s)^{\alpha-1} J_j^{a,b}(s)\mathrm{d}s$$

Following [C. Li, F. Zeng, and F. Liu, 2012, FCAA], $\{\hat{J}_j^{a,b,\alpha}(t)\}_{j=0}^{N}$ satisfies the three term relation:

$$\hat{J}_0^{a,b,\alpha}(t) = \frac{(t+1)^\alpha}{\Gamma(\alpha+1)};$$

$$\hat{J}_1^{a,b,\alpha}(t) = \frac{a+b+2}{2}\left(\frac{t(t+1)^\alpha}{\Gamma(\alpha+1)} - \alpha\frac{(t+1)^{\alpha+1}}{\Gamma(\alpha+2)}\right) + \frac{a-b}{2}\hat{J}_0^{a,b,\alpha}(t);$$

$$\hat{A}_j^{a,b}\hat{J}_{j+1}^{a,b,\alpha}(t) = \hat{B}_j^{a,b}\hat{J}_j^{a,b,\alpha}(t) + \hat{C}_j^{a,b}\hat{J}_{j-1}^{a,b,\alpha}(t) + \hat{D}_j^{a,b}(t+1)^\alpha,$$

# Fractional derivative of Jacobian polynomial

Using the relation

$$\frac{d^m}{dx^m} J_j^{a,b}(t) = d_{j,m}^{a,b} J_{j-m}^{a+m,b+m}(t), \quad m \geqslant j, \quad m \in \mathbb{N},$$

the fractional derivative of a Jacobi polynomial can be expressed as

$$(J_j^{a,b})^{(\alpha)}(t) \triangleq {}_{-1}D_t^{\alpha} J_j^{a,b}(t) = d_{j,n}^{a,b} \hat{J}_{j-n}^{a+n,b+n,n-\alpha}(t), \quad n-1 < \alpha < n. \tag{2}$$

# Spectral collocation method

Assume $u(t)$ is a polynomial of degree no greater than $N$.
Then we have

$$u(t) = \sum_{i=0}^{N} \hat{u}_i J_i^{a,b}(t),$$

Hence, the fractional derivative of $u(t)$ is expressed as

$$_{-1}D_t^{\alpha} u(t) = \sum_{j=0}^{N} \hat{u}_j (J_j^{a,b})^{(\alpha)}(t).$$

# Matrix form

Let $\{t_i\}_{i=0}^{N}$ be a set of Gauss quadrature points, e.g., the L-G-L points. Define the fractional differentiation matrix $\mathcal{D}^{\alpha}$ as

$$\mathcal{D}_{ij}^{\alpha} = (J_j^{a,b})^{(\alpha)}(t_i),$$

Denote

$$\mathbf{U} = (u(t_0), u(t_1), \cdots, u(t_N))^T, \hat{\mathbf{U}} = (\hat{u}(t_0), \hat{u}(t_1), \cdots, \hat{u}(t_N))^T,$$

Then, equation (1) can be written as matrix form,

$$\mathcal{D}^{\alpha}\hat{\mathbf{U}} = \lambda \mathbf{U} + F$$

# Matrix form

Suppose $\mathcal{J}$ is a $(N+1)\times(N+1)$ matrix defined by $\mathcal{J}_{ij} = J_j^{a,b}(t_i)$, then $\mathbf{U} = \mathcal{J}\hat{\mathbf{U}}$.

The matrix equation can be expressed as modal form

$$\mathcal{D}^\alpha \hat{\mathbf{U}} = \lambda \mathcal{J} \hat{\mathbf{U}} + F$$

and a nodal form,

$$\mathcal{D}^\alpha \mathcal{J}^{-1} \mathbf{U} = \lambda \mathbf{U} + F$$

Combined with the initial condition, both matrix equations can be solved uniquely.

# Numerical example 1

Consider (1) with $\lambda = 0$,

$$ {}_{-1}D_t^{\alpha} u(t) = f(t), \quad t \in [-1, 1]. $$

We choose $u(t) = \sin_M(t+1)$ as our test example, where $\sin_M(x)$ is Taylor of expansion of $\sin(x)$ up to degree $M$.

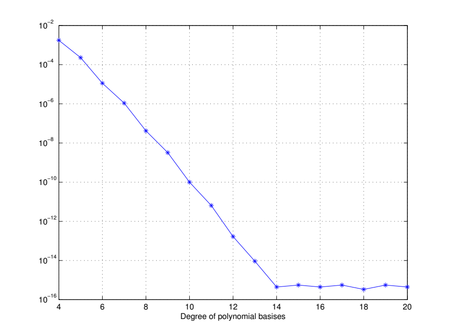



Figure: Maximum errors of the spectral collocation method with $\alpha = 0.5$.

# Spectral Element Method for the TFDE

Let $\Omega = [t_l, t_r]$, which is divided into $K$ elements:

$$t_l = t_0 < t_1 < \cdots < t_K = t_r, \Omega_i = [t_{i-1}, t_i],\ h_i = t_i - t_{i-1}, 1 \leqslant i \leqslant K.$$

For $t \in [t_i, t_{i+1}]$ and $n - 1 < \alpha \leqslant n$, we have

$$\begin{aligned} {}_0D_t^\alpha u(t) = & \frac{1}{\Gamma(n-\alpha)} \left( \sum_{k=1}^{i} \int_{\Omega_k} (t-s)^{n-1-\alpha} \frac{\partial^n u(s)}{\partial s^n} ds \right) \\ & + \frac{1}{\Gamma(n-\alpha)} \left( \int_{t_i}^{t} (t-s)^{n-1-\alpha} \frac{\partial^n u(s)}{\partial s^n} ds \right). \end{aligned}$$

The last term can be evaluated using (2). This induces a stiffness matrix, $M_0$, which comes from $\mathcal{D}^\alpha$.

For each $j$, we define the following integral,

$$\tilde{J}_j^{a,b,\alpha}(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_{-1}^{1} (t-s)^{\alpha-1} J_j^{a,b}(s) ds, \quad t > 1.$$

How to compute $\tilde{J}_j^{a,b,\alpha}(t)$? Exact integral or numerical integral?

1. If $t \gg s$, Gauss quadrature;
2. If $t = 1$, formula (2) and Jacobian-Gauss quadrature;
3. If $t$ close to 1, accuracy of both Gauss and Jacobian-Gauss quadrature would be reduced.

To address this, we return to the three terms recursive relation for Jacobian polynomials. We obtain a new three terms recursive relation for $\tilde{J}_j^{a,b,\alpha}(t)$:

$$\tilde{J}_0^{a,b,\alpha}(t) = \frac{(t+1)^\alpha - (t-1)^\alpha}{\Gamma(\alpha+1)},$$
$$\tilde{J}_1^{a,b,\alpha}(t) = \frac{a+b+2}{2\Gamma(2+\alpha)}\left((t-\alpha)(t+1)^\alpha - (t+\alpha)(t-1)^\alpha\right) + \frac{a-b}{2}\tilde{J}_0^{a,b,\alpha}(t),$$
$$\tilde{A}_j^{a,b}\tilde{J}_{j+1}^{a,b,\alpha}(t) = \tilde{B}_j^{a,b}\tilde{J}_j^{a,b,\alpha}(t) + \tilde{C}_j^{a,b}\tilde{J}_{j-1}^{a,b,\alpha}(t) + \tilde{D}_j^{a,b}(t+1)^\alpha + \tilde{E}_j^{a,b}(t-1)^\alpha,$$

where $\tilde{A}_j^{a,b}, \tilde{B}_j^{a,b}, \tilde{C}_j^{a,b}, \tilde{D}_j^{a,b}, \tilde{E}_j^{a,b}$ are the coefficients.

Then, the integral over element of $(J_j^{a,b})^{(n)}(s)$ can be computed using

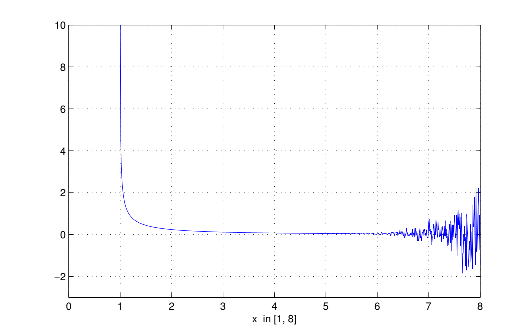
$$\begin{aligned}\tilde{D}^{\alpha} J_j^{a,b}(t) &\triangleq \frac{1}{\Gamma(n-\alpha)} \int_{-1}^{1} (t-s)^{n-\alpha-1} (J_j^{a,b})^{(n)}(s) \mathrm{d}s \\ &= d_{j,n}^{a,b} \tilde{J}_{j-n}^{a+n,b+n,n-\alpha}(t), \quad t > 1.\end{aligned} \tag{3}$$

How well does this work?

We compute

$$\tilde{J}_{14}^{a,b,\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_{-1}^{1} (t-s)^{\alpha-1} J_{14}^{a,b}(s) ds,$$

for $t \in [1,8]$, $\alpha = 0.5$. The value of $\tilde{J}_{14}^{a,b,\alpha}(t)$ is shown in the figure.



# Hybrid strategy

- smooth part : Gauss quadrature;
- near singular part : three term recurrence relation.

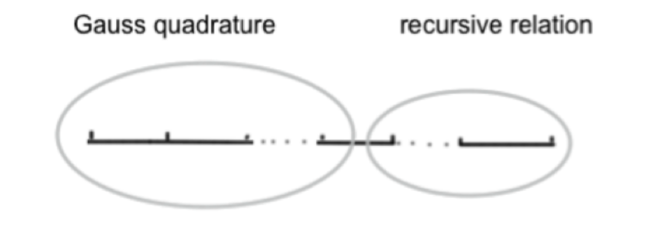



Figure: Hybrid strategy

Denote $\boldsymbol{U}_k$ as the vector of function values at all Gauss points in element $k$,

$$\boldsymbol{U}_k = (u_{k,0}, u_{k,1}, \cdots, u_{k,N})^T$$

Then,

$$M_0\boldsymbol{U}_k + M_1\boldsymbol{U}_{k-1} + \cdots + M_{k-1}\boldsymbol{U}_1 = \lambda\boldsymbol{U}_k + \boldsymbol{f}_k, \quad k = 1, 2, \cdots, K.$$

$$(\lambda\mathcal{I} - M_0)\boldsymbol{U}_k = M_1\boldsymbol{U}_{k-1} + \cdots + M_{k-1}\boldsymbol{U}_1 - \boldsymbol{f}_k, \quad k = 1, 2, \cdots, K.$$

Combined with the initial condition for each element, the equation can be solved uniquely.

# Analysis

For the approximation of fractional derivative, we have the following lemma,

## Lemma 2.1

*(Q. Xu, J. S. Hesthaven,2012) Suppose $u(x)$ is a smooth function defined in $\Omega = [a, b]$. $\Omega_h$ is a discretization of the domain with interval weigth $h$, $u_h(x)$ is a piecewise interpolation of $u$ in $P_h^N = span\{1, x, \cdots, x^N\}$. $\forall x \in I_i$, $u_h$ interpolate $u$ at all Gauss points. Then we have*

$$\|\,{}_aD_x^\alpha u - {}_aD_x^\alpha u_h\|_{L_2} \leqslant \begin{cases} C, & n > N \\ Ch^{N+1-n}, & n \leqslant N, N-n \ odd \\ Ch^{N+1-\alpha}, & n \leqslant N, N-n \ even \end{cases}$$

*where $n - 1 < \alpha < n$, $C$ is a constant independent of $x$.*

# Convergence

Since $u$ is solved element by element, for $0 < \alpha < 1$ the following result is obtained.

## Proposition 2.1

*Suppose $u$ is a smooth function, $u_h$ is the numerical solution of equation (1) using spectral elements method based on Gauss points, $0 < \alpha < 1$, $e = u - u_h$,*

$$\|e\|_2 \leqslant \begin{cases} Ch^N, & N \text{ is even} \\ Ch^{N+1-\alpha}, & N \text{ is odd} \end{cases}$$

# Numerical examples

Consider the following equation,

$$ {}_0D_t^\alpha u(t) = -u(t) + f(t), \quad t \in \Omega = [0, 3\pi], $$

We choose the exact solution to be $u(t) = \sin_M(t)$.

| | $N = 1$ | | | | | |
|---|---|---|---|---|---|---|
| | $\alpha = 0.1$ | | $\alpha = 0.4$ | | $\alpha = 0.8$ | |
| K | $\|e_h\|_2$ | Order | $\|e_h\|_2$ | Order | $\|e_h\|_2$ | Order |
| 20 | 6.70e-04 | 1.68 | 5.15e-03 | 1.50 | 2.74e-02 | 1.18 |
| 40 | 2.09e-04 | 1.69 | 1.82e-03 | 1.51 | 1.21e-02 | 1.18 |
| 60 | 1.05e-04 | 1.66 | 9.87e-04 | 1.50 | 7.51e-03 | 1.19 |
| 80 | 6.54e-05 | - | 6.42e-04 | - | 5.34e-03 | - |
| | $N = 2$ | | | | | |
| | $\alpha = 0.1$ | | $\alpha = 0.4$ | | $\alpha = 0.8$ | |
| K | $\|e_h\|_2$ | Order | $\|e_h\|_2$ | Order | $\|e_h\|_2$ | Order |
| 20 | 3.33e-06 | 2.94 | 2.42e-05 | 2.89 | 1.88e-04 | 2.27 |
| 40 | 4.33e-07 | 2.85 | 3.27e-06 | 2.90 | 3.90e-05 | 2.25 |
| 60 | 1.36e-07 | 2.75 | 1.01e-06 | 2.84 | 1.56e-05 | 2.26 |
| 80 | 6.18e-08 | - | 4.45e-07 | - | 8.18e-06 | - |

# Numerical examples

| | $N=3$ | | | | | |
|---|---|---|---|---|---|---|
| | $\alpha=0.1$ | | $\alpha=0.4$ | | $\alpha=0.8$ | |
| K | $\|e_h\|_2$ | Order | $\|e_h\|_2$ | Order | $\|e_h\|_2$ | Order |
| 20 | 3.10e-08 | 3.94 | 2.82e-07 | 3.81 | 2.40e-06 | 3.33 |
| 40 | 2.01e-09 | 3.95 | 2.02e-08 | 3.81 | 2.39e-07 | 3.29 |
| 60 | 4.06e-10 | 3.94 | 4.31e-09 | 3.80 | 6.30e-08 | 3.27 |
| 80 | 1.31e-10 | - | 1.45e-09 | - | 2.46e-08 | - |
| | $N=4$ | | | | | |
| | $\alpha=0.1$ | | $\alpha=0.4$ | | $\alpha=0.8$ | |
| K | $\|e_h\|_2$ | Order | $\|e_h\|_2$ | Order | $\|e_h\|_2$ | Order |
| 20 | 2.31e-10 | 4.96 | 1.73e-09 | 4.83 | 1.54e-08 | 4.41 |
| 40 | 7.42e-12 | 4.96 | 6.08e-11 | 4.74 | 7.28e-10 | 4.35 |
| 60 | 9.93e-13 | 4.96 | 8.91e-12 | 4.45 | 1.25e-10 | 4.31 |
| 80 | 2.38e-13 | - | 2.48e-12 | - | 3.61e-11 | - |
| | $N=5$ | | | | | |
| | $\alpha=0.1$ | | $\alpha=0.4$ | | $\alpha=0.8$ | |
| K | $\|e_h\|_2$ | Order | $\|e_h\|_2$ | Order | $\|e_h\|_2$ | Order |
| 20 | 1.51e-12 | 5.82 | 1.26e-11 | 5.85 | 1.22e-10 | 5.40 |
| 40 | 2.66e-14 | - | 2.20e-13 | 5.72 | 2.89e-12 | 5.33 |
| 60 | 2.89e-15 | - | 2.16e-14 | - | 3.34e-13 | 5.12 |
| 80 | 1.23e-14 | - | 5.88e-15 | - | 7.66e-14 | - |

# Parareal algorithm for ODEs

The parallel-in-time approach, called parareal method, was proposed in (Lions, Maday, and Turinici, 2001).
Consider the following systems of ODEs:

$$
\begin{aligned}
&u_t = f(u,t), \quad t \in \Theta := (0,T],\\
&u(0) = u_0,
\end{aligned}
$$

The parareal algorithm is defined using two propagation operators.

- A coarse solver $\mathcal{G}(t_2, t_1, u_1)$, provides a cheap but inaccurate approximation to $u(t_n)$ with $u(t_{n-1}) = u_{n-1}$,
- A fine solver $\mathcal{F}(t_2, t_1, u_1)$, provides an expensive but accurate approximation of $u(t_n)$ from $u(t_{n-1}) = u_{n-1}$.

First, the time domain is decomposed into $K$ elements:

$$0 = T_0 < \cdots < T_n < \cdots < T_K = T, \quad T_n = n\Delta T, \quad \Delta T = \frac{T}{K}.$$

Assume that $u_n^k$ is the $k'th$ iterated approximation to $u_n$. The process of parareal algorithm is shown as,

1. Initial guess : compute $u_n^0$ using coarse solver,
2. Assuming that $u_n^k$ is known, compute the solution in parallel using the fine solver for each element,
3. The parareal iteration

$$u_{n+1}^{k+1} = \mathcal{G}u_n^{k+1} + \mathcal{F}u_n^k - \mathcal{G}u_n^k.$$

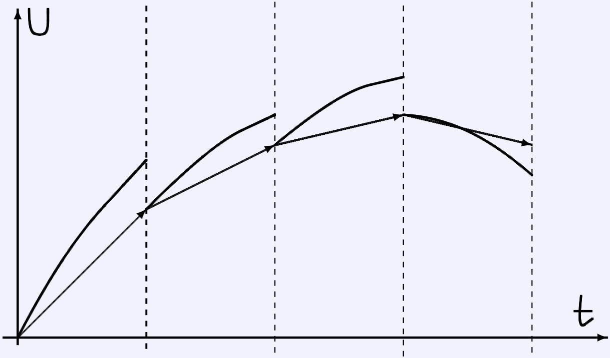
4. The algorithm is terminated if $|u_n^k - u_n^{k-1}| \leqslant \tau$.

# Parareal method for TFDE

1. Initial guess using coarse solver, ( serial)

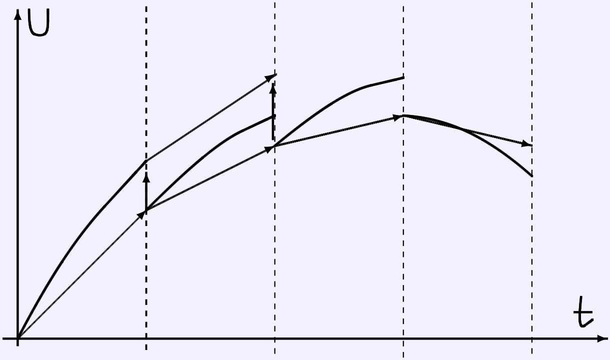
# Parareal method for TFDE

2. Using fine solver in each element, ( parallel )



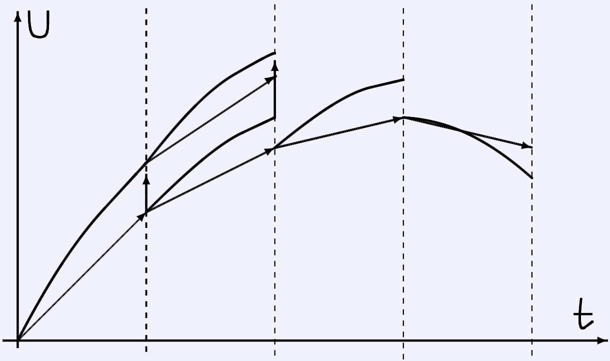
# Parareal method for TFDE

3. Compute the new solution using coarse solver



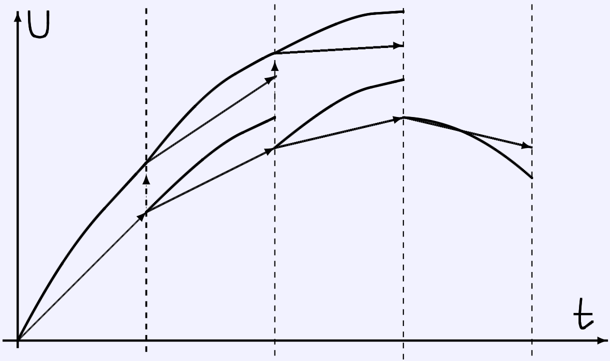
# Parareal method for TFDE

Correct the solution



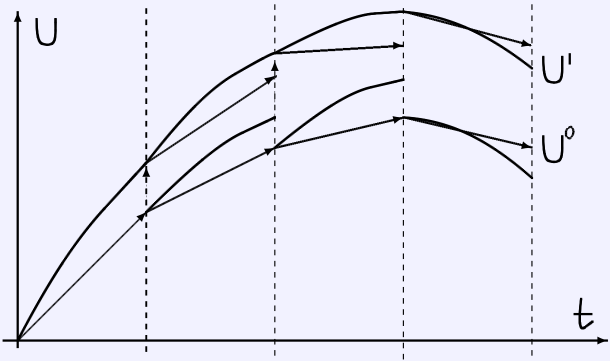
# Parareal method for TFDE

Advance iteration



# Parareal method for TFDE

Advance iteration



# Parareal method for TFDE

Consider the fractional ordinary differential equation with $\alpha < 1$

$$\begin{cases} {}_0D_t^{\alpha} u(t) = \lambda u(t), \quad t \in \Omega, \\ u(0) = u_0, \end{cases}$$

Both the coarse solver and the fine solver are defined based on our spectral element method in this talk.

- Coarse grid : $N_c + 1$ collocation points.
- Fine grid: $N_f + 1$ collocation points.
- Memory part : $\boldsymbol{U}_{0:n} = \{\boldsymbol{U}_0, \boldsymbol{U}_1, \cdots, \boldsymbol{U}_n\}$.
- Denote $v_n$ as the value of $u(t)$ at right interface of element $n$, that's $v_n = \boldsymbol{U}_{n,N} = \boldsymbol{U}_{n+1,0}$.

**Challenge 1**: Dealing with the memory part.
We propose the following parareal method for TFDE,

$$\boldsymbol{U}_{n+1}^{k+1} = \mathcal{G}\boldsymbol{U}_{0:n}^{k+1} + \mathcal{F}\boldsymbol{U}_{0:n}^{k} - \mathcal{G}\boldsymbol{U}_{0:n}^{k}, \quad 0 \leqslant n \leqslant N-1.$$

**Challenge 1**: Dealing with the memory part.
We propose the following parareal method for TFDE,

$$\boldsymbol{U}_{n+1}^{k+1} = \mathcal{G}\boldsymbol{U}_{0:n}^{k+1} + \mathcal{F}\boldsymbol{U}_{0:n}^{k} - \mathcal{G}\boldsymbol{U}_{0:n}^{k}, \quad 0 \leqslant n \leqslant N-1.$$

**Challenge 2**: Convergence.
The spectral element method above,

- fine solution converges to exact solution as $\Delta t \to 0$ or $K \to \infty$.
- Will this parareal solution converges to fine solution as iteration number $k$ increase?

Now we consider the values at each interface of elements, $v_n$.

We have

$$\boldsymbol{U}_n = \sum_{j=1}^{n-1} (\lambda \mathcal{I} - M_0)^{-1} M_j \boldsymbol{U}_{n-j}, \quad n = 1, 2, \cdots, K.$$

For the coarse grid, we define an operator $\mathcal{R}$, s.t.

$$v_n^k = \sum_{j=1}^{n} \mathcal{R}_j v_{n-j}^k$$

where $k$ denotes $k'th$ iteration, $v_0$ is provided by the initial condition.

The fine integration operator $\mathcal{R}_j^{\mathcal{F}}$ can be defined accordingly.

Introduce the parareal method for $v_n^{k+1}$,

$$v_n^{k+1} = \sum_{j=1}^{n} \mathcal{R}_j v_{n-j}^{k+1} + \sum_{j=1}^{n} \mathcal{R}_j^{\mathcal{F}} v_{n-j}^{k} - \sum_{j=1}^{n} \mathcal{R}_j v_{n-j}^{k}, \quad 0 \leqslant n \leqslant K-1.$$

Denote $\boldsymbol{v}^k = (v_0^k, v_1^k, \cdots, v_K^k)^T$

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \mathcal{R}_1 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & 0 \\ \mathcal{R}_{K+1} & \cdots & \mathcal{R}_1 & 0 \end{pmatrix}$$

$$\mathcal{N} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \mathcal{R}_1^{\mathcal{F}} - \mathcal{R}_1 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & 0 \\ \mathcal{R}_{K+1}^{\mathcal{F}} - \mathcal{R}_{K+1} & \cdots & \mathcal{R}_1^{\mathcal{F}} - \mathcal{R}_1 & 0 \end{pmatrix}$$

then we get the matrix form,

$$\boldsymbol{v}^{k+1} = \mathcal{M}\boldsymbol{v}^{k+1} + \mathcal{N}\boldsymbol{v}^k, \quad 0 \leqslant n \leqslant K-1.$$

Assume $u(t_n) =$ fine solution at $t_n$, then $u(t_n) = \sum_{j=1}^{n} \mathcal{R}_j^{\mathcal{F}} u(t_{n-j})$.

Denote $e_n^k = u(t_n) - v_n^k$ and $\boldsymbol{e}^k = (e_0^k, e_1^k, \cdots, e_K^k)^T$, then

$$
\begin{aligned}
e_n^{k+1} &= u(t_n) - v_n^{k+1} \\
&= \sum_{j=1}^{n} \mathcal{R}_j^{\mathcal{F}} u(t_{n-j}) - \sum_{j=1}^{n} \mathcal{R}_j v_{n-j}^{k+1} - \sum_{j=1}^{n} \mathcal{R}_j^{\mathcal{F}} v_{n-j}^{k} + \sum_{j=1}^{n} \mathcal{R}_j v_{n-j}^{k} \\
&= \sum_{j=1}^{n} (\mathcal{R}_j^{\mathcal{F}} - \mathcal{R}_j) e_{n-j}^{k} + \sum_{j=1}^{n} \mathcal{R}_j e_{n-j}^{k+1}
\end{aligned}
$$

In matrix form, we have

$$
\boldsymbol{e}^{k+1} = \mathcal{M} \boldsymbol{e}^{k+1} + \mathcal{N} \boldsymbol{e}^{k}
$$

Since $\mathcal{M}, \mathcal{N}$ are strictly lower triangular matrix,

$$\begin{aligned} \boldsymbol{e}^{k+1} &= \mathcal{N}(\sum_{j=0}^{K} \mathcal{M}^j)\boldsymbol{e}^k \\ &= \mathcal{N}^{k+1}(\sum_{j=0}^{K} \mathcal{M}^j)^{k+1}\boldsymbol{e}^0 \end{aligned}$$

Assume the convergence order of coarse solver is $p+1$, then $(\mathcal{R}^{\mathcal{F}} - \mathcal{R}) \sim (\Delta t)^{p+1}$, $\|\mathcal{N}^k \boldsymbol{e}^0\| \sim \left(K(\Delta t)^{p+1}\right)^k \sim (\Delta t)^{kp}$.

If $|(\sum_{j=0}^{K} \mathcal{M}^j)^k|$ is bounded, the parareal solution converges to the fine solution with convergence order $(\Delta t)^{kp}$,($k$ is number of iterations).

For the case $N_c = 1$. Our method reduce to $L_1$ scheme. The distribution of $\mathcal{R}_j$ with $K = 20, \alpha = 0.5$ is shown in the following figure.

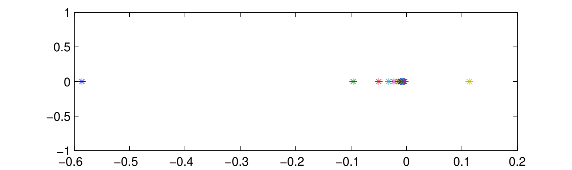



Figure: The distribution of $\mathcal{R}_j$

1. For $N_c = 1, \lambda < 0$, $|(\sum_{j=0}^{K} \mathcal{M}^j)^k|$ is bounded.
2. For a high order coarse solver $(N_c \geqslant 2)$, convergence of parareal method can be verified numerically.

Assume $u_e(t)$ is the exact solution. Let $\eta_n = u_e(t_n) - u(t_n)$, $\epsilon_n^k = u_e(t_n) - v_n^k$, then $\|\epsilon^k\| \leqslant C(\|e^k\| + \|\eta\|)$.

We consider

$$ {}_0D_t^{\alpha} u(t) = -u(t) + f(t), \quad t \in [0, T]. $$

To test the numerical accuracy, we choose the exact solution $u_e(t) = \sin_{60}(t)$. $f(t)$ can be calculated according to this setting.

**Example 3.1**

*Let $T = 2\pi$.The number of processors $N_p = 10$, and $N_F = 12$. $N_C = 2, 4, 6$. The absolute error $\max|v^k - u_e|$ is shown as,*

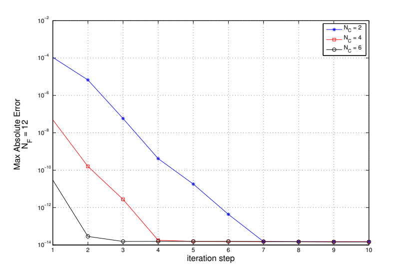



Figure: The accuracy of the parareal method in Example 3.1.

## Example 3.2

*Let* $f(t) = \sin(t)$, $T = 15\pi, \alpha = 0.5$. *The number of processors* $N_p = 20$, *and* $N_F = 22$. $N_C = 1$. $e^k = \max|v^k - u|$.

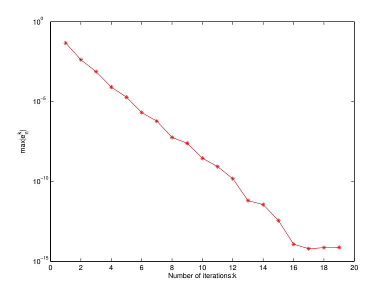



Figure: Convergence

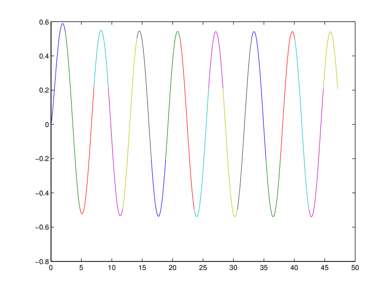


Figure: The parareal solution

## Example 3.3

*Let* $f(t) = \sin(t)$,$T = 15\pi, \alpha = 0.1, 0.5, 0.9$. *The number of processors* $N_p = 20$, *and* $N_F = 16$. $N_C = 1$. *The error* $e^k = \max|v^k - u|$.

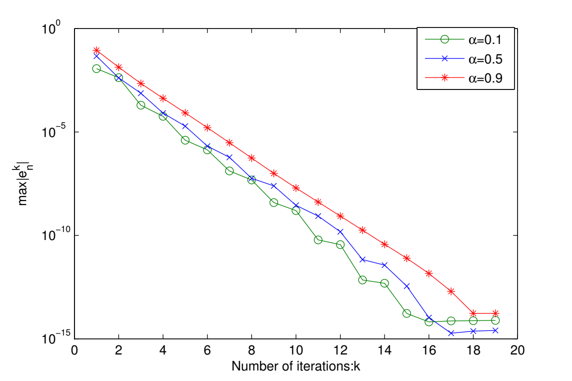



Figure: The error for different $\alpha$

## Example 3.4

*Let* $f(t) = \sin(t)$,$T = 15\pi, \alpha = 0.6$. *The number of processors* $N_p = 20$, *and* $N_F = 16.N_C = 1, 2, 3, 4$. *The error* $e^k = \max|v^k - u|$.

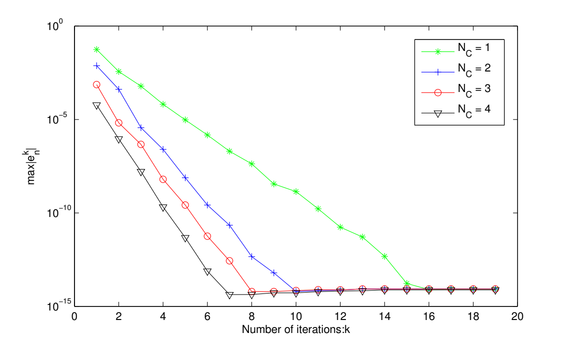



Figure: The error for different coarse solvers

# Parareal method for subdiffusion equation

We consider the following equation with periodic boundary condition,

$$ {}_0D_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in [0, 2\pi], t \in [0, T]. $$

with initial condition $u(x,0) = \sin(x)$. The exact solution is $u(x,t) = E_{\alpha}(-t^{\alpha})\sin(x)$, where $E_{\alpha}(t)$ is Mittag-Leffler function with one parameter.
Using Fourier transform in space, this equation becomes a system of decoupled ODEs for each Fourier mode $\omega$.

$$ {}_0D_t^{\alpha} \hat{u} = -\omega^2 \hat{u}. $$

# Parareal method for subdiffusion equation

## Example 4.1

*Let* $T = 2, \alpha = 0.2$. *The number of processors* $N_p = 30$, *and* $N_F = 16$. $N_C = 2$. *Number of Fourier modes* $m = 40$. *The error* $e^k = \max |v^k - u|$.

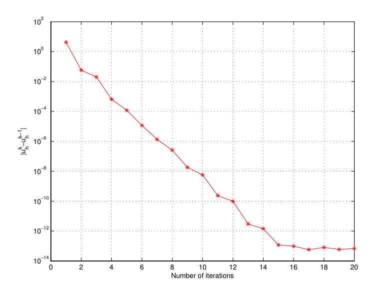



Figure: Convergence

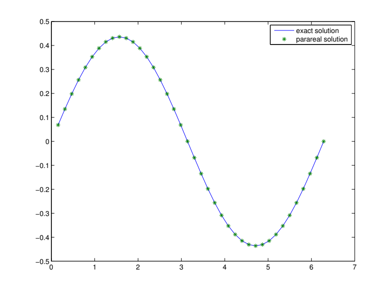



Figure: The parareal solution

# Parareal method for subdiffusion equation

We consider the parareal method for long time integration of subdiffusion equation.

**Example 4.2**

*Let* $T = 100, \alpha = 0.2, 0.5, 0.8$*. The number of processors* $N_p = 30$*, and* $N_F = 16.N_C = 2$*, Number of Fourier modes* $m = 60$*.*

We consider the error between parareal solution and fine solution.

# Parareal method for subdiffusion equation

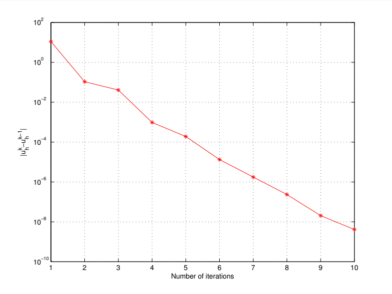



Figure: Convergence for $\alpha = 0.2$

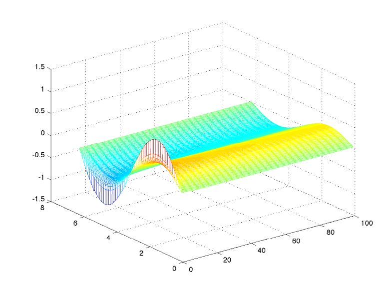


Figure: The parareal solution for $\alpha = 0.2$

# Parareal method for subdiffusion equation

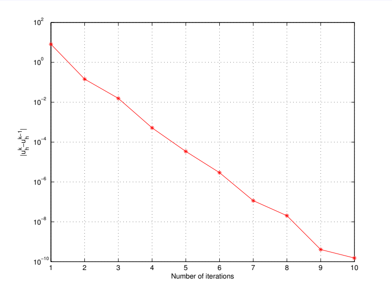



Figure: Convergence for $\alpha = 0.5$

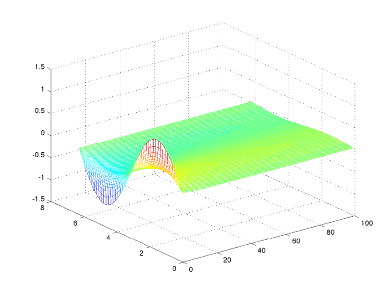


Figure: The parareal solution for $\alpha = 0.5$

# Parareal method for subdiffusion equation

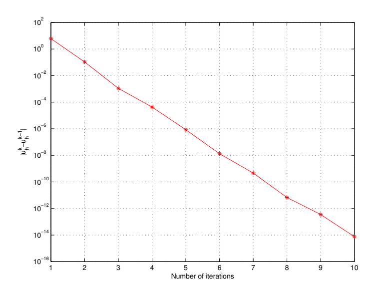



Figure: Convergence for $\alpha = 0.8$

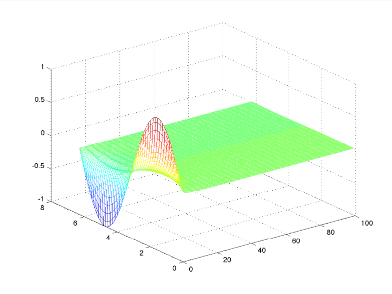



Figure: The parareal solution for $\alpha = 0.8$

## Summary

1. hybrid strategy for near singular integration of memory part;
2. spectral element method for TFDE and the convergence;
3. parareal method for time fractional differential equation;
4. convergence of parareal method for TFDE.

## Future work

1. Analysis of high order coarse solver;
2. Parareal method for fractional diffusion-wave equation.

Thanks for your attention!