Matrix-based approaches as an emerging framework for numerical solution of initial and boundary value problems for ordinary and partial differential equations of arbitrary real order

Igor Podlubny

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http://www.tuke.sk/podlubny/

A talk
in five short movements

Overture

Slovakia

Norway
cotland
Inverness
Peterhead
Edinburgh
Estonia
Latvia
Denmark
Lithuania
Belfast
Dumfries
Dublin
United Kingdom
Belarus
Wexford
Birmingham
Norwich
Netherlands
Poland
Cork
Stroud
London
Germany
Southampton
Brighton
Belgium
Czech Republic
England
Slovakia
Ukraine
Austria
Hungary
Moldova
France
Switzerland
Romania
Croatia
Serbia and Montenegro
Italy
Bulgaria
Georgia
Spain
Portugal
Greece
Turkey
Syria
Tunisia
Iraq
Morocco
Jordan

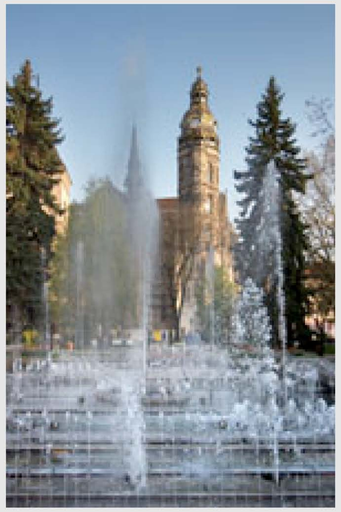
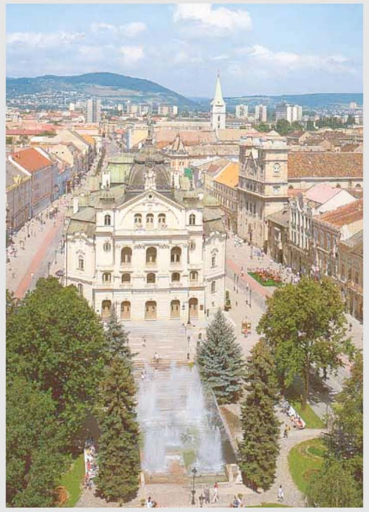
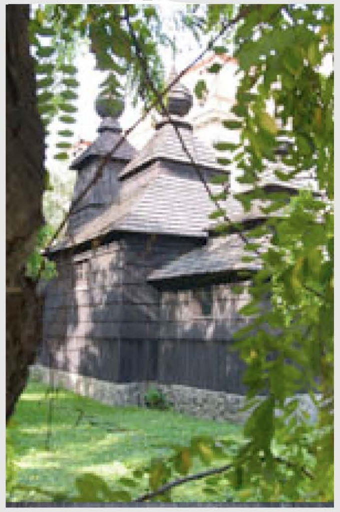
Is it in the middle of nowhere?

Boston Marathon: 1897
Yonkers Marathon: 1907
Kosice Marathon: 1924

KOŠICE2013
EUROPEAN CAPITAL OF CULTURE

#3 in CNN's Top Travel destinations

Kosice and surroundings

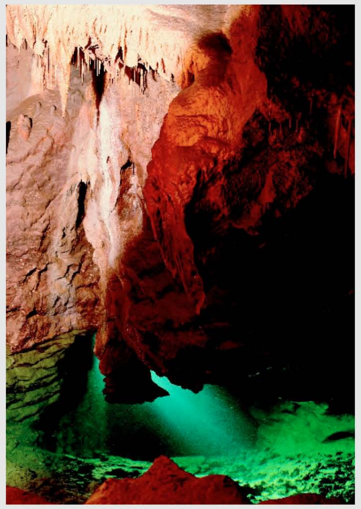





Tatra Mountains



Caves



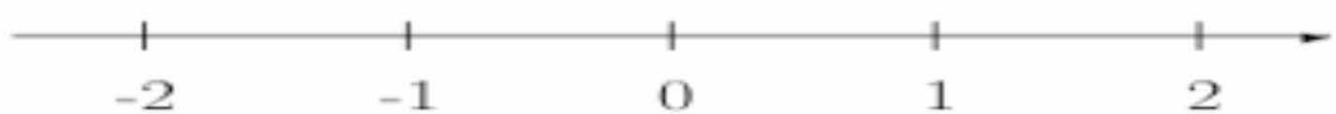


Main idea of fractional calculus: Interpolation of operators

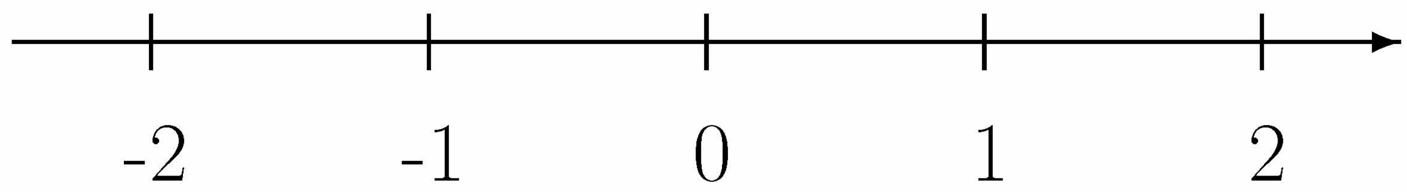
$$f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \frac{d^3 f}{dt^3}, \quad \cdots$$

$$f, \quad \int f(t)dt, \quad \int dt \int f(t)dt, \quad \int dt \int dt \int f(t)dt, \quad \ldots$$

$$\ldots, \quad \frac{d^{-2} f}{dt^{-2}}, \quad \frac{d^{-1} f}{dt^{-1}}, \quad f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \ldots$$



From integer to non-integer



$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n}$$

$$x^n = e^{n \ln x}$$

$$n! = 1 \cdot 2 \cdot 3 \cdot \ \ldots \cdot (n-1) \cdot n,$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad x > 0,$$

$$\Gamma(n+1) = 1 \cdot 2 \cdot 3 \cdot \ \ldots \cdot n = n!$$

Fractional Calculus: a response to S&T needs

static models	dynamical models	fractional order modeling
geometry, algebra	differential and integral calculus	fractional calculus

Fractional Calculus in WoK: 136 subject areas (applications)

ACOUSTICS
AGRICULTURAL ECONOMICS & POLICY
AGRICULTURAL ENGINEERING
AGRONOMY
ANESTHESIOLOGY
ASTRONOMY & ASTROPHYSICS
AUTOMATION & CONTROL SYSTEMS
BIOCHEMICAL RESEARCH METHODS
BIOCHEMISTRY & MOLECULAR BIOLOGY
BIOLOGY
BIOPHYSICS
BIOTECHNOLOGY & APPLIED MICROBIOLOGY
BUSINESS
BUSINESS, FINANCE
CARDIAC & CARDIOVASCULAR SYSTEMS
CELL BIOLOGY
CHEMISTRY, ANALYTICAL
CHEMISTRY, APPLIED
CHEMISTRY, INORGANIC & NUCLEAR
CHEMISTRY, MULTIDISCIPLINARY
CHEMISTRY, ORGANIC
CHEMISTRY, PHYSICAL
COMPUTER SCIENCE, ARTIFICIAL INTELLIGENCE
COMPUTER SCIENCE, CYBERNETICS
COMPUTER SCIENCE, HARDWARE & ARCHITECTURE
COMPUTER SCIENCE, INFORMATION SYSTEMS
COMPUTER SCIENCE, INTERDISCIPLINARY APPLICATIONS
COMPUTER SCIENCE, SOFTWARE ENGINEERING
COMPUTER SCIENCE, THEORY & METHODS
CONSTRUCTION & BUILDING TECHNOLOGY
CRIMINOLOGY & PENOLOGY
CRYSTALLOGRAPHY
DENTISTRY, ORAL SURGERY & MEDICINE
ECOLOGY
ECONOMICS
EDUCATION & EDUCATIONAL RESEARCH
EDUCATION, SCIENTIFIC DISCIPLINES
ELECTROCHEMISTRY
ENERGY & FUELS
ENGINEERING, AEROSPACE
ENGINEERING, BIOMEDICAL
ENGINEERING, CHEMICAL
ENGINEERING, CIVIL
ENGINEERING, ELECTRICAL & ELECTRONIC
ENGINEERING, ENVIRONMENTAL
ENGINEERING, GEOLOGICAL
ENGINEERING, INDUSTRIAL
ENGINEERING, MANUFACTURING
ENGINEERING, MECHANICAL
ENGINEERING, MULTIDISCIPLINARY
ENGINEERING, OCEAN
ENGINEERING, PETROLEUM
ENVIRONMENTAL SCIENCES
ENVIRONMENTAL STUDIES
EVOLUTIONARY BIOLOGY
FOOD SCIENCE & TECHNOLOGY
GASTROENTEROLOGY & HEPATOLOGY
GENETICS & HEREDITY
GEOCHEMISTRY & GEOPHYSICS
GEOLOGY
GEOSCIENCES, MULTIDISCIPLINARY
HEMATOLOGY
HISTORY & PHILOSOPHY OF SCIENCE
HOSPITALITY, LEISURE, SPORT & TOURISM
IMAGING SCIENCE & PHOTOGRAPHIC TECHNOLOGY
INSTRUMENTS & INSTRUMENTATION
INTERNATIONAL RELATIONS
LIMNOLOGY
MANAGEMENT
MATERIALS SCIENCE, BIOMATERIALS
MATERIALS SCIENCE, CERAMICS
MATERIALS SCIENCE, CHARACTERIZATION & TESTING
MATERIALS SCIENCE, COATINGS & FILMS
MATERIALS SCIENCE, COMPOSITES
MATERIALS SCIENCE, MULTIDISCIPLINARY
MATERIALS SCIENCE, TEXTILES
MATHEMATICAL & COMPUTATIONAL BIOLOGY
MATHEMATICS
MATHEMATICS, APPLIED
MATHEMATICS, INTERDISCIPLINARY APPLICATIONS
MECHANICS
MEDICAL INFORMATICS
MEDICINE, RESEARCH & EXPERIMENTAL
METALLURGY & METALLURGICAL ENGINEERING
METEOROLOGY & ATMOSPHERIC SCIENCES
MINERALOGY
MINING & MINERAL PROCESSING
MULTIDISCIPLINARY SCIENCES
MYCOLOGY
NANOSCIENCE & NANOTECHNOLOGY
NEUROSCIENCES
NUCLEAR SCIENCE & TECHNOLOGY
OCEANOGRAPHY
ONCOLOGY
OPERATIONS RESEARCH & MANAGEMENT SCIENCE
OPHTHALMOLOGY
OPTICS
ORNITHOLOGY
PERIPHERAL VASCULAR DISEASE
PHARMACOLOGY & PHARMACY
PHYSICS, APPLIED
PHYSICS, ATOMIC, MOLECULAR & CHEMICAL
PHYSICS, CONDENSED MATTER
PHYSICS, FLUIDS & PLASMAS
PHYSICS, MATHEMATICAL
PHYSICS, MULTIDISCIPLINARY
PHYSICS, NUCLEAR
PHYSICS, PARTICLES & FIELDS
PHYSIOLOGY
PLANNING & DEVELOPMENT
PLANT SCIENCES
POLITICAL SCIENCE
POLYMER SCIENCE
PSYCHOLOGY, EXPERIMENTAL
PSYCHOLOGY, MATHEMATICAL
PUBLIC, ENVIRONMENTAL & OCCUPATIONAL HEALTH
RADIOLOGY, NUCLEAR MEDICINE & MEDICAL IMAGING
REMOTE SENSING
RESPIRATORY SYSTEM
ROBOTICS
SOCIAL SCIENCES, INTERDISCIPLINARY
SOCIAL SCIENCES, MATHEMATICAL METHODS
SOCIOLOGY
SOIL SCIENCE
SPECTROSCOPY
SPORT SCIENCES
STATISTICS & PROBABILITY
SURGERY
TELECOMMUNICATIONS
THERMODYNAMICS
TRANSPORTATION SCIENCE & TECHNOLOGY
URBAN STUDIES
WATER RESOURCES

The current map of the fractional calculus

ALGERIA
ARGENTINA
AUSTRALIA
AUSTRIA
BELGIUM
BRAZIL
BULGARIA
BYELARUS
CANADA
CHILE
CROATIA
CUBA
CZECH REP
DENMARK
EGYPT
ENGLAND
ESTONIA
FINLAND
FRANCE
GERMANY
GREECE
GUADELOUPE
HUNGARY
INDIA
IRAN
IRELAND
ISRAEL
ITALY
JAPAN
JORDAN
KUWAIT
LEBANON
LITHUANIA
MACEDONIA
MALAYSIA
MEXICO
MOROCCO
NETHERLANDS
NEW ZEALAND
NIGERIA
NORTH IRELAND
NORWAY
PAKISTAN
P.R. CHINA
POLAND
PORTUGAL
QATAR
ROMANIA
RUSSIA
SAUDI ARABIA
SCOTLAND
SERBIA
MONTENEG
SINGAPORE
SLOVAKIA
SLOVENIA
SOUTH AFRICA
SOUTH KOREA
SPAIN
SWEDEN
SWITZERLAND
TAIWAN
THAILAND
TUNISIA
TURKEY
TURKMENISTAN
U ARAB EMIRATES
UKRAINE
USA
UZBEKISTAN
VENEZUELA
VIETNAM
WALES
YUGOSLAVIA

Data Source: Web of Knowledge

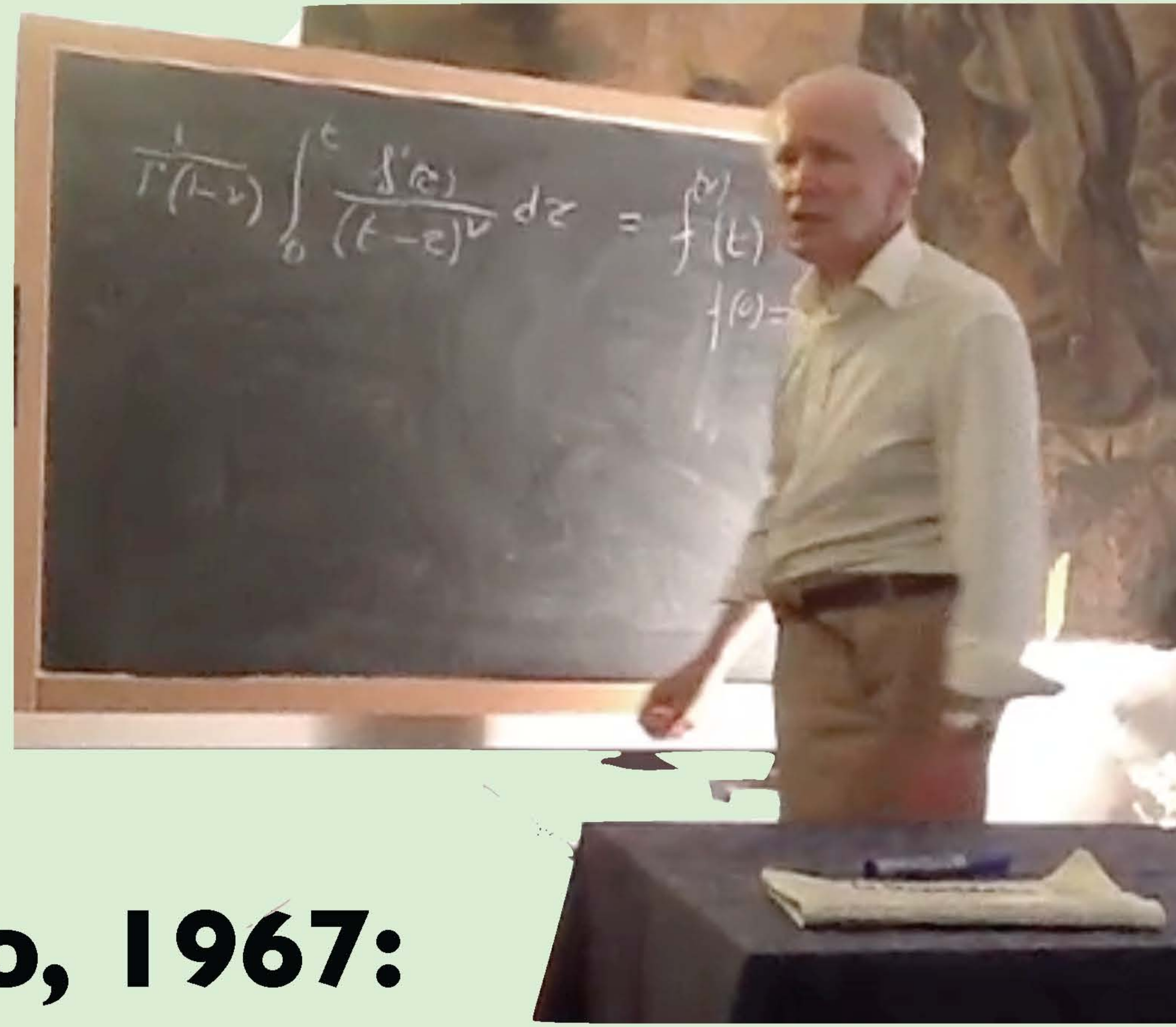
Movement I: Scherzo

3 definitions

Riemann–Liouville:

(goes back to Letnikov, 1870)

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int\limits_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}}, \qquad (n-1 \leq \alpha < n) $$



Caputo, 1967:

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha-n+1}}, \qquad (n-1 \le \alpha < n) $$

Grünwald–Letnikov, 1860s

(goes back to Liouville, 1830s)

$$D^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{n} f(t-kh)$$

For good functions RL, C, and GL definitions are equivalent

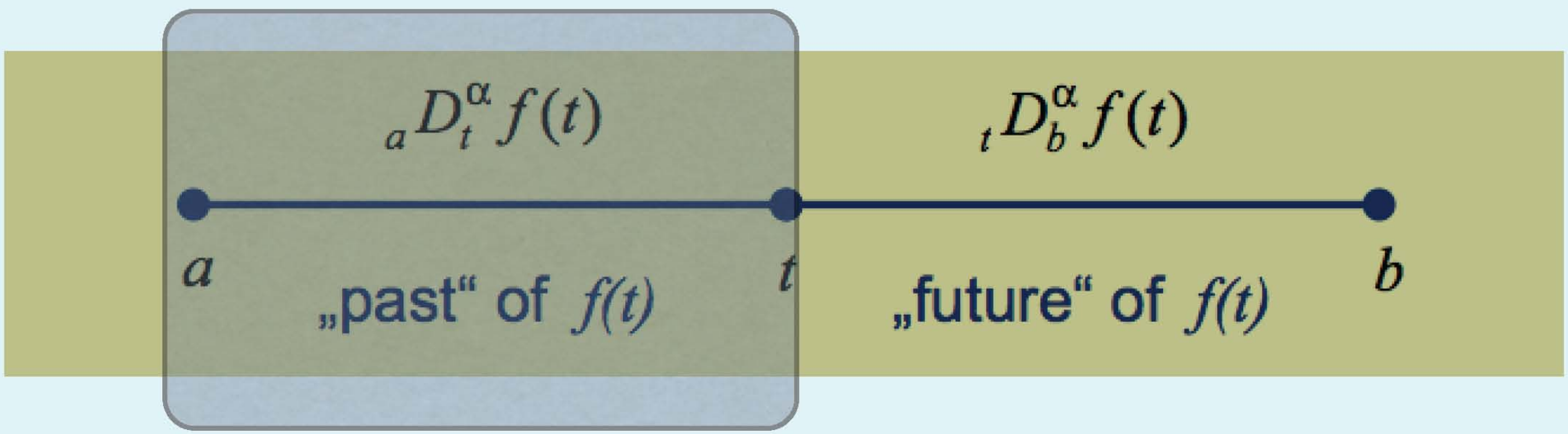
For ${}_aD_t^{\alpha} f(t)$ with $n-1 < \alpha \leq n$ "good" means

$$f \in C^{(n)}[a,b], \quad f^{(k)}(a) = 0 \ (k = 0, \ldots, n-1)$$

3 flavors

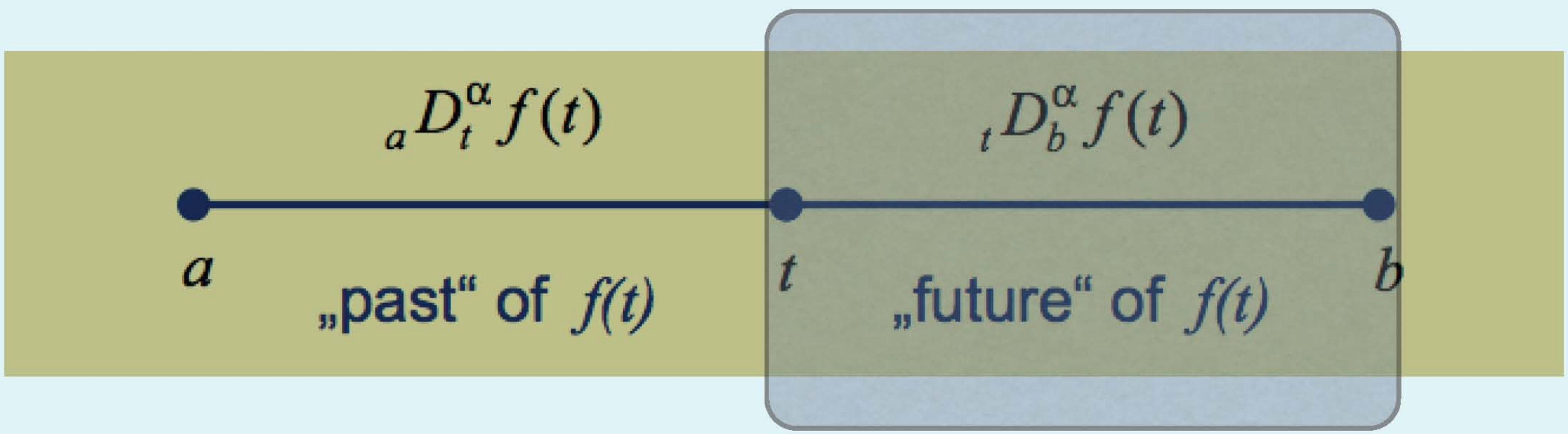
Left-sided

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\, d\tau}{(t-\tau)^{\alpha-n+1}} $$



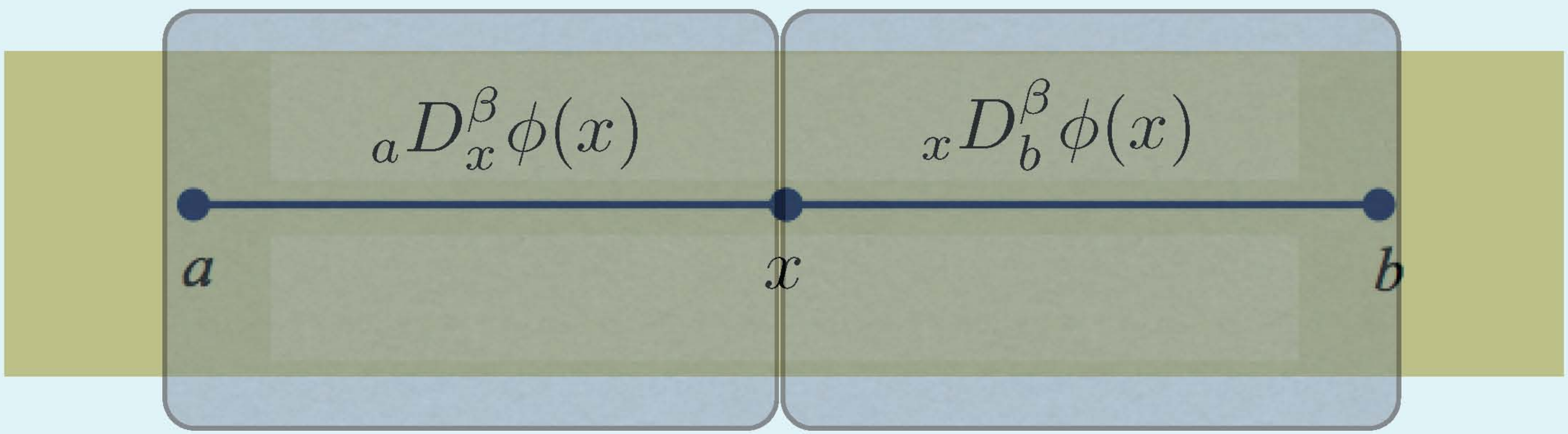
Right-sided

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b \frac{f(\tau)d\tau}{(\tau - t)^{\alpha-n+1}}$$



Symmetric

$$\frac{d^\beta \phi(x)}{d|x|^\beta} = D_R^\beta \phi(x) = \frac{1}{2}\left({}_aD_x^\beta \phi(x) + {}_xD_b^\beta \phi(x) \right)$$



3 grades

Constant non-integer order (CO)

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}}, $$

$$ (n-1 \leq \alpha < n) $$

Variable order (VO)

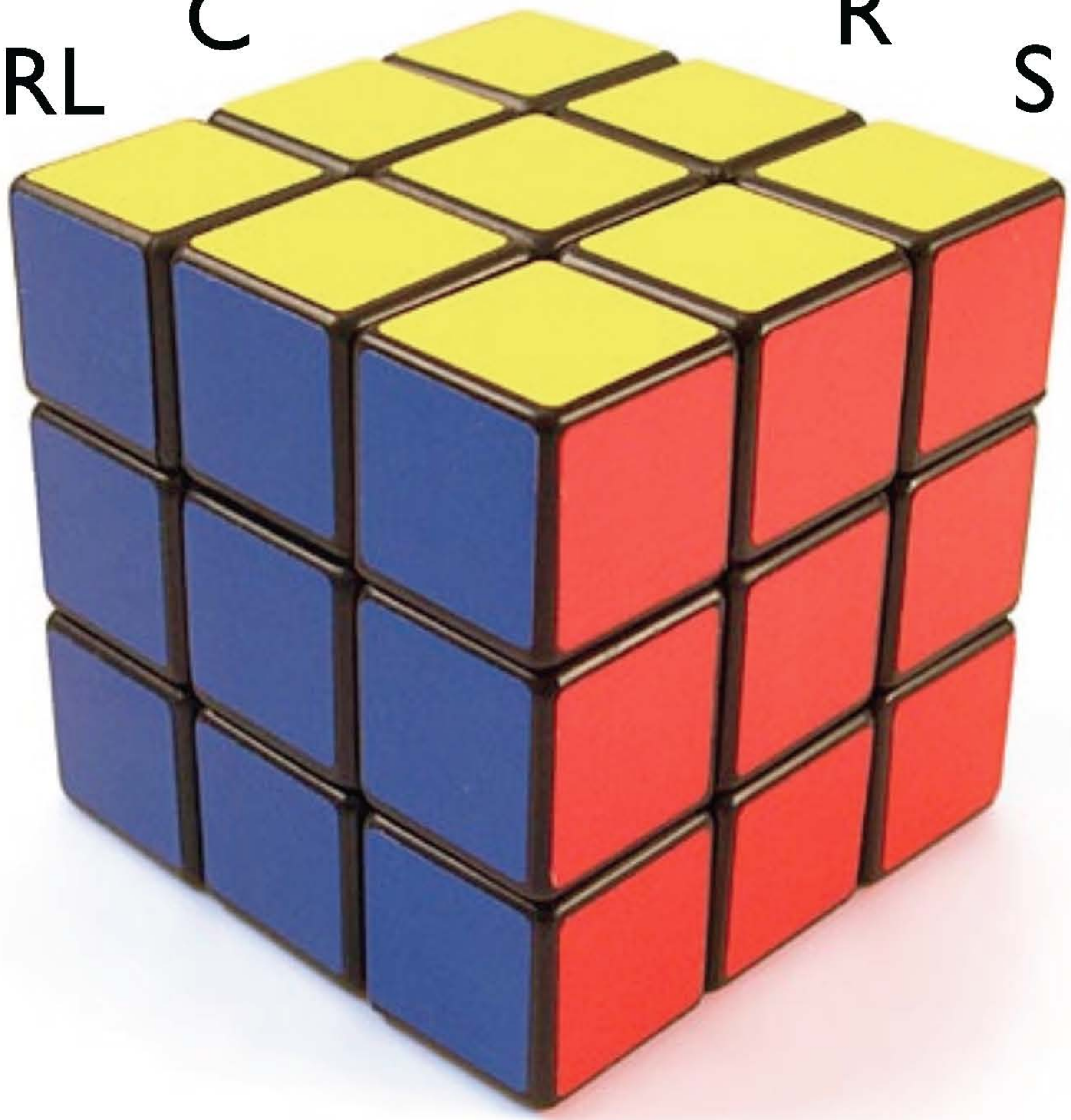
$$ {}_{0}^{C}D_{t}^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \int_{0}^{t} \frac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha(t)+1-n}}, $$

$$ (n - 1 \leq \alpha(t) < n) $$

Distributed order (DO)

$$ {}_aD_t^{\varphi(\alpha)} f(t) = \int_c^d \varphi(\alpha)\, {}_aD_t^{\alpha} f(t)\, d\alpha $$

$$ \int_c^d \varphi(\alpha)\, d\alpha = 1 $$


Definitions
RL
C
GL
Flavors
L
R
S
Grades
CO
VO
DO

Intelligent fitting of data with the help of solutions of differential equations

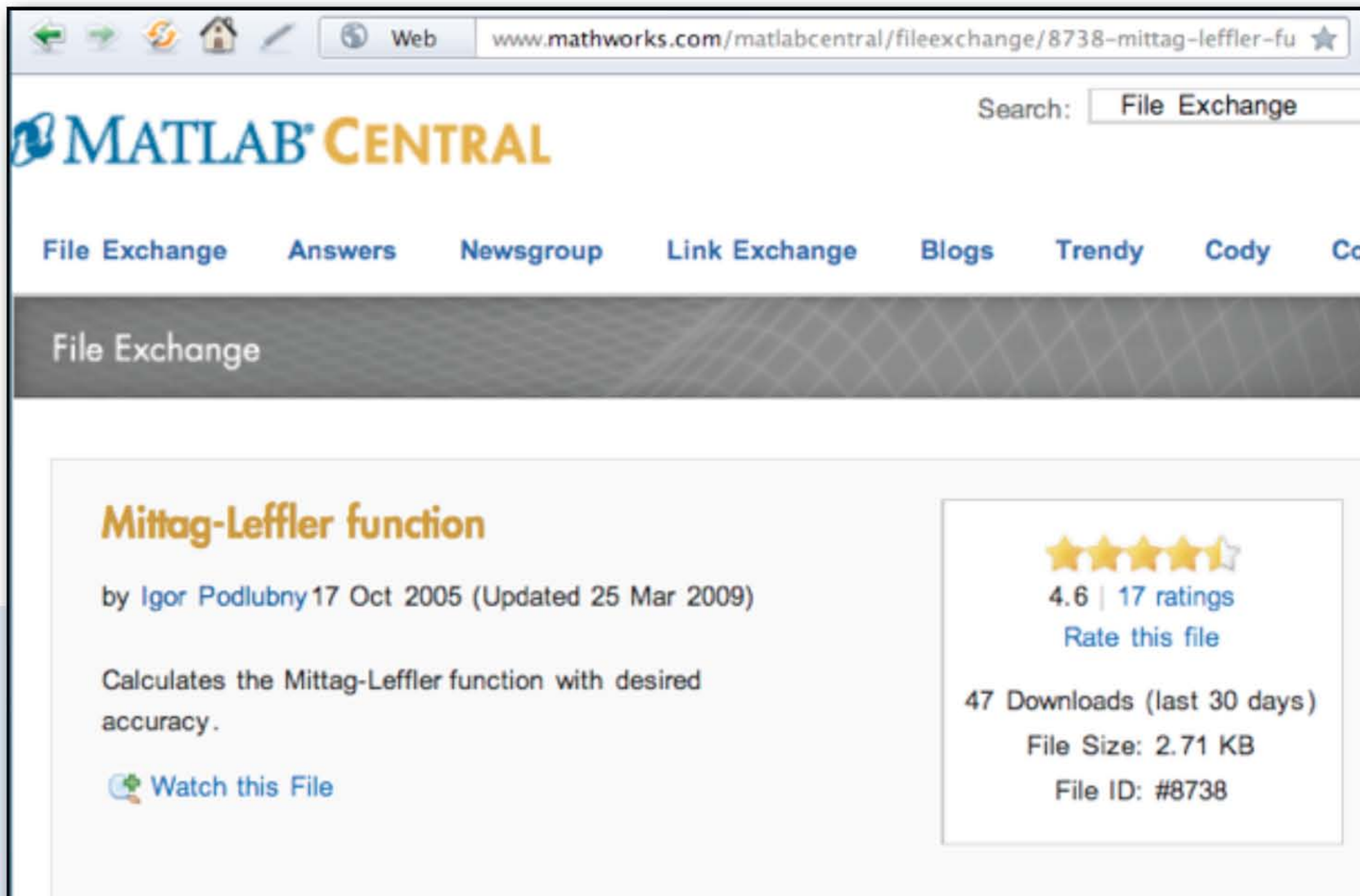
$y = kx + b$	$y'' = 0$
$y = a\sin(wx) + b\cos(wx)$	$y'' + w^2 y = 0$
$y = Ce^{kx}$	$y' - ky = 0$
$y = Ae^{kx}\sin(wx) + Be^{kx}\cos(wx)$	$a_2 y'' + a_1 y' + a_0 y = 0$

Instead of postulating the type of the fitting function, we can postulate the type of the differential equation; its coefficients must be determined.

$$Ay'' + By' + Cy = 0$$

The Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \quad \beta > 0)$$



Fitting data using the Mittag-Leffler function

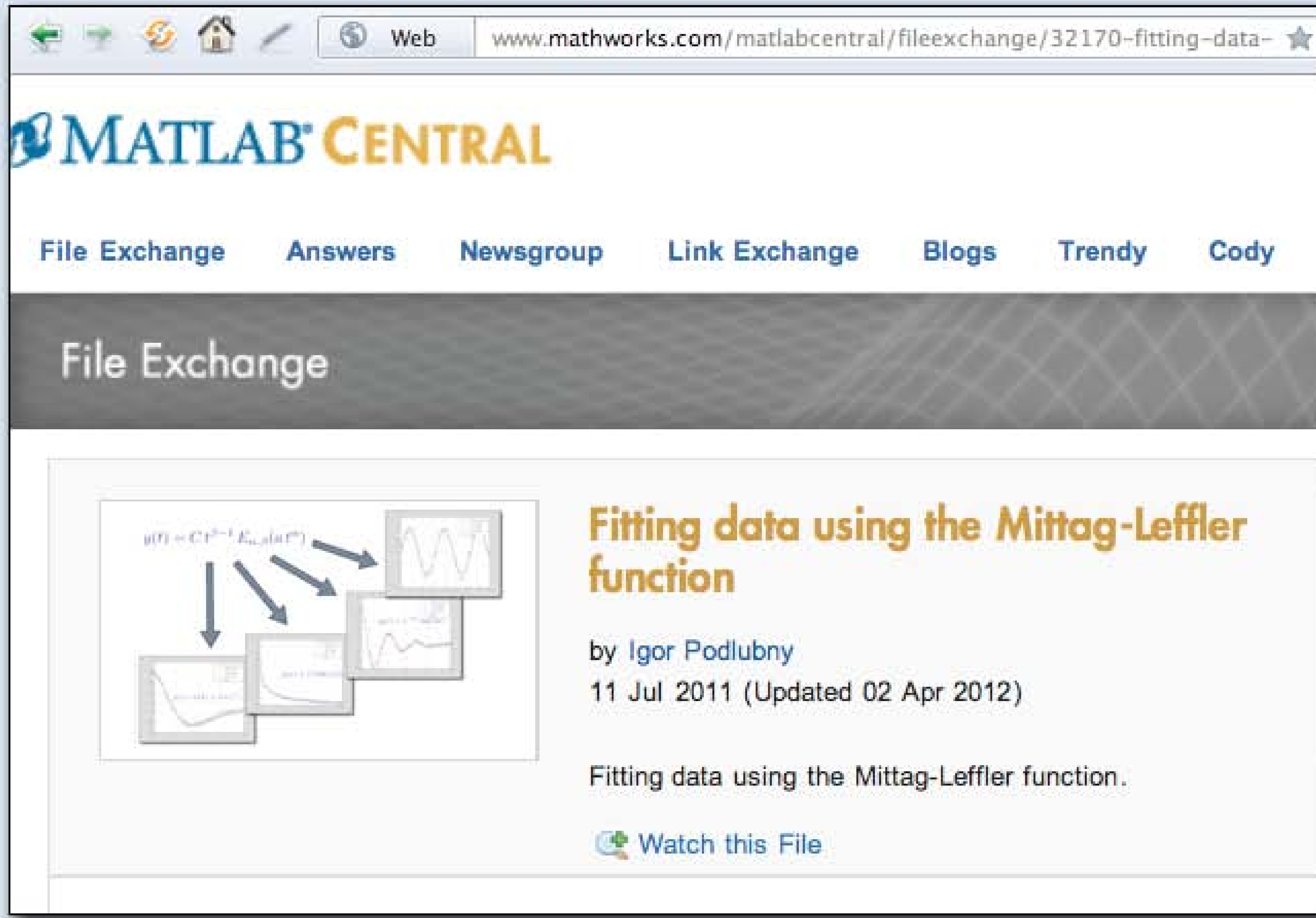
$$y = y_0\, e^{kt} \qquad\qquad y'(t) - k\, y(t) = 0, \quad y(0) = y_0$$

$$y = y_0\, t^{\beta-1} E_{\alpha,\beta}(a\, t^{\alpha})$$

$$y = y_0\, E_{\alpha,1}(a\, t^{\alpha}) \qquad {}_0^C D_t^{\alpha} y(t) - k\, y(t) = 0, \quad y(0) = y_0$$

Fitting the experimental data with the M-L function immediately gives the basic FDE describing the process.

Just supply your data...



Just supply your data...

Identification of Parameters of a Half-Order System

INCLUDED WITH YOUR SUBSCRIPTION

Petras, I.; Sierociuk, D.; Podlubny, I.
Signal Processing, IEEE Transactions on
Volume: 60 , Issue: 10
Topic(s): Signal Processing & Analysis
Digital Object Identifier: 10.1109/TSP.2012.2205920
Publication Year: 2012 , Page(s): 5561 - 5566
IEEE JOURNALS & MAGAZINES

| Quick Abstract | PDF (845 KB) | HTML

Experimental Evidence of Variable-Order Behavior of Ladders and Nested Ladders

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Sierociuk, D.; Podlubny, I.; Petras, I.
Control Systems Technology, IEEE Transactions on
Volume: 21 , Issue: 2
Topic(s): Signal Processing & Analysis
Digital Object Identifier: 10.1109/TCST.2012.2185932
Publication Year: 2013 , Page(s): 459 - 466
Cited by 3
IEEE JOURNALS & MAGAZINES

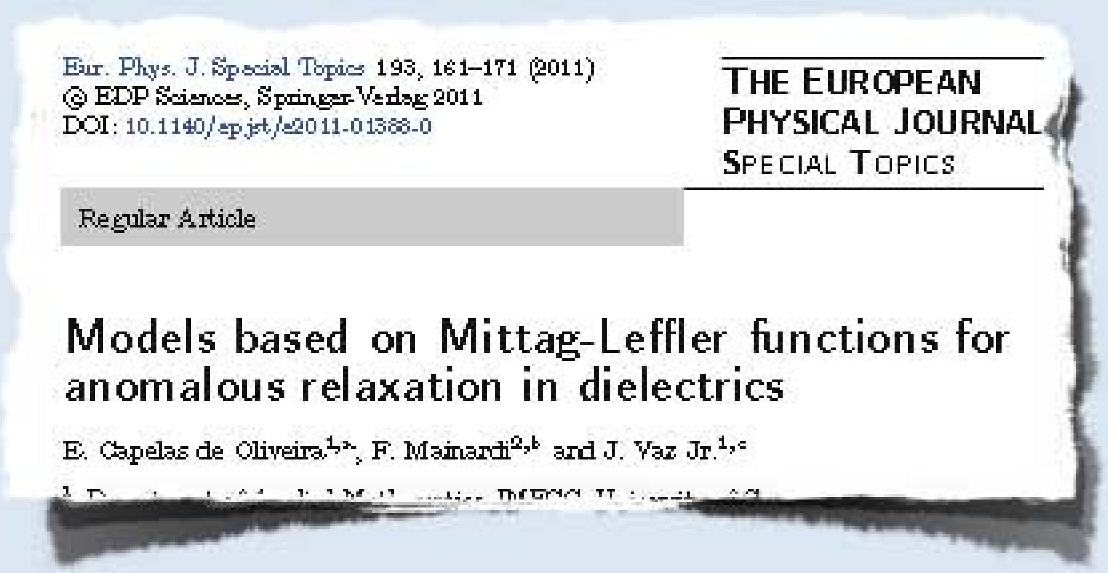
| Quick Abstract | PDF (1977 KB)

The Queen Function

"In fact, ... , functions of Mittag-Leffler type enter as solutions of many problems dealt with fractional calculus so that they like to refer to the **Mittag-Leffler function** to as the

Queen function of Fractional Calculus,

in contrast with its role of a **Cinderella** function played in the past."


Eur. Phys. J. Special Topics 193, 161–171 (2011)
© EDP Sciences, Springer-Verlag 2011
DOI: 10.1140/epjst/e2011-01388-0

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics

E. Capelas de Oliveira[1,a], F. Mainardi[2,b] and J. Vaz Jr.[1,c]

Mittag-Leffler function:

a complete replacement for the exponential function

Automatica 45 (2009) 1965–1969

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

ELSEVIER

Technical communique

Mittag–Leffler stability of fractional order nonlinear dynamic systems☆

Yan Li[a], YangQuan Chen[b,*], Igor Podlubny[c]

[a] Institute of Applied Math, School of Mathematics and System Sciences, Shandong University, Jinan 250100, PR China
[b] Center for Self-Organizing and Intelligent Systems (CSOIS), Electrical and Computer Engineering Department, Utah Sta
[c] Department of Applied Informatics and Process Control, Faculty BERG, Technical University of Kosice, B. Nemcovej 3, 0

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Fitting of experimental data using the Mittag-Leffler function

Igor Podlubny
Ivo Petráš
Tomáš Škovránek
Technical University of Kosice, Slovakia

ICCC 2012, Podbanské, High Tatras, Slovakia, May 28–31, 2012

Ann. Inst. Statist. Math.
Vol. 42, No. 1, 157–161 (1990)

ON MITTAG-LEFFLER FUNCTIONS AND RELATED DISTRIBUTIONS

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(Received August 29, 1988; revised February 27, 1989)

Abstract. The distribution $F_\alpha(x) = 1 - E_\alpha(-x^\alpha)$, $0 < \alpha \leq 1$; $x \geq 0$, where $E_\alpha(x)$ is the Mittag-Leffler function is studied here with respect to its Laplace transform. Its infinite divisibility and geometric infinite divisibility are proved, along with many other properties. Its relation with stable distribution is established. The Mittag-Leffler process is defined and some of its properties are deduced.

Key words and phrases: Completely monotone function, Laplace transform, infinite divisibility, geometric infinite divisibility, stable process.

Movement 2: Allegro

Operational Matrices ...

... and Fourier series

... and Taylor series

... and Orthogonal Polynomials

- Legendre
- Jacobi
- Chebyshev
- Bernoulli
- Bernstein
- ...

... and Walsh functions

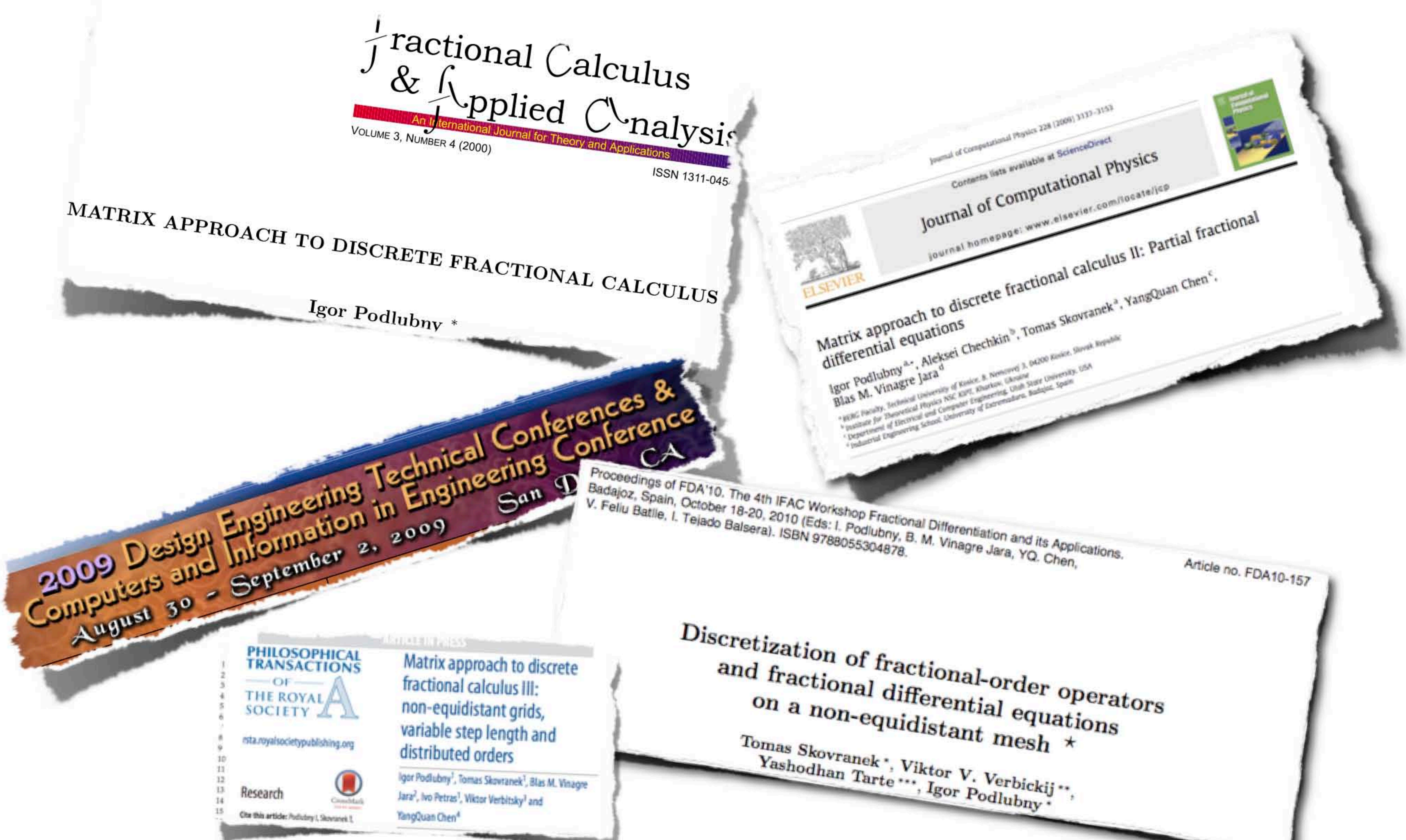
... and Wavelets

... or, Block-Pulse Matrices

... and, in general, any suitable basis functions

Matrix approach to discrete fractional calculus:

OFDEs: 2000, PFDEs: 2008, SOFDEs: 2009, NEG: 2010; VODEs: 2012; DODEs: 2012; Large Steps: 2012



Triangular strip matrices (TSM)

Lower TSM:

$$L_N = \begin{bmatrix} \omega_0 & 0 & 0 & 0 & \cdots & 0 \\ \omega_1 & \omega_0 & 0 & 0 & \cdots & 0 \\ \omega_2 & \omega_1 & \omega_0 & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ldots & \ldots \\ \omega_{N-1} & \ddots & \omega_2 & \omega_1 & \omega_0 & 0 \\ \omega_N & \omega_{N-1} & \ddots & \omega_2 & \omega_1 & \omega_0 \end{bmatrix},$$

Upper TSM:

$$U_N = \begin{bmatrix} \omega_0 & \omega_1 & \omega_2 & \ddots & \omega_{N-1} & \omega_N \\ 0 & \omega_0 & \omega_1 & \ddots & \ddots & \omega_{N-1} \\ 0 & 0 & \omega_0 & \ddots & \omega_2 & \ddots \\ 0 & 0 & 0 & \ddots & \omega_1 & \omega_2 \\ \cdots & \cdots & \cdots & \cdots & \omega_0 & \omega_1 \\ 0 & 0 & 0 & \cdots & 0 & \omega_0 \end{bmatrix},$$

If two TSMs are of the same type, then: $CD = D\,C$.

Generating functions for TSMs

$$\varrho(z) = \sum_{k=0}^{\infty} \omega_k z^k \quad \longrightarrow \quad \mathrm{trunc}_N\left(\varrho(z)\right) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{N} \omega_k z^k = \varrho_N(z)$$

Function $\varrho(z)$ generates a sequence os lower TSMs:

$$L_N, \qquad N = 1, 2, \ldots$$

or upper TSMs

$$U_N, \qquad N = 1, 2, \ldots$$

Properties:

$$\mathrm{trunc}_N\left(\gamma\lambda(z)\right) = \gamma\, \mathrm{trunc}_N\left(\lambda(z)\right)$$

$$\mathrm{trunc}_N\left(\lambda(z) + \mu(z)\right) = \mathrm{trunc}_N\left(\lambda(z)\right) + \mathrm{trunc}_N\left(\mu(z)\right)$$

$$\mathrm{trunc}_N\left(\lambda(z)\mu(z)\right) = \mathrm{trunc}_N(\mathrm{trunc}_N\left(\lambda(z)\right)\, \mathrm{trunc}_N\left(\mu(z)\right))$$

Operations with TSMs

$$A_N = \sum_{k=0}^{N} a_k (E_1^-)^k = \lambda_N(E_1^-), \quad B_N = \sum_{k=0}^{N} b_k (E_1^-)^k = \mu_N(E_1^-),$$

$$\lambda_N(z) = \text{trunc}_N\left(\lambda(z)\right), \qquad \mu_N = \text{trunc}_N\left(\mu(z)\right)$$

Addition and subtraction:

$$A_N \pm B_N \longleftrightarrow \text{trunc}_N\left(\lambda(z) \pm \mu(z)\right)$$

Multiplication by a constant:

$$\gamma A_N \longleftrightarrow \text{trunc}_N\left(\gamma\lambda(z)\right)$$

Product of TSMs:

$$A_N B_N \longleftrightarrow \text{trunc}_N\left(\lambda(z)\mu(z)\right)$$

Matrix inversion:

$$(A_N)^{-1} \longleftrightarrow \text{trunc}_N\left(\lambda^{-1}(z)\right)$$

Left-sided R-L derivatives

$$ {}_aD^{\alpha}_{t_k} f(t) \approx \frac{\nabla^{\alpha} f(t_k)}{h^{\alpha}} = h^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f_{k-j}, \qquad k = 0, 1, \dots, N. $$

$$ \begin{bmatrix} h^{-\alpha}\nabla^{\alpha} f(t_0) \\ h^{-\alpha}\nabla^{\alpha} f(t_1) \\ h^{-\alpha}\nabla^{\alpha} f(t_2) \\ \vdots \\ h^{-\alpha}\nabla^{\alpha} f(t_{N-1}) \\ h^{-\alpha}\nabla^{\alpha} f(t_N) \end{bmatrix} = \frac{1}{h^{\alpha}} \begin{bmatrix} \omega_0^{(\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 \\ \omega_N^{(\alpha)} & \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix} $$

$$ \omega_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}, \qquad j = 0, 1, \dots, N. $$

$$B_N^\alpha = \frac{1}{h^\alpha}\begin{bmatrix} \omega_0^{(\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 \\ \omega_N^{(\alpha)} & \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \end{bmatrix}$$

$$\beta_\alpha(z) = h^{-\alpha}(1-z)^\alpha.$$

$$B_N^\alpha B_N^\beta = B_N^\beta B_N^\alpha = B_N^{\alpha+\beta},$$

$$ {}_aD_t^\alpha(\, {}_aD_t^\beta f(t)) = \, {}_aD_t^\beta(\, {}_aD_t^\alpha f(t)) = \, {}_aD_t^{\alpha+\beta} f(t),$$

$$f^{(k)}(a) = 0, \quad k = 1, 2, \ldots, r-1,$$

$$r = \max\{n, m\}$$

Left-sided R-L integration

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad (a < t < b), $$

$$ I_N^\alpha = \left(B_N^\alpha\right)^{-1}. $$

$$ I_N^\alpha \longleftrightarrow \varphi_N(z) = \mathrm{trunc}_N \left(\beta_\alpha^{-1}(z)\right) = \mathrm{trunc}_N \left(h^\alpha (1-z)^{-\alpha}\right). $$

$$ I_N^\alpha = h^\alpha \begin{bmatrix} \omega_0^{(-\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{N-1}^{(-\alpha)} & \ddots & \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 \\ \omega_N^{(-\alpha)} & \omega_{N-1}^{(-\alpha)} & \ddots & \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} \end{bmatrix} $$

Other generating functions for TSM

Christian Lubich's formulas:

$$\omega_1^{(\alpha)}(z) = (1-z)^\alpha$$

$$\omega_2^{(\alpha)}(z) = (\frac{3}{2} - 2z + \frac{1}{2}z^2)^\alpha,$$

$$\omega_3^{(\alpha)}(z) = (\frac{11}{6} - 3z + \frac{3}{2}z^2 - \frac{1}{3}z^2)^\alpha,$$

$$\omega_4^{(\alpha)}(z) = (\frac{25}{12} - 4z + \frac{3}{2}z^2 - \frac{4}{3}z^3 + \frac{1}{4}z^4)^\alpha,$$

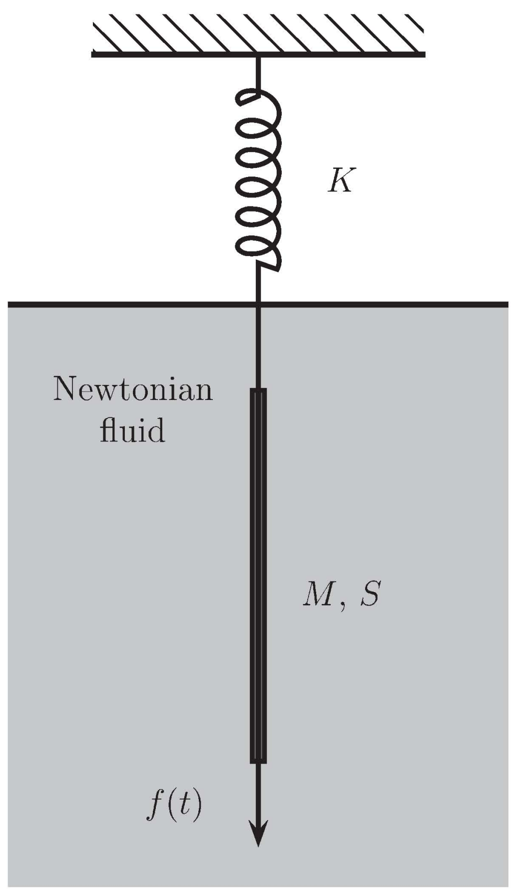
$$\omega_5^{(\alpha)}(z) = (\frac{137}{60} - 5z + 5z^2 - \frac{10}{3}z^3 + \frac{5}{4}z^4 - \frac{1}{5}z^5)^\alpha,$$

$$\omega_6^{(\alpha)}(z) = (\frac{147}{60} - 6z + \frac{15}{2}z^2 - \frac{20}{3}z^3 + \frac{15}{4}z^4 - \frac{6}{5}z^5 + \frac{1}{6})^\alpha$$

Expand these functions in Taylor series and use the coefficients for generating the corresponding TSMs.

Movement 3: Rondo

Example (Bagley–Torvik equation)



$$a\, y''(t) + b\, {}_0D_t^{3/2} y(t) + c\, y(t) = f(t)$$

$$y(0) = 0, \qquad y'(0) = 0$$

$$\left(a\, B_n^{(2)} + b\, B_n^{(3/2)} + c\, B_n^{(0)}\right) Y_n = F_n$$

Example: Riesz kernel

$$\frac{1}{\Gamma(1-\alpha)} \int_{-1}^{1} \frac{y(\tau)\, d\tau}{|t-\tau|^{\alpha}} = 1, \quad (-1 < t < 1),$$

Exact solution:

$$y(t) = \pi^{-1}\Gamma(1-\alpha) \cos\left(\frac{\alpha\pi}{2}\right) (1-t^2)^{(\alpha-1)/2}.$$

Numerical solution:

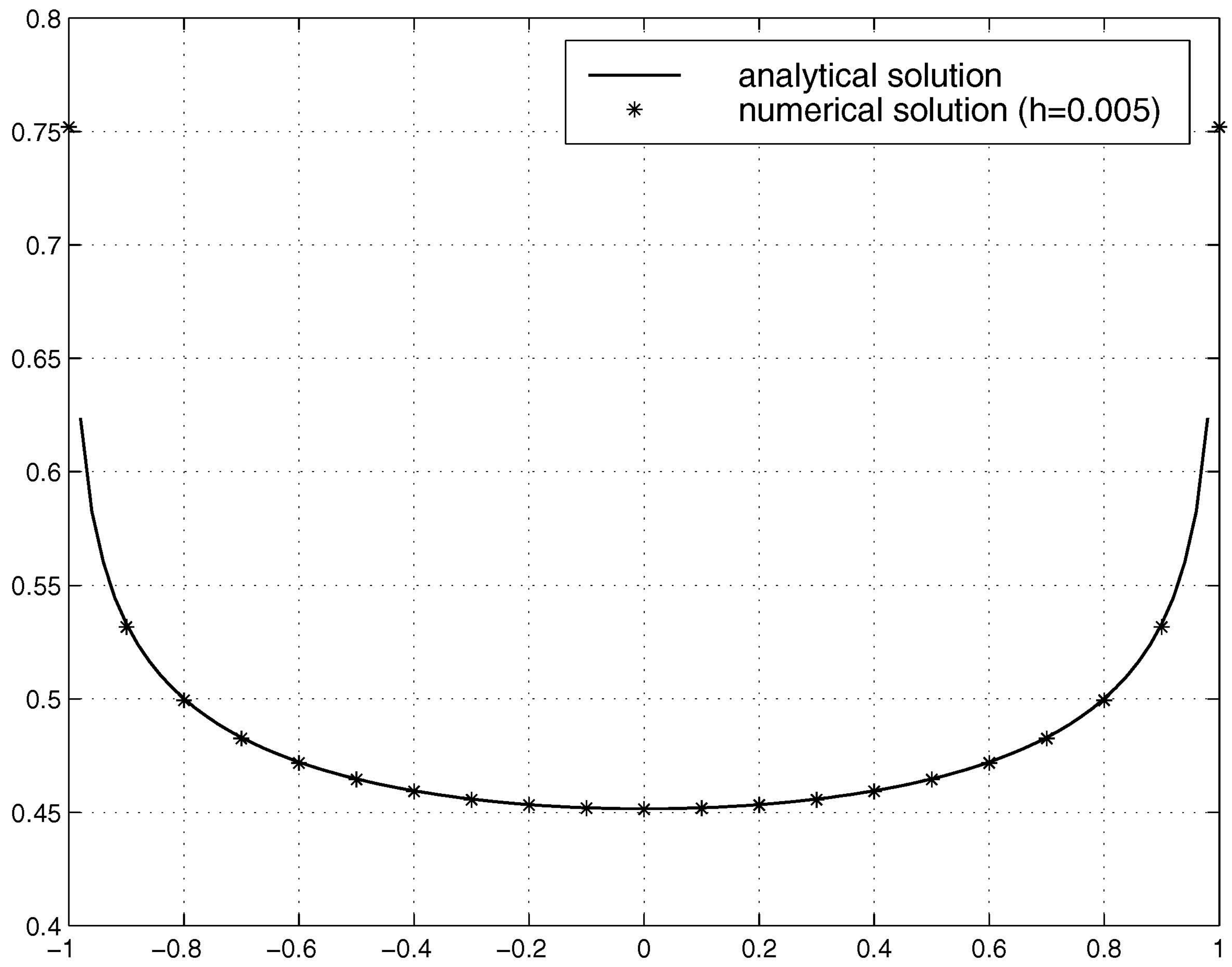
$${}_{-1}D_t^{-(1-\alpha)} y(t) + {}_{t}D_1^{-(1-\alpha)} y(t) = 1,$$

$$(B_N^{-(1-\alpha)} + F_N^{-(1-\alpha)}) Y_N = F_N$$

Historically the first example of numerical solution of equations with left-sided and right-sided fractional-order operators

Example: Riesz kernel

$$\alpha = 0.8$$



Example (Caputo derivatives)

$$y^{(\alpha)}(t) + y(t) = 1,$$
$$y(0) = 0, \quad y'(0) = 0,$$

Exact solution:

$$y(t) = t^{\alpha} E_{\alpha,\alpha+1}(-t^{\alpha}).$$

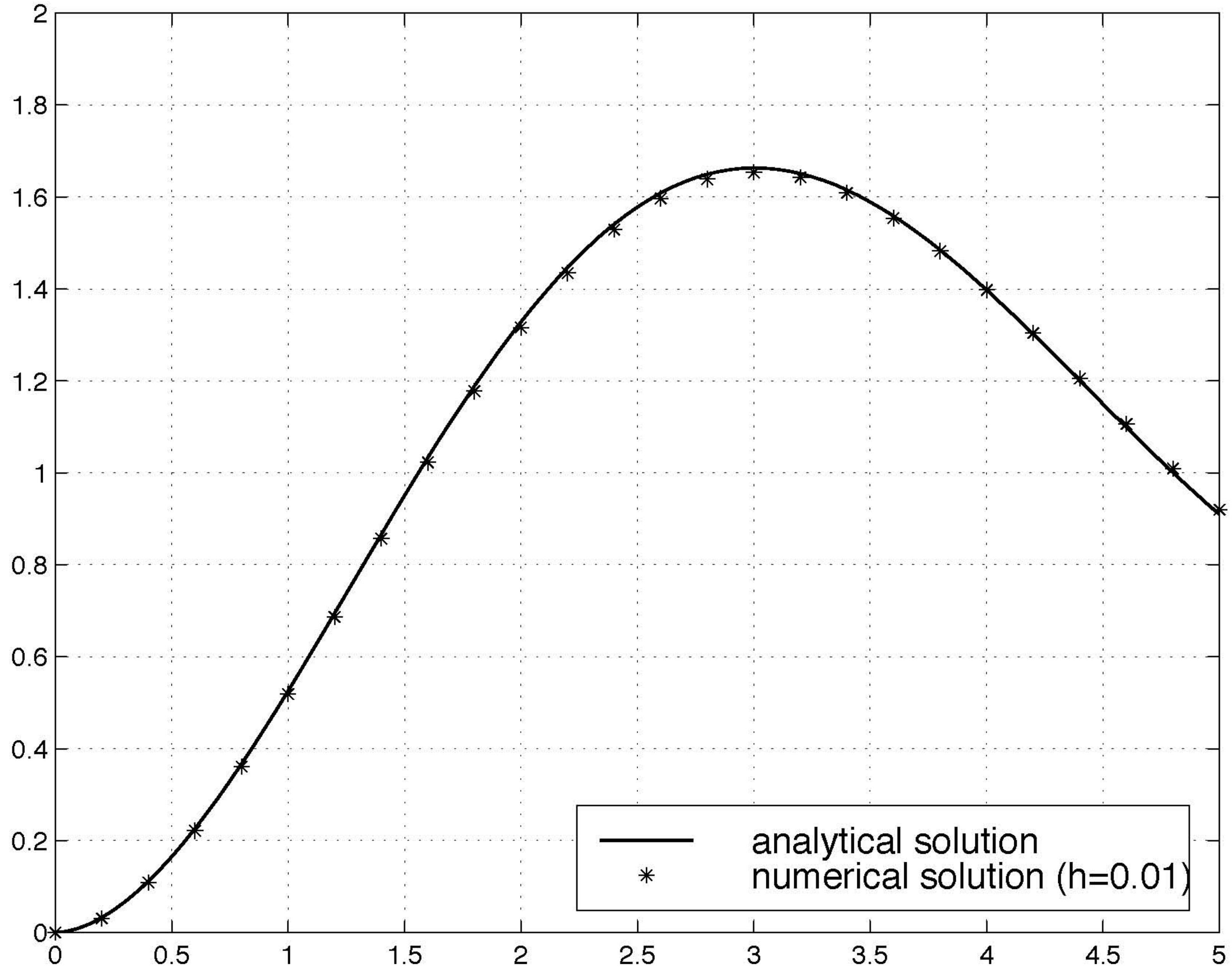
Numerical solution:

$$\{B_{N-2}^{\alpha} + E_{N-2}\}\{S_{0,1} Y_N\} = S_{0,1} F_N.$$

and from the initial conditions we have:

$$y_0 = y_1 = 0$$

Example (Caputo derivatives)



Nonlinear FDEs? Not a problem

Diethelm K., Weilbeer M.: A numerical approach for Joulin's model of appoint source initiated flame.
Fractional Calculus and Applied Analysis, (7):2, 191-212, 2004

problem

$$R(t)D_*^{1/2}R(t) = R(t)\ln R(t) + Eq(t), \qquad R(0) = 0. \tag{1}$$

Here $D_*^{1/2}$ denotes the Caputo differential operator of order 1/2, defined by (see, e.g., [6])

$$D_*^{1/2}y(t) = \frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-1/2}y'(s)\,\mathrm{d}s.$$

Moreover the function q describes a point source energy that depends on the time, and therefore it is assumed to be nonnegative, continuous and integrable

The nonlinear algebraic system is solved by Newton's method

Movement 4: Ritonello

Kronecker matrix product

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1q} \\ b_{21} & b_{22} & \dots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pq} \end{bmatrix}$$

Kronecker matrix product:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{bmatrix}$$

Kronecker matrix product

Important properties:

- if A and B are band matrices, then $A \otimes B$ is also a band matrix,
- if A and B are lower (upper) triangular, then $A \otimes B$ is also lower (upper) triangular.

Kronecker matrix product

Kronecker products with identity matrices: example:

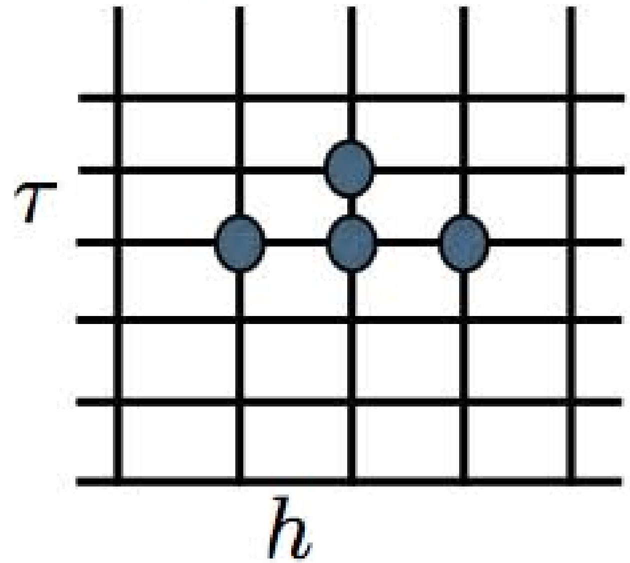
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$E_2 \otimes A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} \end{bmatrix}$$

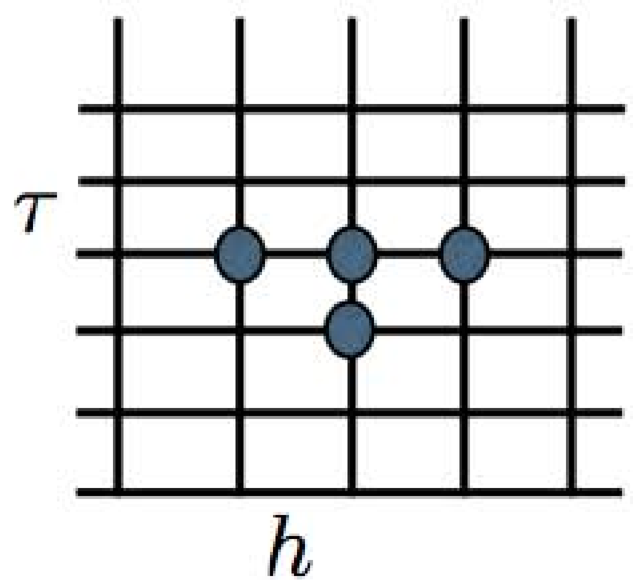
$$A \otimes E_3 = \begin{bmatrix} a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\ a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} & 0 & 0 & a_{23} \end{bmatrix}$$

Discretization schemes

Integer orders:

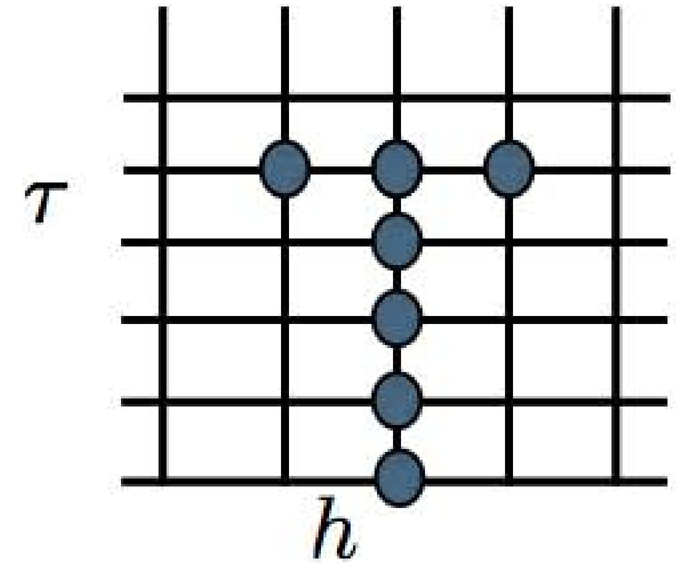


$$\frac{\partial U}{\partial t} \quad \frac{\partial^2 U}{\partial x^2}$$

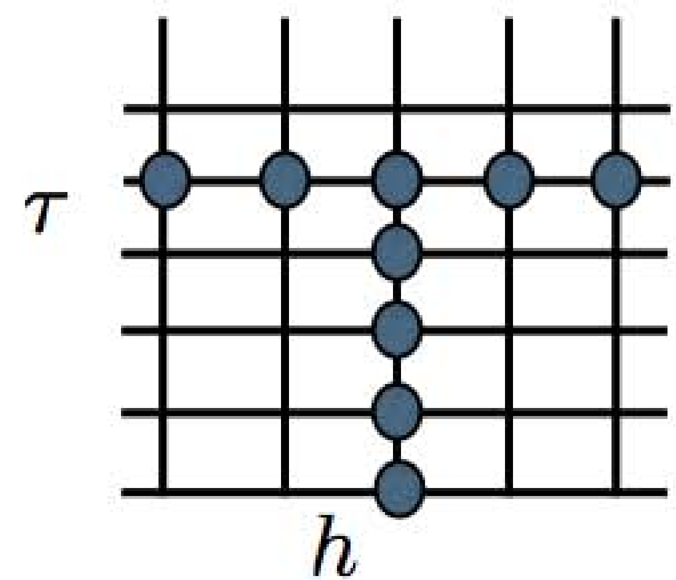


$$\frac{\partial^2 U}{\partial x^2} \quad \frac{\partial U}{\partial t}$$

Fractional orders:



$$\frac{\partial^2 U}{\partial x^2} \quad {}_0D_t^{\alpha} U$$



$$\frac{\partial^\beta U}{\partial |x|^\beta} \quad {}_0D_t^{\alpha} U$$

Symmetric Riesz fractional derivative

$$\frac{d^\beta \phi(x)}{d|x|^\beta} = D_R^\beta \phi(x) = \frac{1}{2}\Big({}_aD_x^\beta \phi(x) + {}_xD_b^\beta \phi(x) \Big)$$

Riemann-Liouville

RIESZ POTENTIAL OPERATORS AND INVERSES VIA FRACTIONAL CENTRED DERIVATIVES

MANUEL DUARTE ORTIGUEIRA

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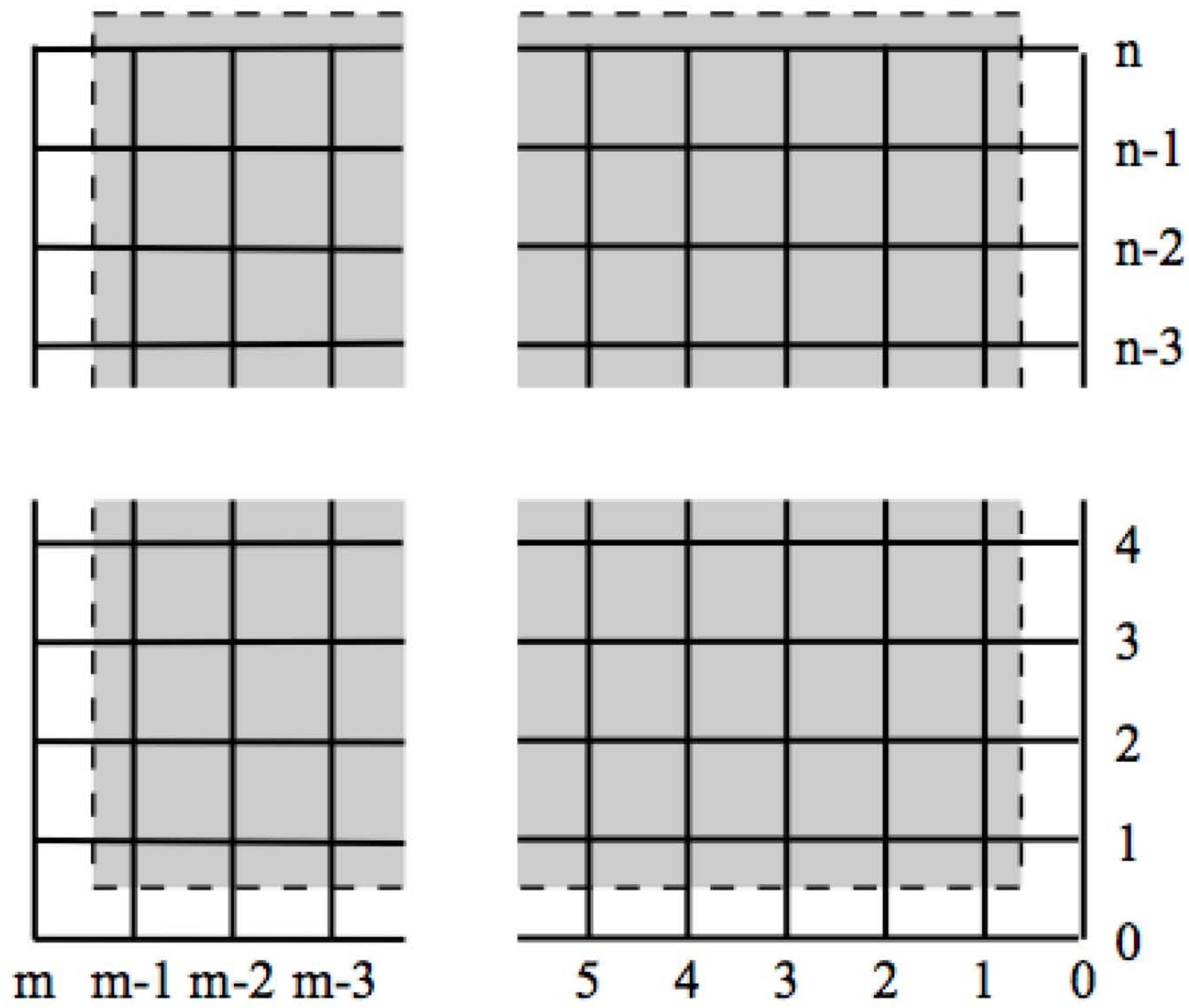
Volume 2006, Article ID 48391, Pages 1–12
DOI 10.1155/IJMMS/2006/48391

$$R_m^{(\beta)} = h^{-\beta} \begin{bmatrix} \omega_0^{(\beta)} & \omega_1^{(\beta)} & \omega_2^{(\beta)} & \omega_3^{(\beta)} & \cdots & \omega_m^{(\beta)} \\ \omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_1^{(\beta)} & \omega_2^{(\beta)} & \cdots & \omega_{m-1}^{(\beta)} \\ \omega_2^{(\beta)} & \omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_1^{(\beta)} & \cdots & \omega_{m-2}^{(\beta)} \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{m-1}^{(\beta)} & \ddots & \omega_2^{(\beta)} & \omega_1^{(\beta)} & \omega_0^{(\beta)} & \omega_1^{(\beta)} \\ \omega_m^{(\beta)} & \omega_{m-1}^{(\beta)} & \ddots & \omega_2^{(\beta)} & \omega_1^{(\beta)} & \omega_0^{(\beta)} \end{bmatrix}$$

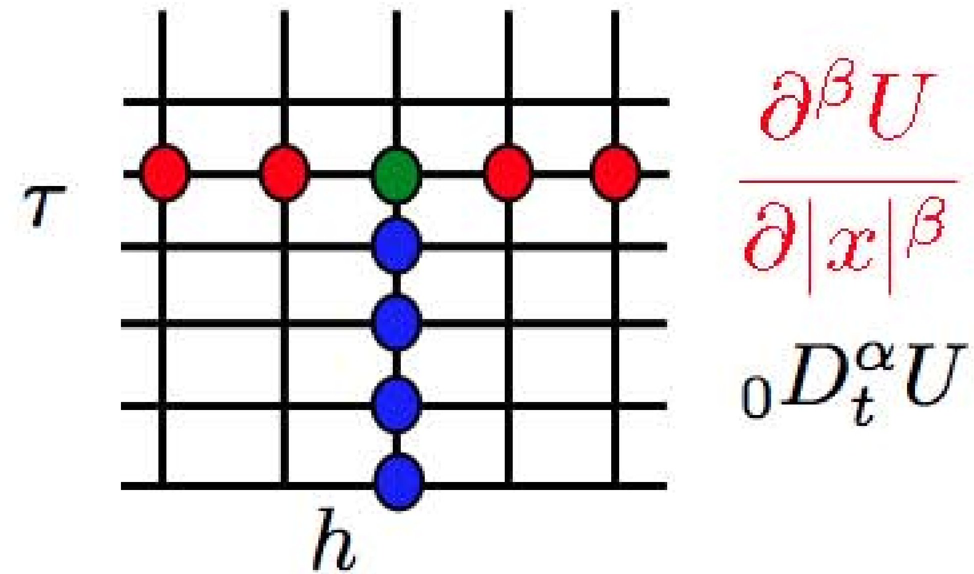
$$\omega_k^{(\beta)} = \frac{(-1)^k\, \Gamma(\beta+1)\, \cos(\beta\pi/2)}{\Gamma(\beta/2-k+1)\,\Gamma(\beta/2+k+1)}$$

Discretization grid



Nodes and their right-to-left, and bottom-to-top numbering.

Discretization using TSMs

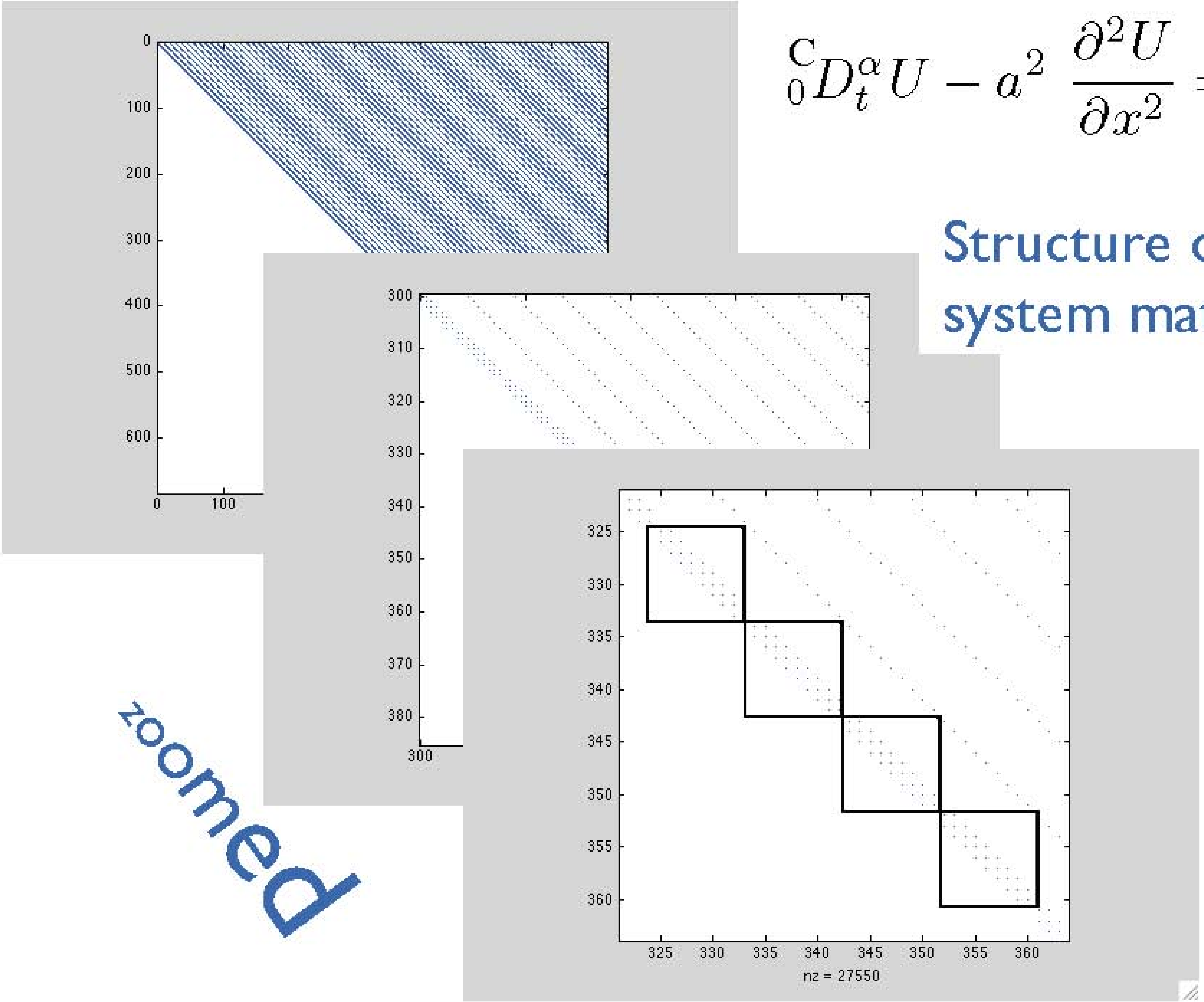


$${}_0D_t^\alpha U - a^2 \frac{\partial^\beta U}{\partial |x|^\beta} = F$$

$$\left\{ B_n^\alpha \otimes E_m - a^2\, E_n \otimes R_m^\beta \right\} u_{nm} = f_{nm}$$

$$ {}_{0}^{\mathrm{C}}D_t^{\alpha} U - a^2 \; \frac{\partial^2 U}{\partial x^2} = F $$

Structure of the system matrix



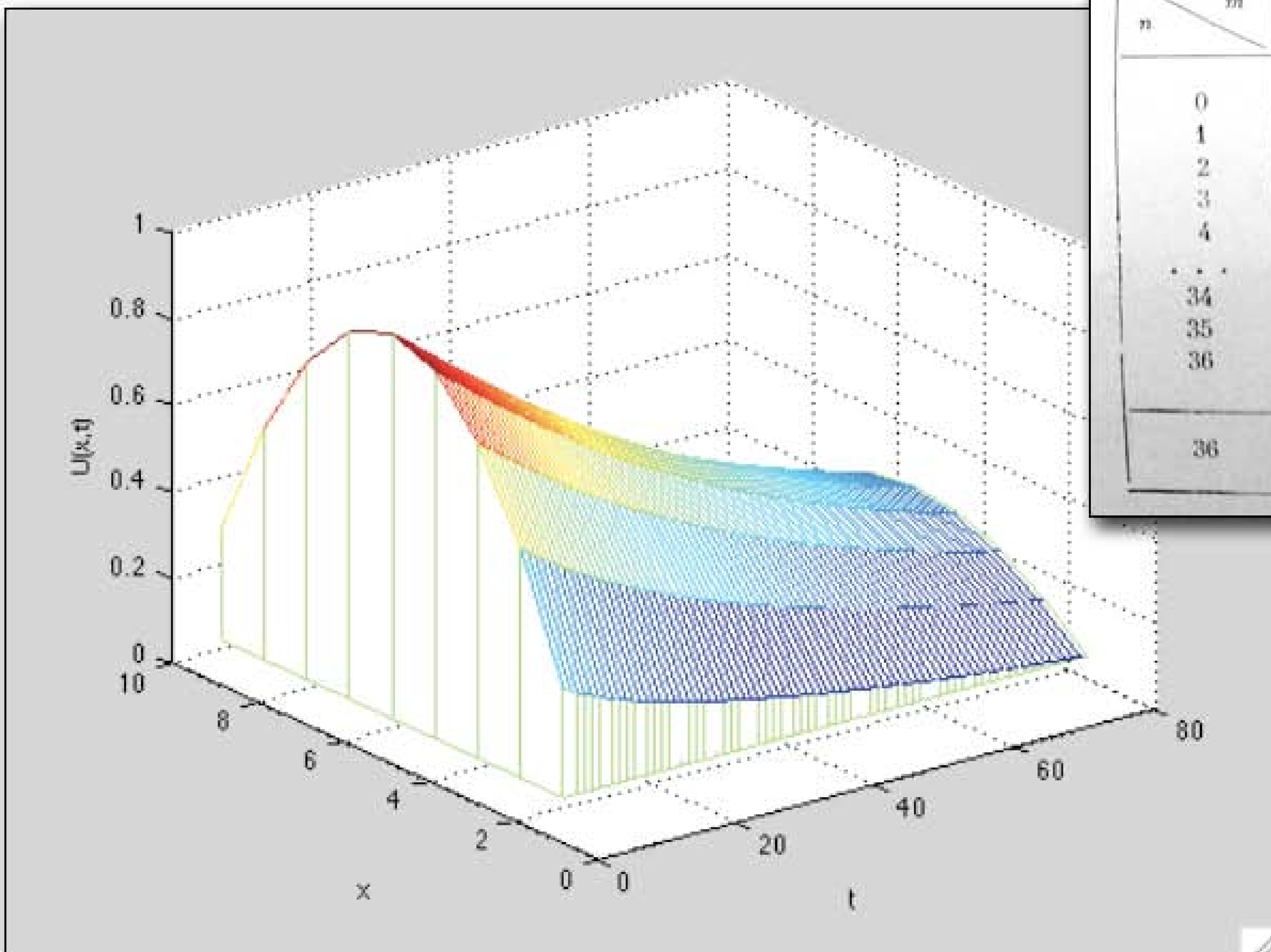
Test example

$$U_t = a^2 U_{xx}$$

$$U(0,t) = 0, \quad U(L,t) = 0$$

$$U(x,0) = 4x(L-x)/L^2$$

W. E. Milne, Numerical Solution of Differential Equations, Wiley, NY, 1953



n \ m	0	1	2	3	4	5
0	0	0,36000	0,64000	0,84000	0,96000	1,00000
1	0	0,34667	0,62637	0,82667	0,94667	0,98667
2	0	0,33556	0,61333	0,81333	0,93333	0,97333
3	0	0,32593	0,60037	0,80000	0,92000	0,96000
4	0	0,31735	0,58790	0,78673	0,90667	0,94667
...	...	...	...	...	...	...
34	0	0,18248	0,34696	0,47731	0,56088	0,58964
35	0	0,17948	0,34127	0,46951	0,55175	0,58005
36	0	0,17653	0,33568	0,46184	0,54276	0,57061
36	(0)	(0,17655)	(0,33572)	(0,46189)	(0,54280)	(1,57066)

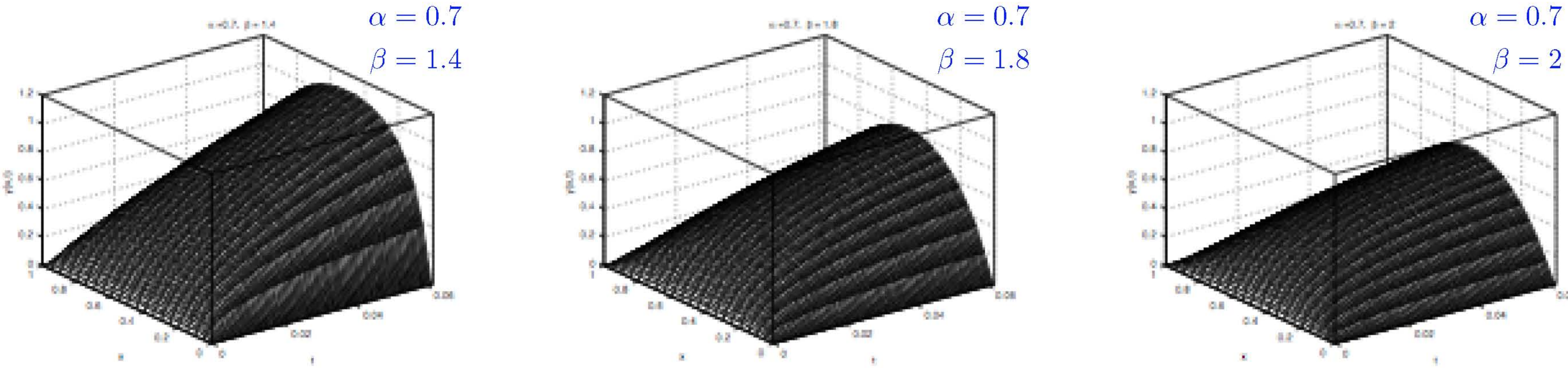
0.3484	0.6269	0.8267	0.9467	0.9867
0.3380	0.6141	0.8135	0.9334	0.9733
0.3287	0.6017	0.8003	0.9201	0.9600
0.3203	0.5897	0.7873	0.9068	0.9467
...	...	...	...	...
0.1843	0.3504	0.4817	0.5659	0.5948
0.1813	0.3447	0.4740	0.5568	0.5853
0.1784	0.3391	0.4664	0.5479	0.5760

Example: Time-space fractional diffusion equation

$$ {}_0^C D_t^\alpha y - \frac{\partial^\beta y}{\partial |x|^\beta} = f(x,t), \qquad \text{(with } f(x,t) \equiv 8\text{)} $$

$$ y(0,t) = 0, \quad y(1,t) = 0; \qquad y(x,0) = 0. $$

$$ \left\{ B_n^{(\alpha)} \otimes E_m - E_n \otimes R_m^{(\beta)} \right\} y_{nm} = f_{nm} $$



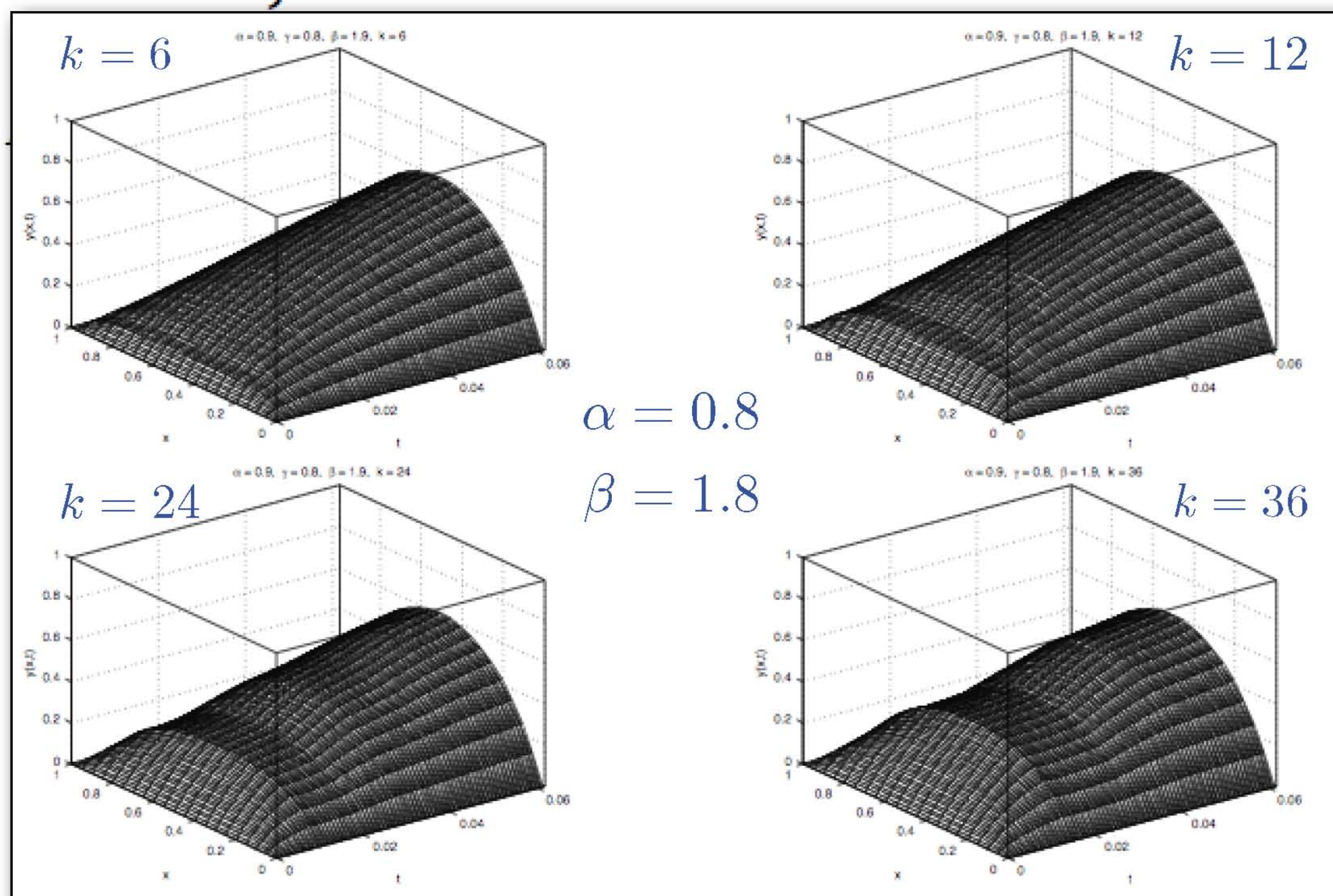
Example: Time-space fractional diffusion equation with delayed fractional derivative

$$\frac{1}{2}\left\{ {}_0^C D_t^{\alpha} y + {}_0^C D_{t-\delta}^{\gamma} y \right\} - \frac{\partial^{\beta} y}{\partial |x|^{\beta}} = f(x,t) \qquad (\text{with } f(x,t) \equiv 8)$$

$$y(0,t) = 0, \quad y(1,t) = 0 \quad y(x,0) = 0$$

$$\left\{ \frac{1}{2} \left(B_n^{(\alpha)} \otimes E_m + {}_{+k}B_n^{(\gamma)} \otimes E_m \right) - E_n \otimes R_m^{(\beta)} \right\} y_{nm} = f_{nm}$$

$${}_{+k}B_n^{(\gamma)} = S_{n+1,\ldots,n+k} \, E_{n+k,k}^{+}$$



Historically the first example of numerical solution of fractional differential equations with delayed fractional derivatives

Non-equidistant grids: they are everywhere

Venice, Italy, August 2010

$$w''(x): \quad \frac{1}{\hat{h}_i}\left(\frac{w_{i+1}-w_i}{h_i} - \frac{w_i - w_{i-1}}{h_{i-1}}\right)$$

$$h_i = x_{i+1} - x_i,$$
$$\hat{h}_i = \hat{x}_i - \hat{x}_{i-1},$$

Change the viewpoint:

Left-sided fractional derivatives:
inverse of left-sided fractional integrals

$$B_N^\alpha = (I_N^\alpha)^{-1}$$

Any approximation of fractional integration after inversion gives an approximation for fractional differentiation on the same grid!

The simplest approach: approximation by a piecewise constant function

$$B_N^\alpha = (I_N^\alpha)^{-1}$$

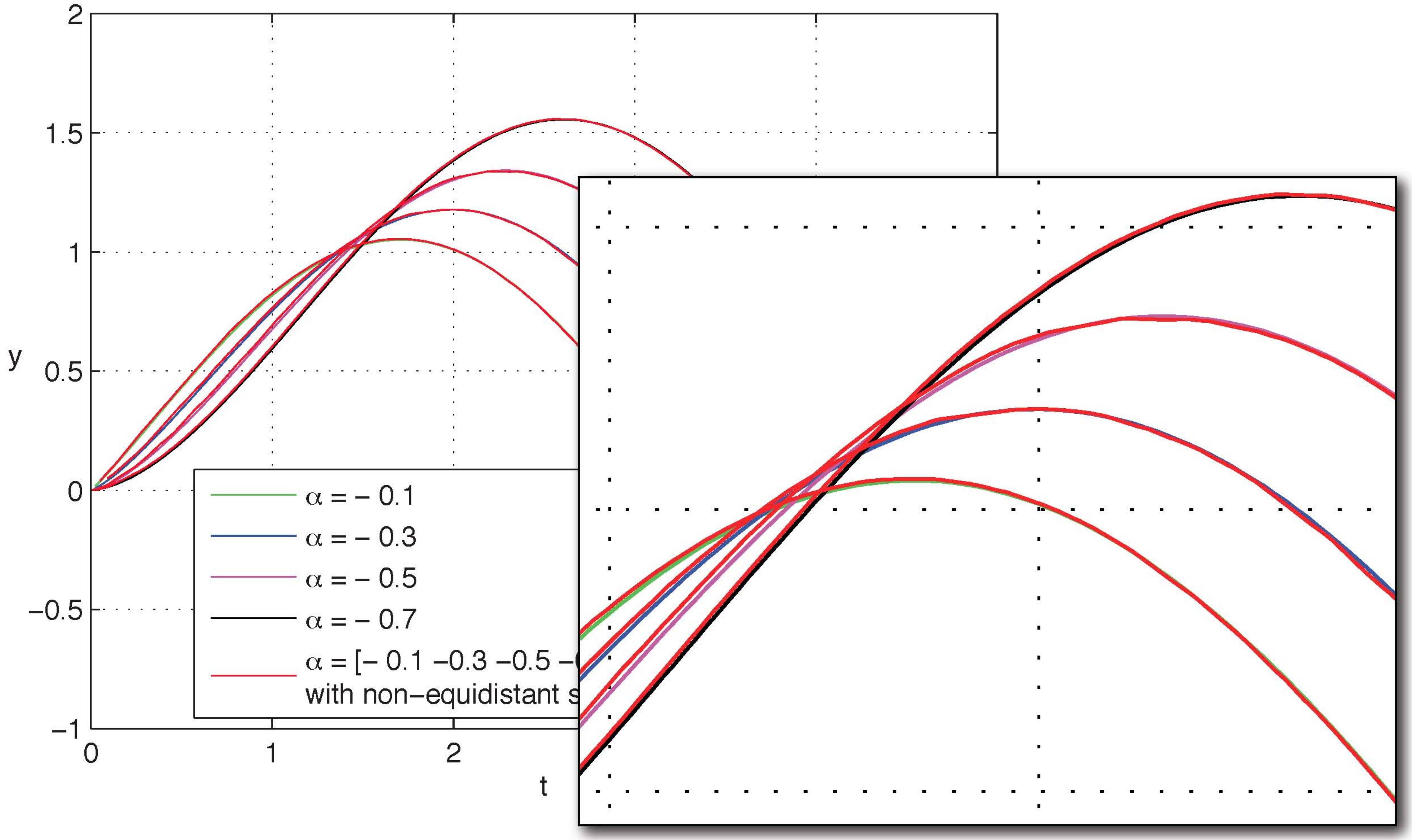
Coefficients of I_N^α

$$I_{k,j} = \frac{(t_k - t_{j-1})^\alpha - (t_k - t_j)^\alpha}{\Gamma(\alpha+1)},$$

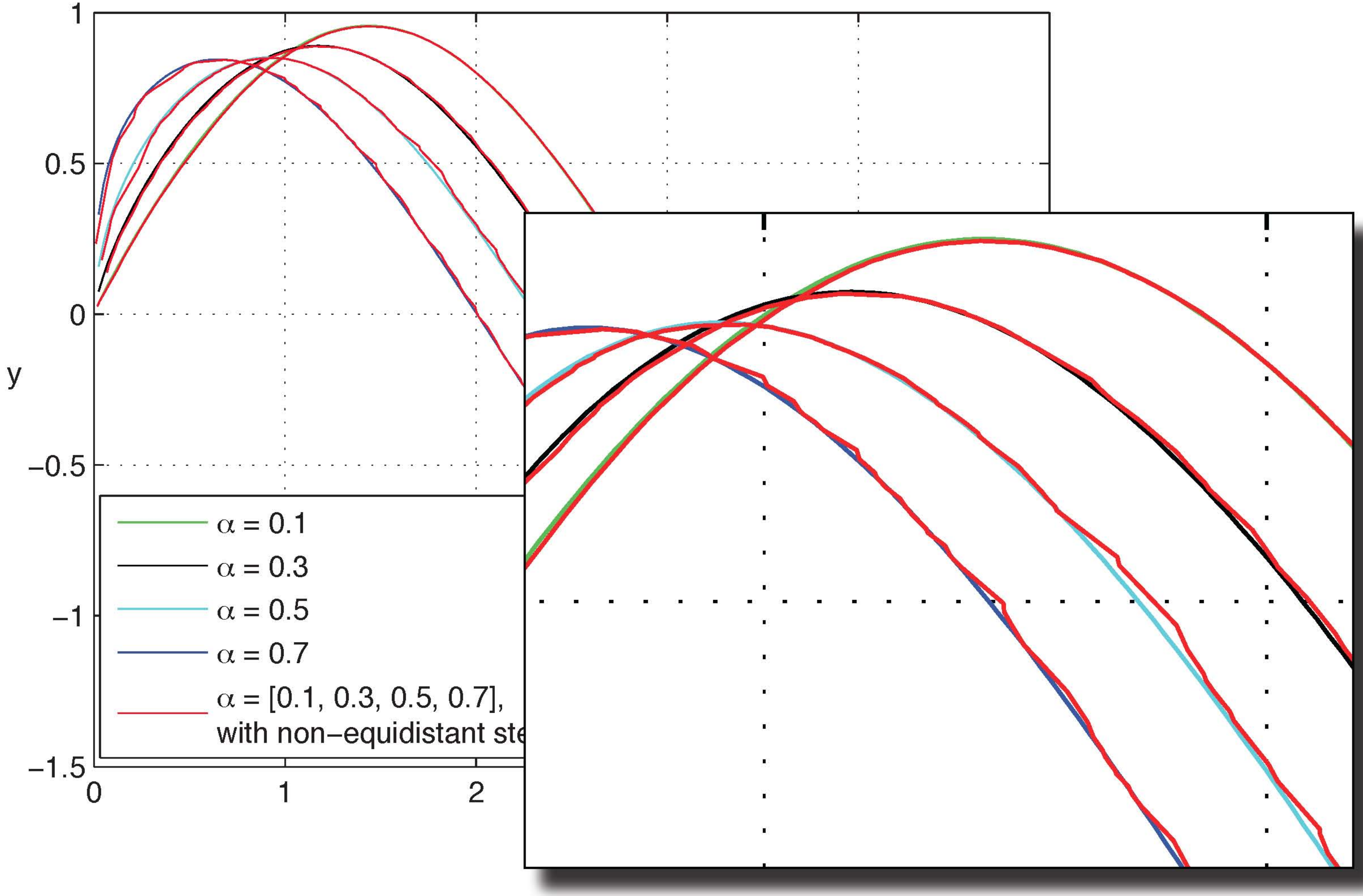
$$j = 1, \ldots, k; \quad k = 1, \ldots, N.$$

For non-equidistant grids, the matrix is not a TSM .

Example: fractional integrals of $\sin(x)$



Example: fractional derivatives of $\sin(x)$



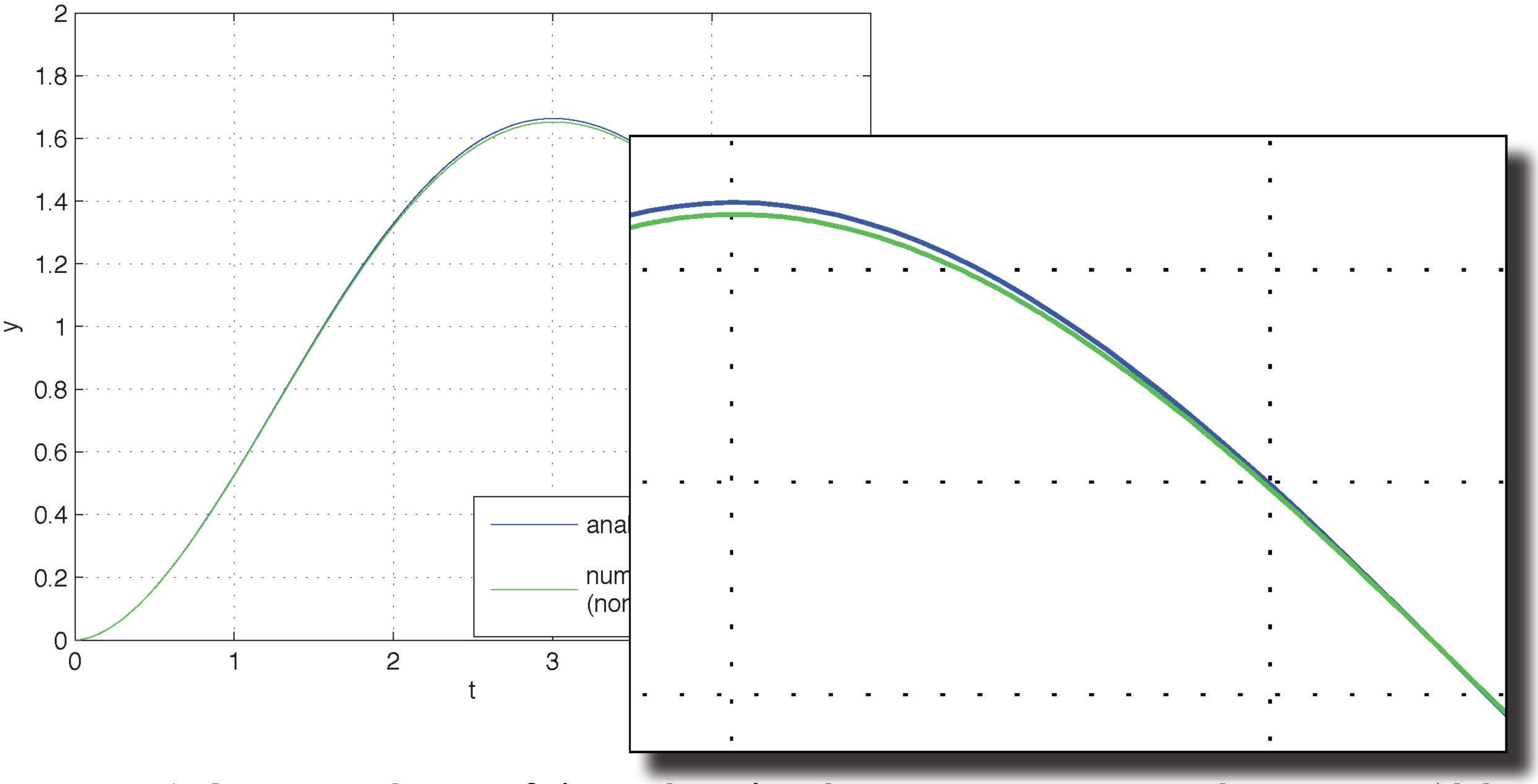
Example: two-term ordinary FDE

$$y^{(\alpha)}(t) + y(t) = 1,$$

$$y(0) = 0, \quad y'(0) = 0.$$

Exact solution:

$$y(t) = t^{\alpha} E_{\alpha,\alpha+1}(-t^{\alpha}).$$

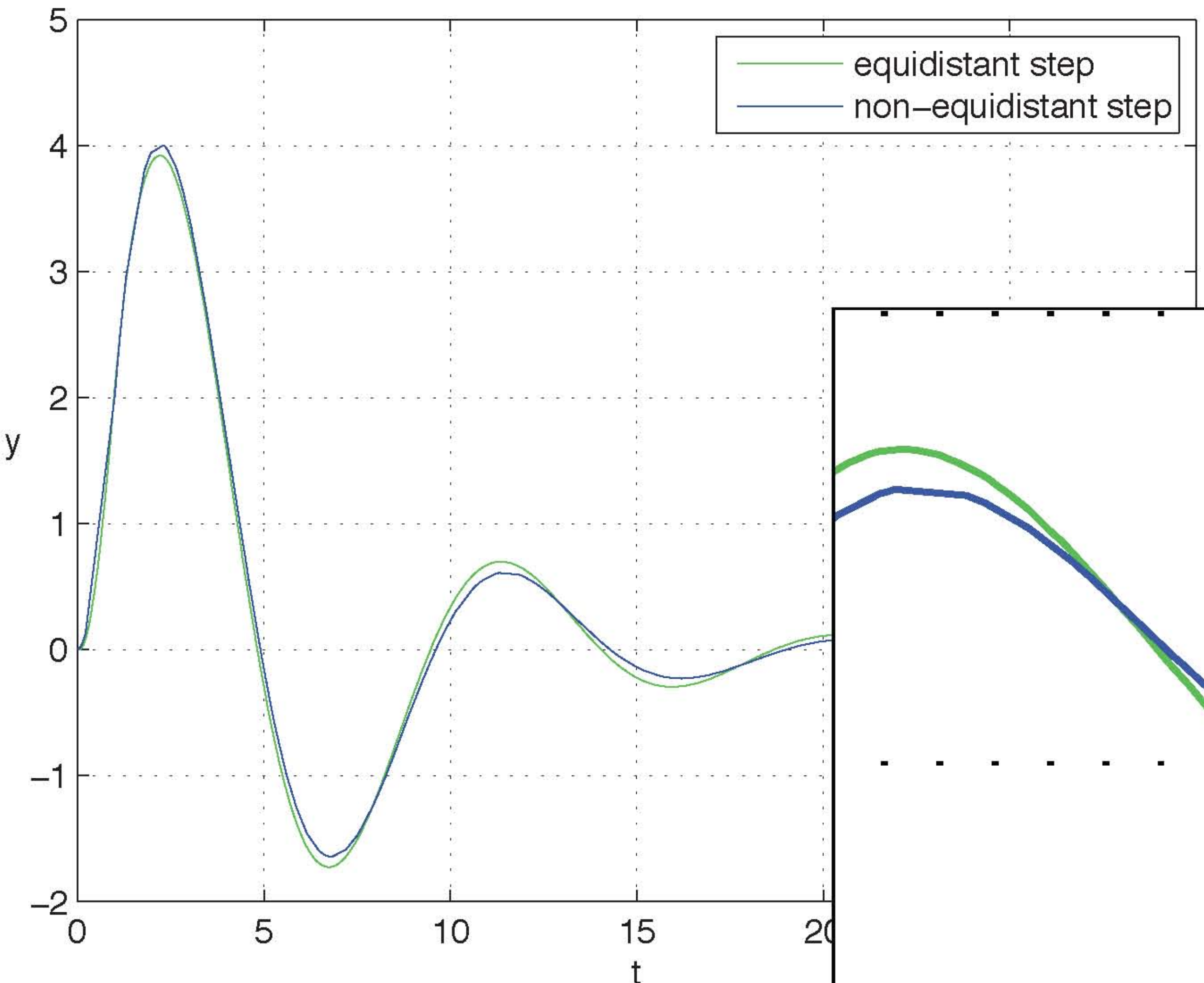


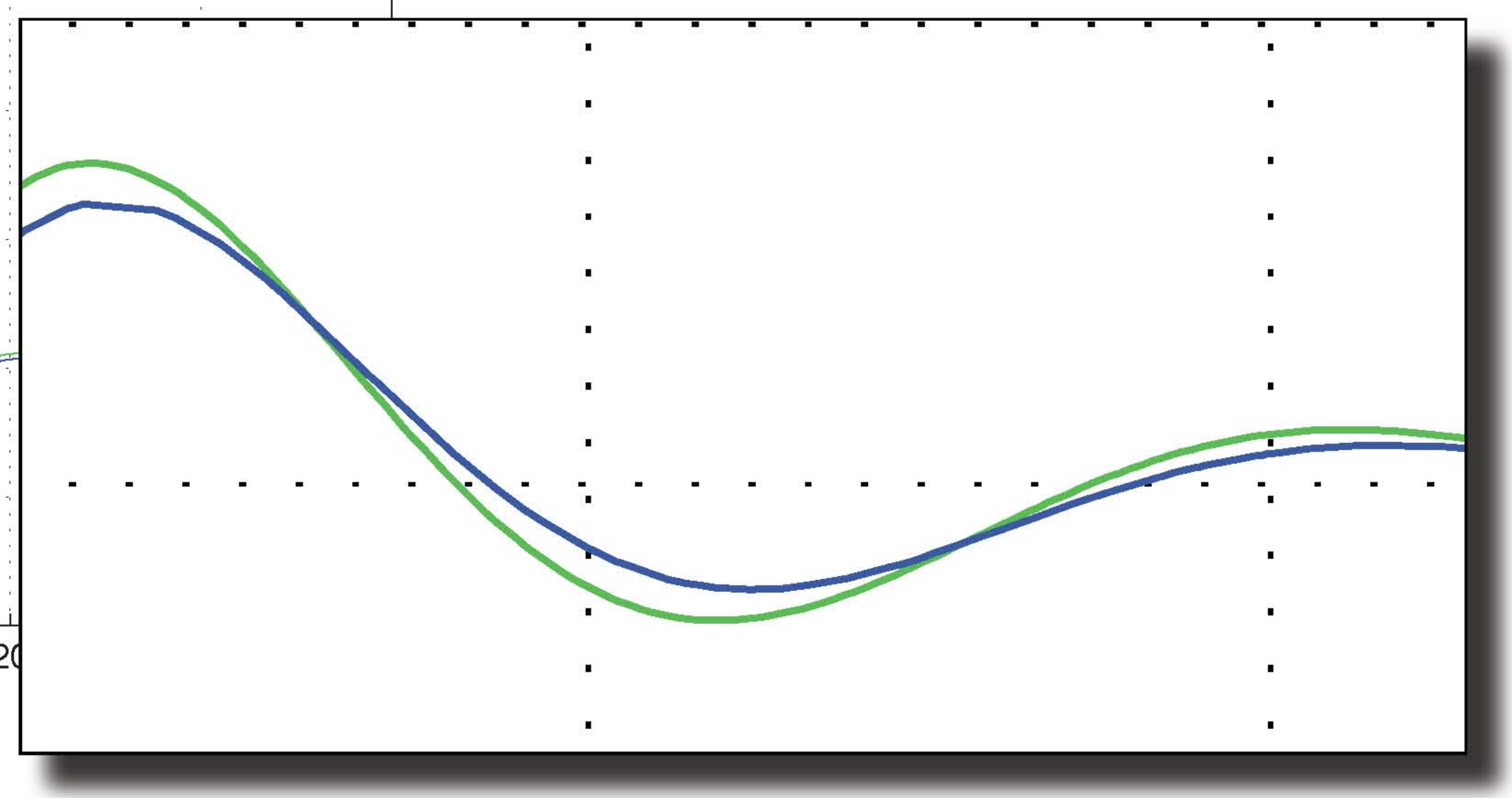
$\alpha = 1.8$, number of (random) discretization nodes $N = 500$

Example: Bagley-Torvik equation

$$Ay''(t) + By^{3/2}(t) + Cy(t) = F(t),$$

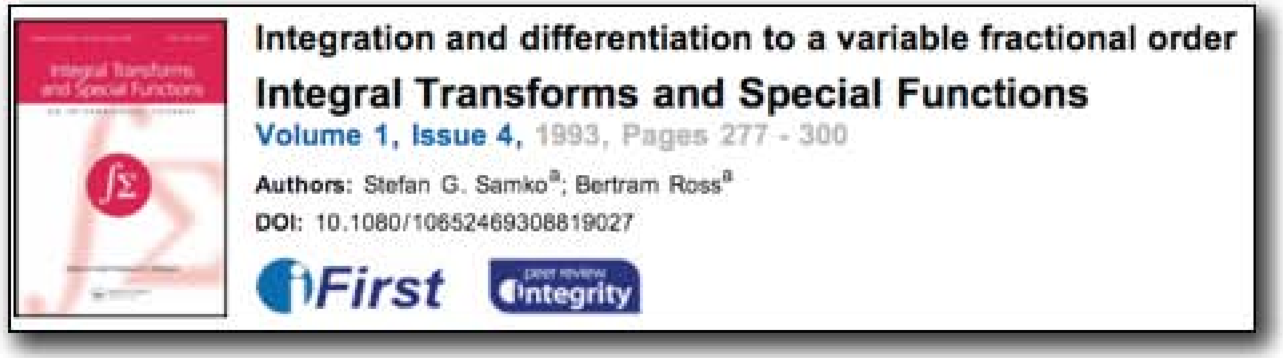
$$F(t) = \begin{cases} 8, & (0 \le t \le 1) \\ 0, & (t > 1) \end{cases}, \quad y(0) = y'(0) = 0.$$





$A = 1,\ B = 1,\ C = 1.$

Variable-order fractional differentiation and integration (VO-FD, VO-FI)


Integration and differentiation to a variable fractional order

Integral Transforms and Special Functions

Volume 1, Issue 4, 1993, Pages 277 - 300

Authors: Stefan G. Samko[a]; Bertram Ross[a]

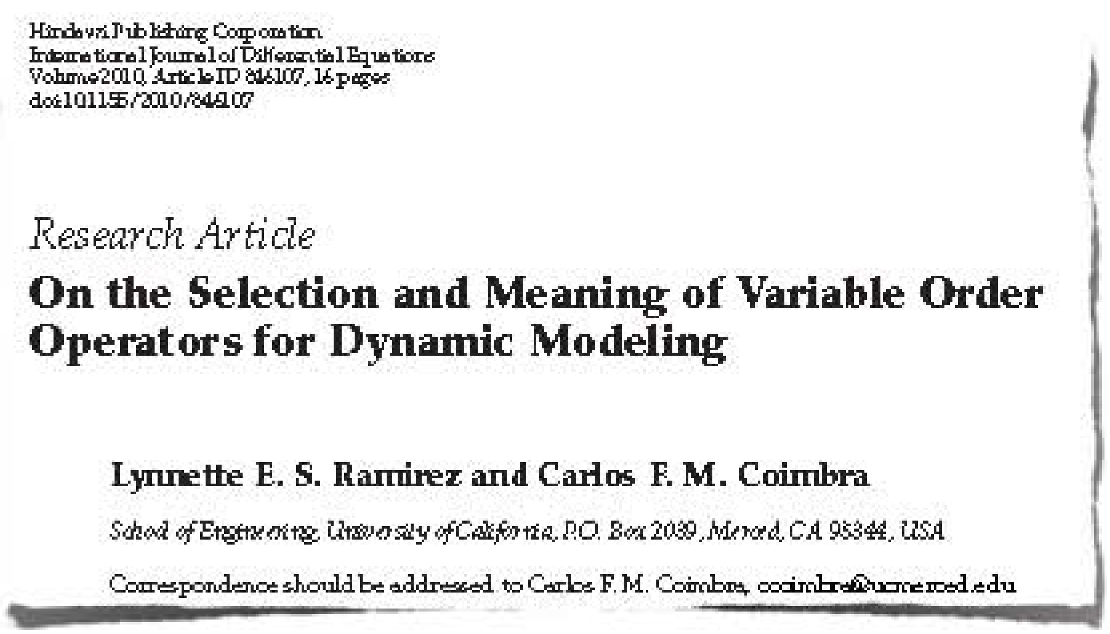
DOI: 10.1080/10652469308819027

iFirst

peer review Integrity

$$\mathfrak{D}_{c+}^{-q(t)} f(t) = \frac{1}{\Gamma[q(t)]} \int_c^t (t-\sigma)^{q(t)-1} f(\sigma) \mathrm{d}\sigma.$$

$$\mathrm{D}_{c+}^{q(t)} f(t) = \frac{1}{\Gamma[1-q(t)]} \frac{\mathrm{d}}{\mathrm{d}t} \int_c^t \frac{f(\sigma)}{(t-\sigma)^{q(t)}} \mathrm{d}\sigma.$$


Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2010, Article ID 846107, 16 pages
doi:10.1155/2010/846107

Research Article

On the Selection and Meaning of Variable Order Operators for Dynamic Modeling

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Correspondence should be addressed to Carlos F. M. Coimbra, ccoimbra@ucmerced.edu

$${}^{1}\mathrm{D}_0^{-q(t)} f(t) = \int_0^t \frac{(t-\sigma)^{q(t)-1}}{\Gamma[q(t)]} f(\sigma) \mathrm{d}\sigma,$$

$${}^{2}\mathrm{D}_0^{-q(t)} f(t) = \int_0^t \frac{(t-\sigma)^{q(\sigma)-1}}{\Gamma[q(\sigma)]} f(\sigma) \mathrm{d}\sigma,$$

$${}^{3}\mathrm{D}_0^{-q(t)} f(t) = \int_0^t \frac{(t-\sigma)^{q(t-\sigma)-1}}{\Gamma[q(t-\sigma)]} f(\sigma) \mathrm{d}\sigma.$$

$${}^{3-5}\mathrm{D}_0^{q(t)} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^t \frac{(t-\sigma)^{q(t,\sigma)-1}}{\Gamma[q(t,\sigma)]} f(\sigma) \mathrm{d}\sigma \right),$$

$${}^{6-8}\mathrm{D}_0^{q(t)} f(t) = \int_0^t \frac{(t-\sigma)^{q(t,\sigma)-1}}{\Gamma[q(t,\sigma)]} f^{(1)}(\sigma) \mathrm{d}\sigma,$$

$${}^{9}\mathrm{D}_0^{-q(t)} f(t) = \frac{1}{\Gamma[1-q(t)]} \int_{0+}^t (t-\sigma)^{-q(t)} f^{(1)}(\sigma) \mathrm{d}\sigma + \frac{(f(0+)-f(0-))t^{-q(t)}}{\Gamma[1-q(t)]},$$

Variable-order Fractional Differentiation (VOFD)

Left-sided

$$ {}_0^C D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(n-\alpha(t))} \int_0^t \frac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha(t)+1-n}}, \quad (n-1 \le \alpha(t) < n) $$

$$ {}_0^{GL} D_t^{\alpha(t)} f(t) = \lim_{\substack{h \to 0 \\ nh = t}} h^{-\alpha(t)} \sum_{k=0}^{n} (-1)^k \binom{\alpha(t)}{k} f(t-kh), $$

$$ (n-1 \le \alpha(t) < n). $$

Right-sided

$$ {}_t^C D_b^{\alpha(t)} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha(t))} \int_t^b \frac{f^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha(t)+1-n}}, \quad (n-1 \le \alpha(t) < n). $$

$$ {}_t^{GL} D_b^{\alpha(t)} f(t) = \lim_{\substack{h \to 0 \\ nh = b-t}} (-1)^n h^{-\alpha(t)} \sum_{k=0}^{n} (-1)^k \binom{\alpha(t)}{k} f(b-t+kh). $$

$$ (n-1 \le \alpha(t) < n) $$

Symmetric

$$ \frac{d^{\beta(x)} \phi(x)}{d|x|^{\beta(x)}} = D_R^{\beta(x)} \phi(x) = \frac{1}{2}\Big({}_a D_x^{\beta(x)} \phi(x) + {}_x D_b^{\beta(x)} \phi(x) \Big), $$

Discretization of left-sided VOFD

$$\left[v_n^{(\alpha_n)} \;\; v_{n-1}^{(\alpha_{n-1})} \;\; \ldots \;\; v_1^{(\alpha_1)} \;\; v_0^{(\alpha_0)}\right]^T = B_n^{(\alpha(t))} \left[v_n \;\; v_{n-1} \;\; \ldots \;\; v_1 \;\; v_0\right]^T$$

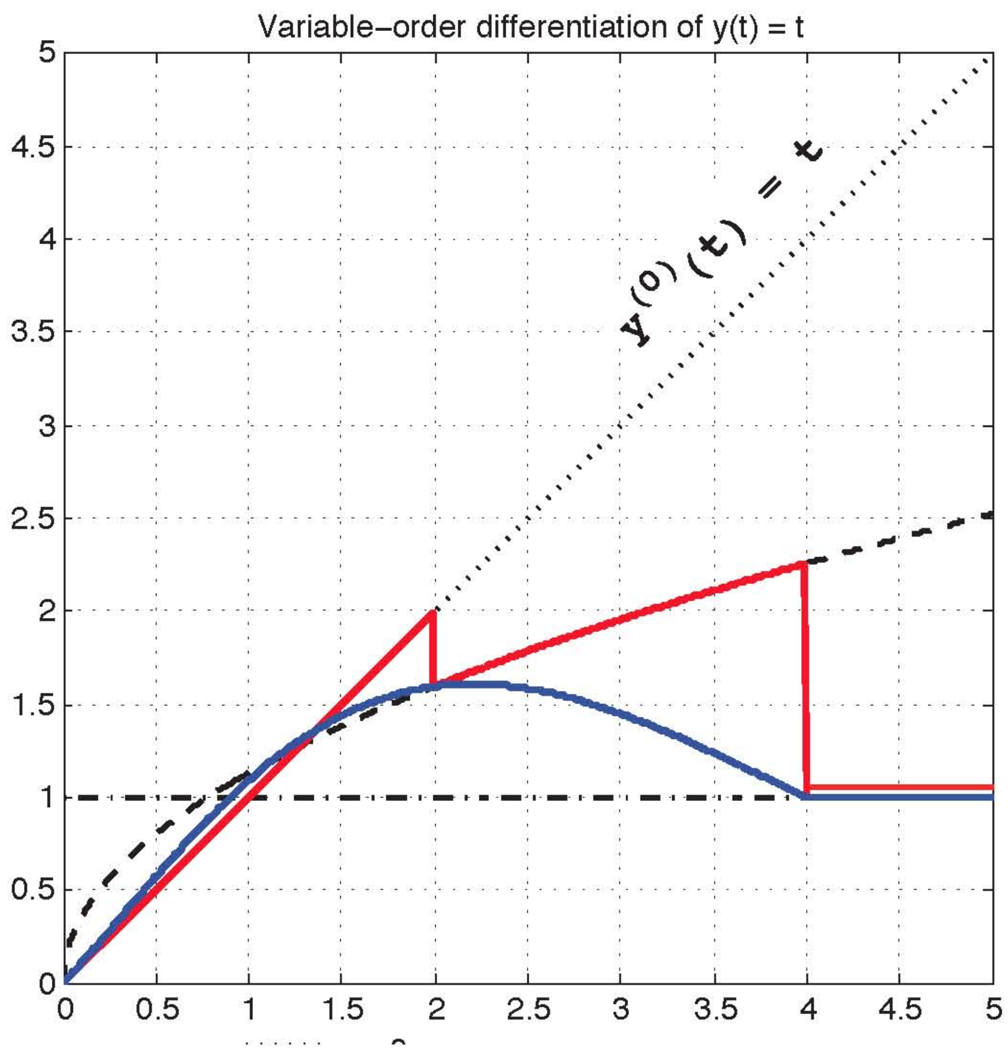
where

$$B_n^{(\alpha(t))} = \begin{bmatrix} \omega_0^{(\alpha_n)} & \omega_1^{(\alpha_n)} & \ddots & \ddots & \omega_{n-1}^{(\alpha_n)} & \omega_n^{(\alpha_n)} \\ 0 & \omega_0^{(\alpha_{n-1})} & \omega_1^{(\alpha_{n-1})} & \ddots & \ddots & \omega_{n-1}^{(\alpha_{n-1})} \\ 0 & 0 & \omega_0^{(\alpha_{n-2})} & \omega_1^{(\alpha_{n-2})} & \ddots & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & \omega_0^{(\alpha_1)} & \omega_1^{(\alpha_1)} \\ 0 & 0 & \cdots & 0 & 0 & \omega_0^{(\alpha_0)} \end{bmatrix}$$

$$\omega_j^{(\alpha_k)} = \tau^{-\alpha_k}(-1)^j \binom{\alpha_k}{j}, \qquad j = 0, 1, \ldots, k; \quad k = 0, 1, \ldots, n.$$

$$\alpha_k = \alpha(t_k), \qquad t_k = k\tau, \quad k = 0, 1, \ldots, n.$$

Example 0: VO-FD of function $y(t) = t$



$$\alpha = \begin{cases} 0, & 0 \le t < 2 \\ 0.5, & 2 \le t < 4 \\ 1, & 4 \le t \le 5 \end{cases}$$

$$\alpha = \begin{cases} 0.25\,t, & 0 \le t < 4 \\ 1, & 4 \le t \le 5 \end{cases}$$

Matlab functions:

- voban
- vofan
- voran

```
VO_alpha = '0 * ((t>0) & (t<2)) + 0.5 * ((t>=2) & (t<4)) + 1 * ((t>=4) & (t<=5))';
Dalphat = voban(VO_alpha, [0 5], h);    % differentiaton matrix
VOFDy = Dalphat * y';   % VO-derivative
```

```
VO_alpha = '0.25*t.*((t>=0)&(t<=4)) + 1*(t>4)';  % variable order
Dalphat = voban(VO_alpha, [0 5], h);    % differentiaton matrix
VOFDy = Dalphat * y';   % VO-derivative
```

Example 1: VO-fractional relaxation equation (1)

$$
{}_0D_t^{\alpha(t)}x(t) + Bx(t) = f(t), \quad (0 < \alpha(t) \le 1),
$$
$$
x(0) = 1,
$$

$\alpha(t) = e^{-At}$ with $A = 0.01$

$B = 0.1 \qquad f(t) = 0,$

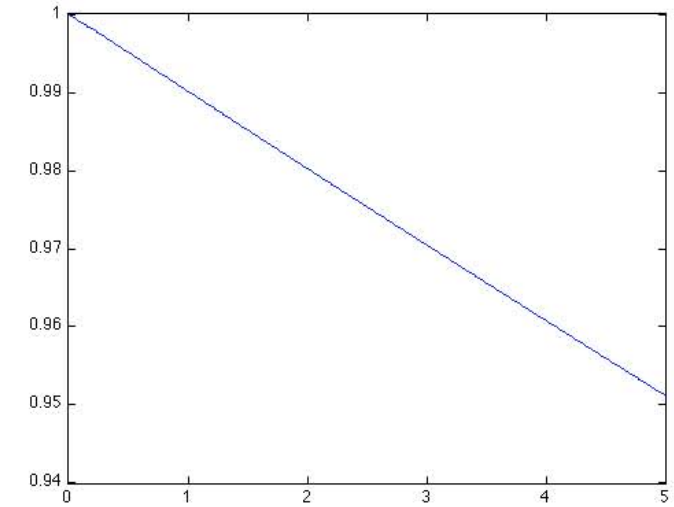


Figure 1: The shape of the variable order $\alpha(t) = e^{-At}$ for $A = 0.01$

"terminal" solutions for $\alpha(t) = 1$ and $\alpha(t) = 0.9512$

$$
{}_0D_t^{\alpha}x(t) + Bx(t) = 0, \quad (0 < \alpha \le 1),
$$
$$
x(0) = 1,
$$

$$
x(t) = E_{\alpha,1}(-Bt^{\alpha})
$$

$\alpha = 1$

$x(t) = E_{1,1}(-Bt) = e^{-Bt}$

$\alpha = e^{-0.05} = 0.9512$

$x(t) = E_{0.9512,1}(-Bt)$

Example 1: VO-fractional relaxation equation (3)

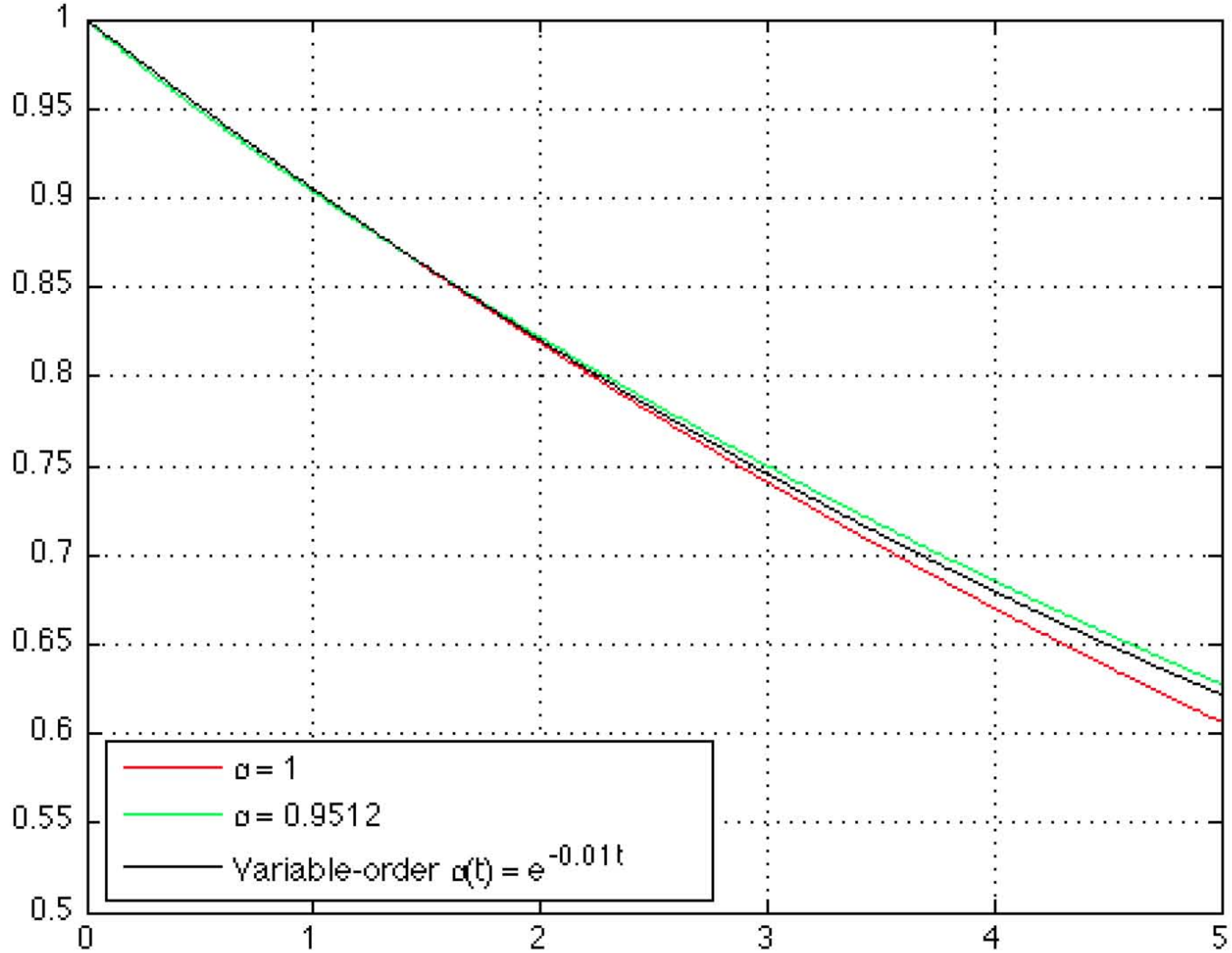


Figure 2: Solutions of problem (14) with $f(t) = 0$ and $B = 0.1$ for $\alpha = 1$ (red line), $\alpha = \exp(-0.05)$ (green line), and $\alpha(t) = \exp(-0.01 \cdot t)$ (black line). The discretization step is $\Delta t = 0.01$.

DO-fractional derivatives

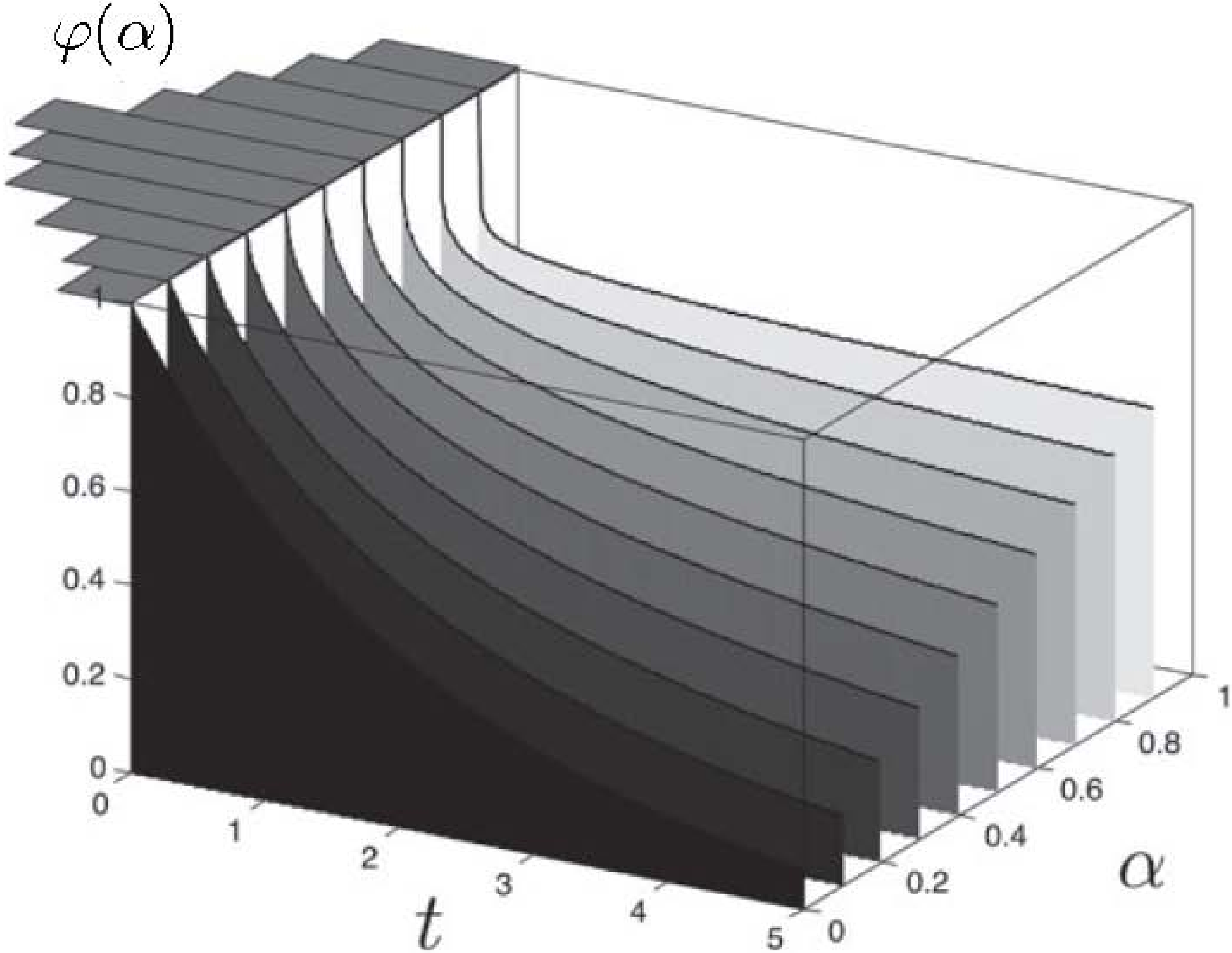
Left-sided
$$ {}_aD_t^{\varphi(\alpha)} f(t) = \int_c^d \varphi(\alpha)\, {}_aD_t^{\alpha} f(t) d\alpha $$

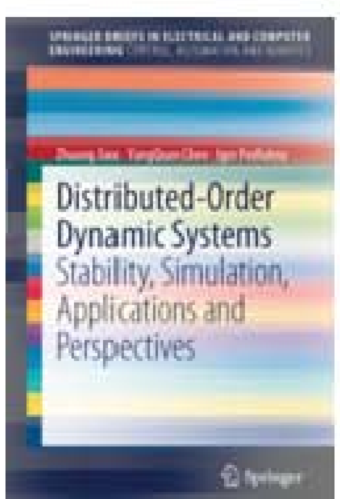
Right-sided
$$ {}_tD_b^{\varphi(\alpha)} f(t) = \int_c^d \varphi(\alpha)\, {}_tD_b^{\alpha} f(t) d\alpha $$

Symmetric
$$ {}_aR_b^{\varphi(\alpha)} f(t) = \int_c^d \varphi(\alpha)\, {}_aR_b^{\alpha} f(t) d\alpha $$

Restriction: $\displaystyle \int_c^d \varphi(\alpha)\, d\alpha = 1$

Interpretation of DO operators



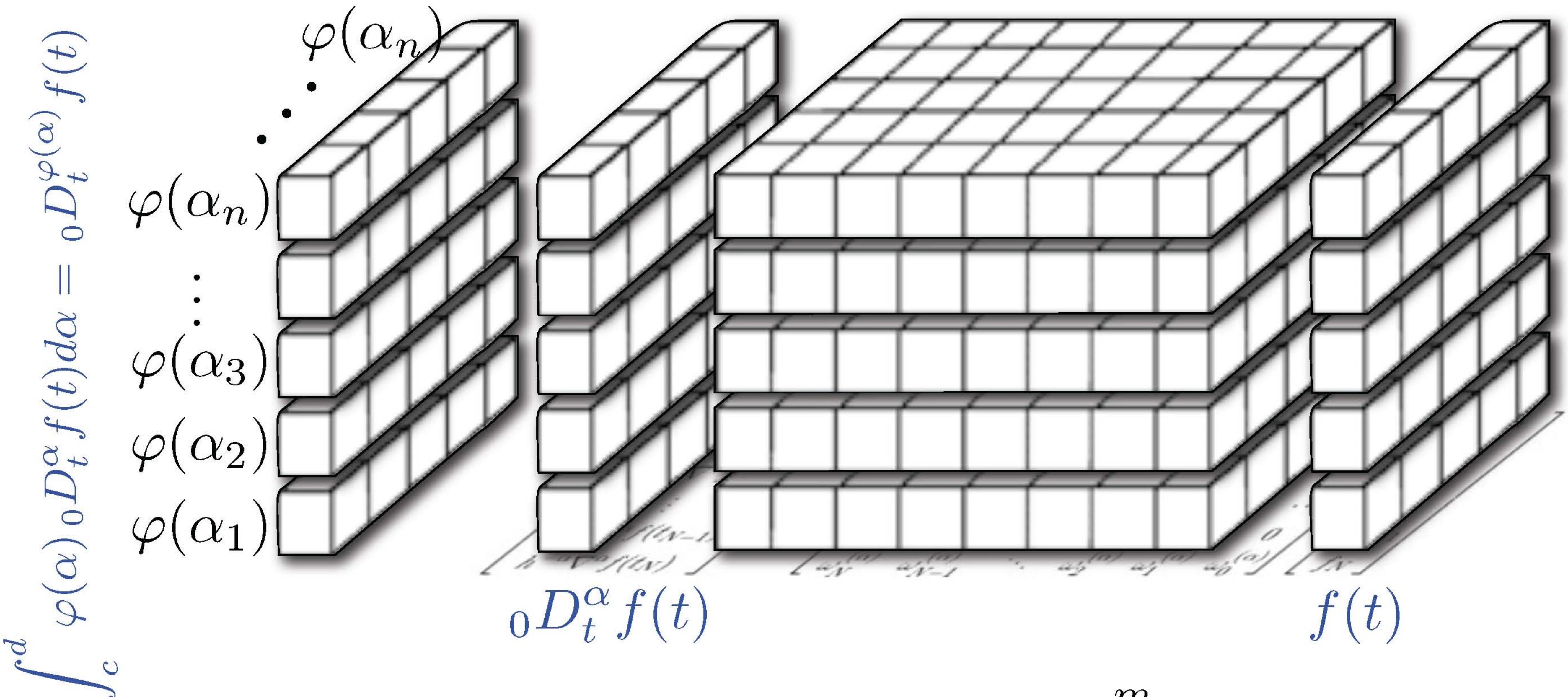


Discretization of DO-FDs:

a piece of cake!

Discretization of DO-FDs:

a piece of cake!



$${}_0D_t^{\varphi(\alpha)} f(t) \approx B_{n,m}^{\varphi(\alpha)} f_n, \qquad B_{n,m}^{\varphi(\alpha)} = \sum_{k=1}^{m} B_n^{\alpha_k} \varphi(\alpha_k) \Delta\alpha_k$$

Movement 5:
"All Together Now!"

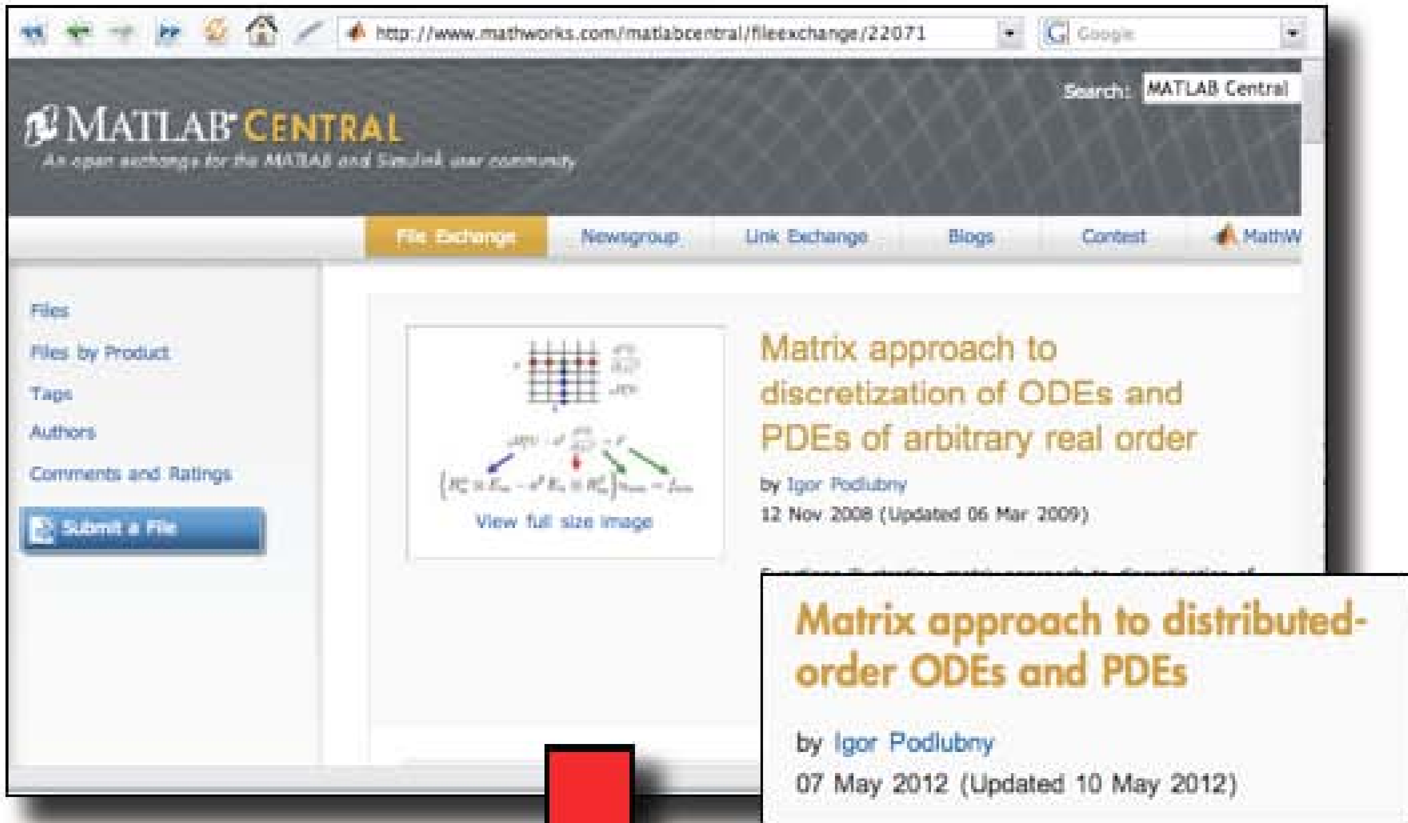
A toolbox for you!

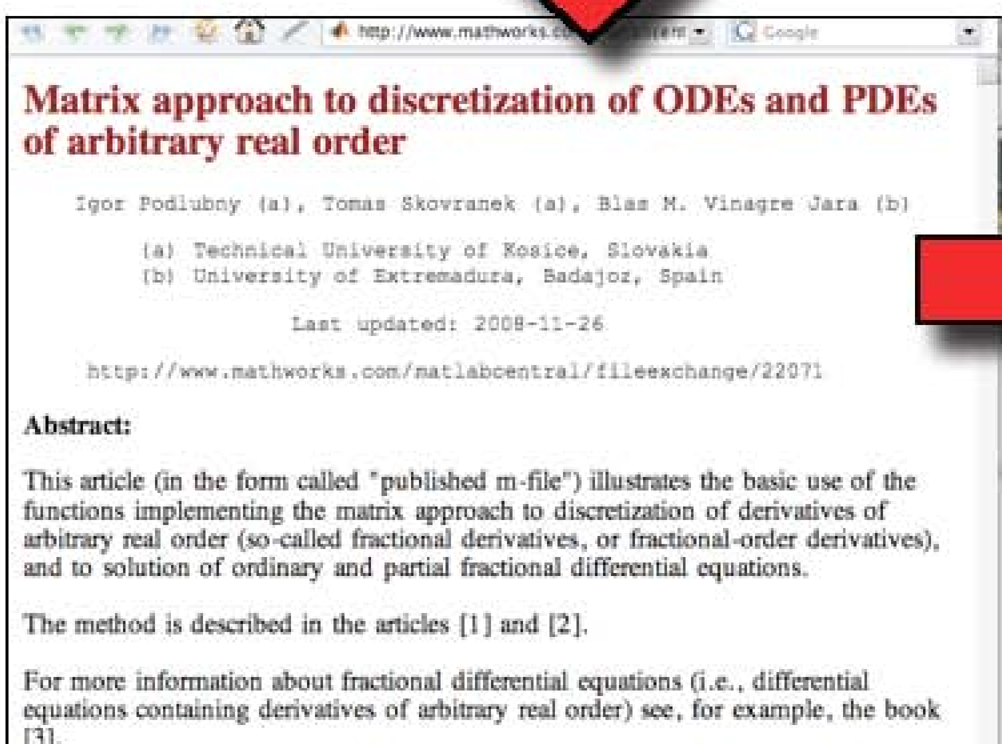
Instant solutions:

(1) add water

(2) microwave

(3) stir





Matrix approach to discretization of ODEs and PDEs of arbitrary real order

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Last updated: 2008-11-26

http://www.mathworks.com/matlabcentral/fileexchange/22071

Abstract:

This article (in the form called "published m-file") illustrates the basic use of the functions implementing the matrix approach to discretization of derivatives of arbitrary real order (so-called fractional derivatives, or fractional-order derivatives), and to solution of ordinary and partial fractional differential equations.

The method is described in the articles [1] and [2].

For more information about fractional differential equations (i.e., differential equations containing derivatives of arbitrary real order) see, for example, the book [3].

Contents

What is in the cup?

- `bcrecur.m`
- `eliminator.m`
- `shift.m`

Constant orders

- `ban.m`
- `fan.m`
- `ranort.m`
- `ransym.m`

Variable orders

- `voban.m`
- `vofan.m`
- `voranort.m`
- `voransym.m`

ready for release

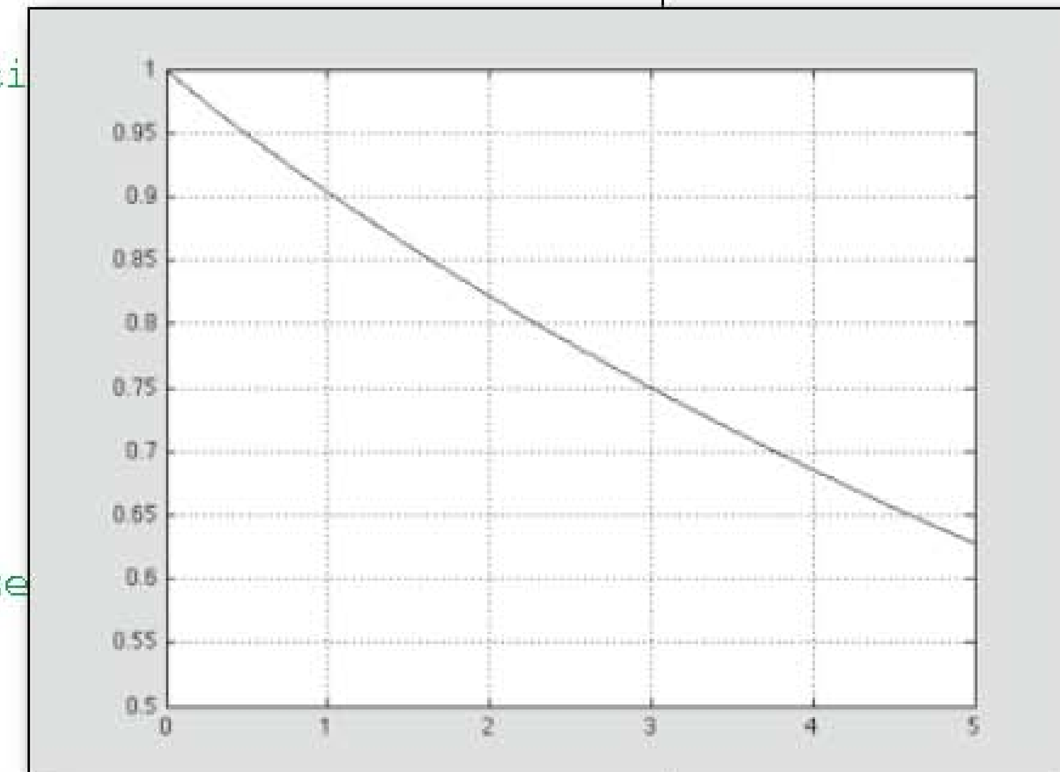
Distributed orders

- `doban.m`
- `dofan.m`
- `doranort.m`
- `doransym.m`

Example TI: CO-fractional relaxation equation

$$ {}_0D_t^{\alpha} x(t) + Bx(t) = 0, \quad (0 < \alpha \leq 1), $$
$$ x(0) = 1, $$

```
h = 0.01;             % step of discretization
t = 0:h:5;            % as in [DOFDS-paper, caption to Fig.6]
N = length(t) + 1;    % number of nodes
B = 0.1;              % coefficient of the equation
                      % as in [DOFDS-paper, caption to Fig.6]
f = '0 + 0*t';        % RHS, as in [DOFDS-paper, caption to Fig.6]
M = zeros(N,N);       % pre-allocate matrix M for the system

alpha=exp(-0.01*5);  % beta = 0.9512,  order of equation

% First, we make the matrix for the entire equation -- this is really easy:
M = ban(alpha, N-1, h) + B*eye(N-1,N-1);

% Then we compute the right-hand side at discretizati
F = eval ([f '-B'], t)';

% Utilize zero initial condition:
M = eliminator(N-1,[1])*M*eliminator(N-1, [1])';
F = eliminator(N-1,[1])*F;

% And solve the system MY=F:
Y = M\F;

% Pre-pend the zero initial value (that one due to ze
Y0 = [0; Y];

% Plot the solution:
plot(t, Y0+1, 'g')
```



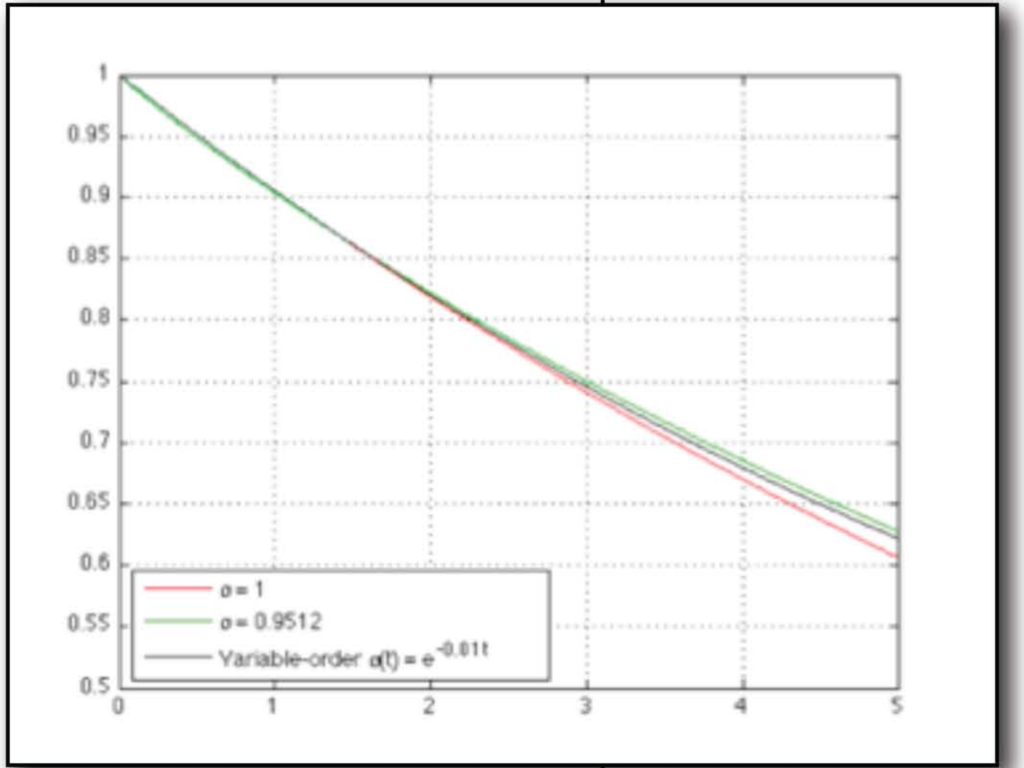
Example T2: VO-fractional relaxation equation

```
h = 0.01;              % step of discretization
t = 0:h:5;             % as in [DOFDS-paper, caption to Fig.6]
N = length(t) + 1;     % number of nodes
B = 0.1;               % coefficient of the equation
                       % as in [DOFDS-paper, caption to Fig.6]
f = '0 + 0*t';         % RHS, as in [DOFDS-paper, caption to Fi
M = zeros(N,N);        % pre-allocate matrix M for the system

% First, we make the matrix for the entire equation -- this is really easy:
M = voban('exp(-0.01*t)', [0 5], h) + B*eye(N-1,N-1);

% Then we compute the right-hand side at discretization nodes:
F = eval ([f '-B'], t)';

% Utilize zero initial condition:
M = eliminator(N-1,[1])*M*eliminator(N-1, [1])';
F = eliminator(N-1,[1])*F;

% And solve the system MY=F:
Y = M\F;

% Pre-pend the zero initial value
% (that one due to zero initial condition)
Y0 = [0; Y];

% Plot the solution:
U = Y0 + 1;
plot(t, U, 'k')
```

$$ {}_0D_t^{\alpha(t)} x(t) + Bx(t) = f(t), \quad (0 < \alpha(t) \leq 1), $$

$$ x(0) = 1, $$

$$ \alpha(t) = e^{-At}, \quad \text{for } A = 0.01 $$

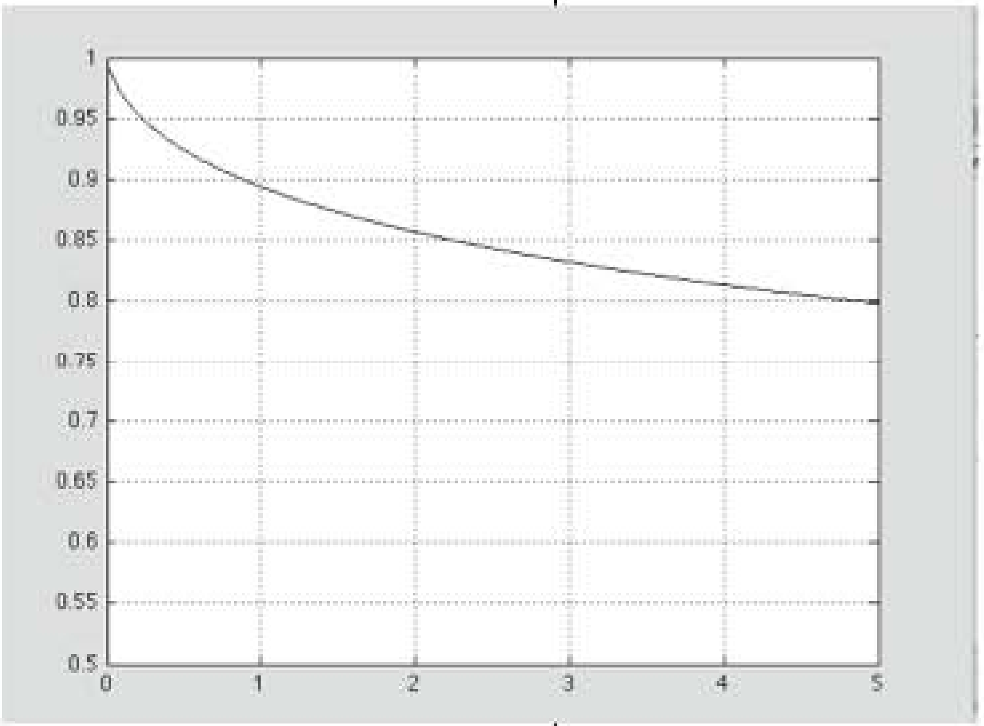


Monday, June 3, 2013

Example T3: DO-fractional relaxation equation

$$ {}_0D_t^{\varphi(\alpha)} x(t) + Bx(t) = f(t), $$
$$ x(0) = 1 $$
$$ \varphi(\alpha) = 6\,\alpha\,(1-\alpha), $$
$$ 0 \le \alpha \le 1 $$

```
h = 0.01;              % step of discretization
t = 0:h:5;             % as in [DOFDS-paper, caption to Fig.6]
N = length(t) + 1;     % number of nodes
B = 0.1;               % coefficient of the equation
                       % as in [DOFDS-paper, caption to Fig.6]
f = '0 + 0*t';         % RHS, as in [DOFDS-paper, caption to Fig.6]
M = zeros(N,N);        % pre-allocate matrix M for the system

% First, we make the matrix for the entire equation -- this is really easy:
M = doban('6*alf.*(1-alf)', [0 1], 0.01, N-1, h) + B*eye(N-1,N-1);

% Then we compute the right-hand side at discretization nodes:
F = eval ([f '-B'], t)';

% Utilize zero initial condition:
M = eliminator(N-1,[1])*M*eliminator(N-1, [1])';
F = eliminator(N-1,[1])*F;

% And solve the system MY=F:
Y = M\F;

% Pre-pend the zero initial value
% (that one due to zero initial condition)
Y0 = [0; Y];

% Plot the solution:
U = Y0 + 1;
plot(t, U, 'k')
```



Example T4: CO-order Bagley-Torvik equation

$$Ay''(t) + By^{3/2}(t) + Cy(t) = F(t), \quad F(t) = \begin{cases} 8, & (0 \le t \le 1) \\ 0, & (t > 1) \end{cases} \qquad y(0) = y'(0) = 0$$

```
% (1) Prepare constants and nodes (this is the longest part of the script):
alpha = 1.5;
A = 1; B = 1; C = 1;  % coefficients of the Bagley-Torvik equation
h = 0.075;            % step of discretization
T = 0:h:30;           % nodes
N = 30/h + 1;         % number of nodes
M = zeros(N,N);       % pre-allocate matrix M for the system
```

```
% (2) Make the matrix for the entire equation -- this is really easy:
M = A*ban(2,N,h) + B*ban(alpha,N,h) + C*eye(N,N);
```

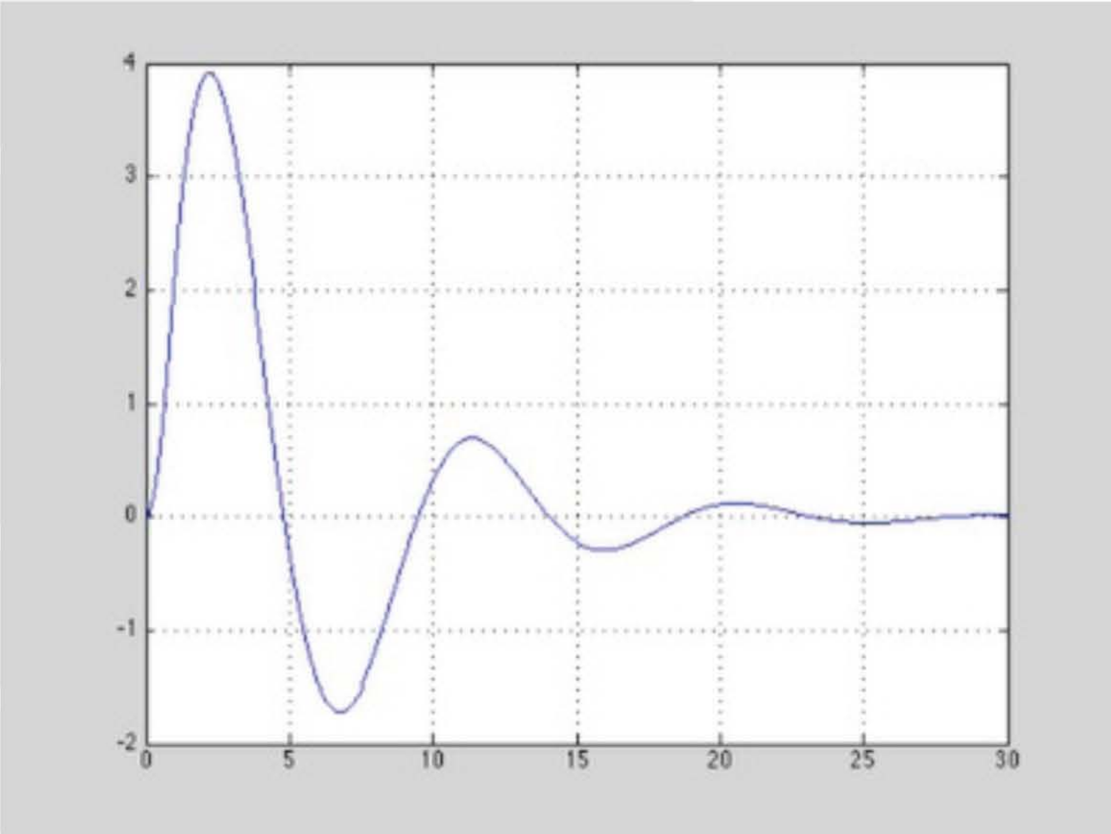
```
% (3) Make right-hand side:
F = 8*(T<=1)';
```

```
% (4) Utilize zero initial conditions:
M = eliminator(N,[1 2])*M*eliminator(N, [1 2])';
F = eliminator(N,[1 2])*F;
```

```
% (5) Solve the system MY=F:
Y = M\F;
```

```
% (6) Pre-pend the zero values (those due to zero
Y0 = [0; 0; Y];
```

```
% Plot the solution:
plot(T,Y0)
```

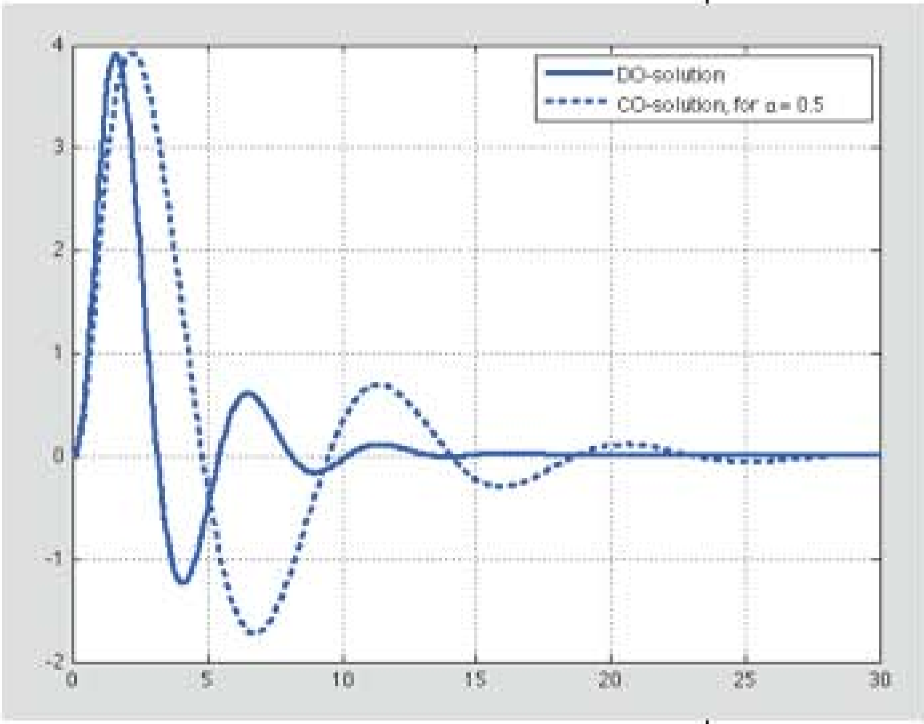


Example T5: DO-order Bagley-Torvik equation

$$Ay''(t) + By^{(\varphi(\alpha))}(t) + Cy(t) = F(t), \qquad F(t) = \begin{cases} 8, & (0 \le t \le 1) \\ 0, & (t > 1) \end{cases}$$

$$y(0) = y'(0) = 0$$

```
% (1) Prepare constants and nodes (this
alpha = 1.5;
A = 1; B = 1; C = 1;   % coefficients of
h = 0.075;             % step of discreti
T = 0:h:30;            % nodes
N = 30/h + 1;          % number of nodes
M = zeros(N,N);        % pre-allocate matrix M for the system

% (2) Make the matrix for the entire equation -- this is really easy:
M = A*ban(2,N,h) + B*doban('6*alf.*(1-alf)', [0 1], 0.01, N, h) + C*eye(N,N);

% (3) Make right-hand side:
F = 8*(T<=1)';

% (4) Utilize zero initial conditions:
M = eliminator(N,[1 2])*M*eliminator(N, [1 2])';
F = eliminator(N,[1 2])*F;

% (5) Solve the system MY=F:
Y = M\F;

% (6) Pre-pend the zero values (those due
% to zero initial conditions)
Y0 = [0; 0; Y];

% Plot the solution:
plot(T,Y0)
```



Example T7: CO-order time- and space-fractional diffusion equation

$$ {}_{0}^{C}D_{t}^{\alpha} y - \frac{\partial^{\beta} y}{\partial |x|^{\beta}} = f(x,t) $$

$$ y(0,t) = 0, \quad y(1,t) = 0; $$

$$ y(x,0) = 0. $$

```
alpha = 0.7; beta = 1.8;
```

```
a2=1;        % coefficient from the diffusion equation
L = 1;       % length of spatial interval
m = 21;      % Number of spatial steps of discretization
n =148;      % Number of steps in time
h = L / (m-1);          % spatial step
tau = h^2 / (6*a2);     % time step
```

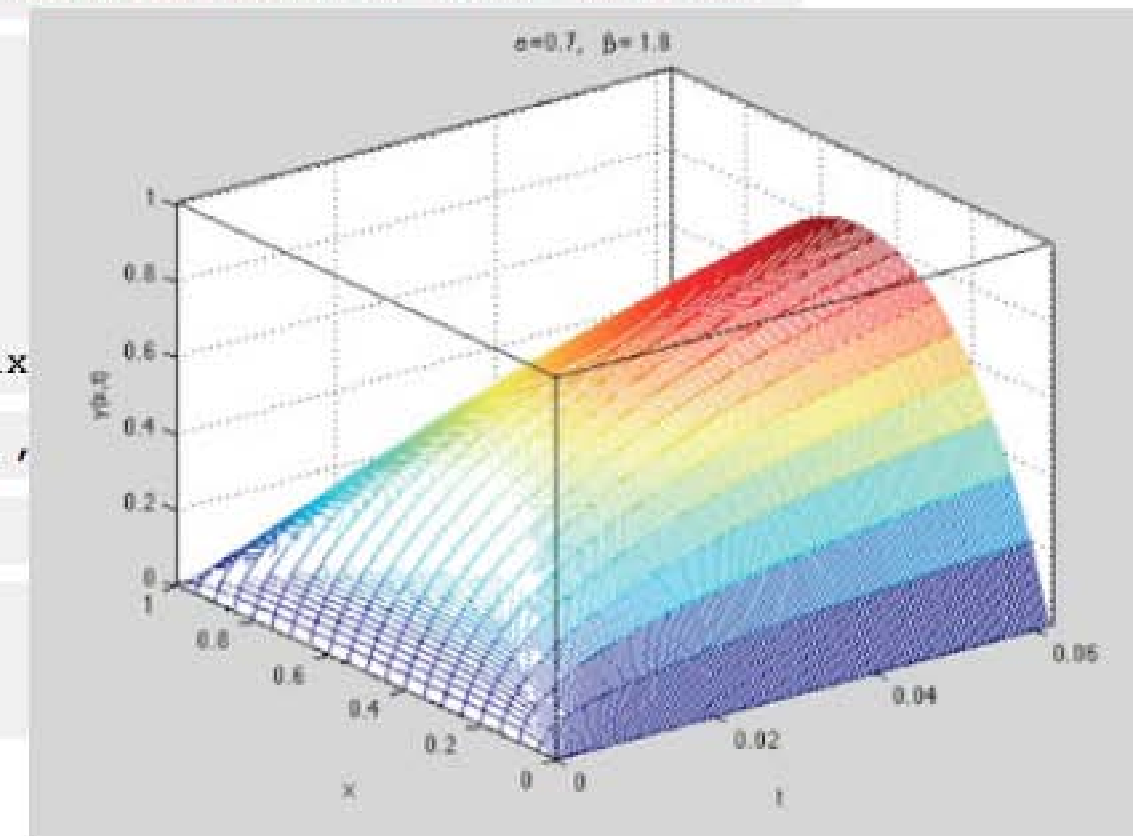
```
B1 = ban(alpha,n-1,tau)';    % alpha-th order derivative with respect to time
TD = kron(B1, eye(m));       % time derivative matrix

B2 = ransym(beta,m,h);       % beta-th order derivative with respect to X
SD = kron(eye(n-1), B2);     % spatial derivative matrix

SystemMatrix = TD - a2*SD;   % matrix corresponding to discretization in space and time
```

```
S = eliminator (m, [1 m]);
SK = kron(eye(n-1), S);
SystemMatrix_without_columns_1_m = SystemMatrix * SK';

S = eliminator (m, [1 m]);
SK = kron(eye(n-1), S);
SystemMatrix_without_rows_columns_1_m = SK * SystemMatrix
```

```
F = 8*ones(size(SystemMatrix_without_rows_columns_1_m,1),
```

```
Y = SystemMatrix_without_rows_columns_1_m\F;
```

```
YS = reshape(Y,m-2,n-1);
YS = fliplr(YS);
U = YS;
```

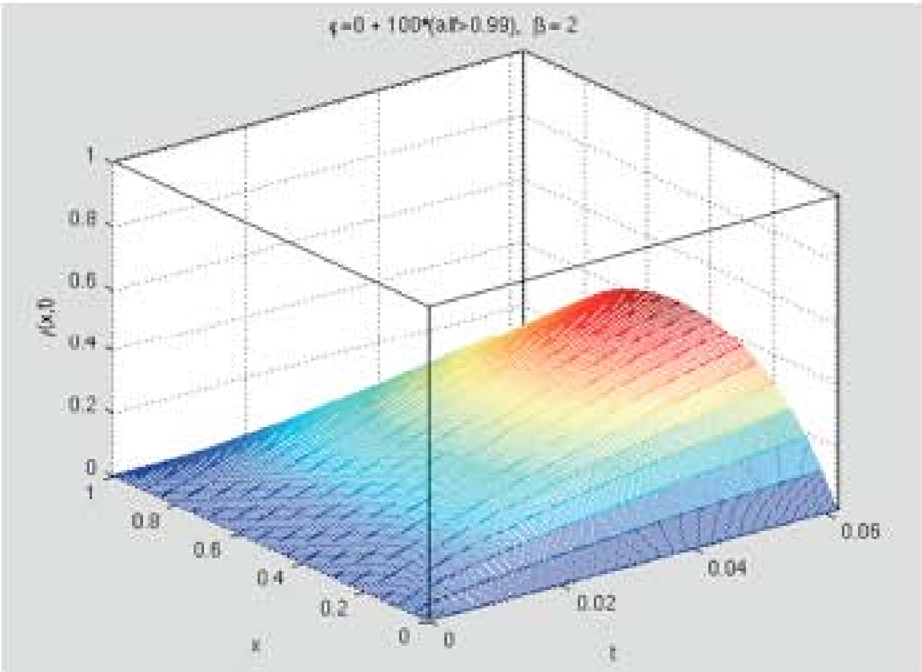


Example T8: DO-diffusion-wave equation

$$ {}_0^C D_t^{\varphi(\alpha)} y - \frac{\partial^\beta y}{\partial |x|^\beta} = f(x,t) $$

$$ y(0,t) = 0, \quad y(1,t) = 0; \qquad\qquad y(x,0) = 0. $$

$$ \varphi(\alpha) = 1 $$



DO → CO

$$ \varphi(\alpha) = \delta(\alpha - \lambda) $$

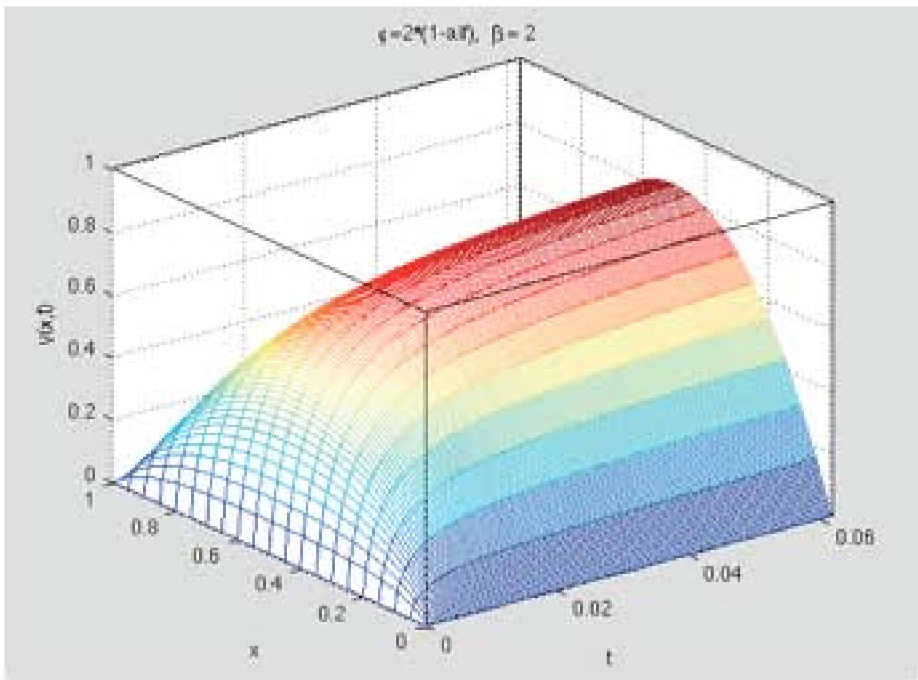
$$ \downarrow $$

$$ {}_0 D_t^\lambda $$

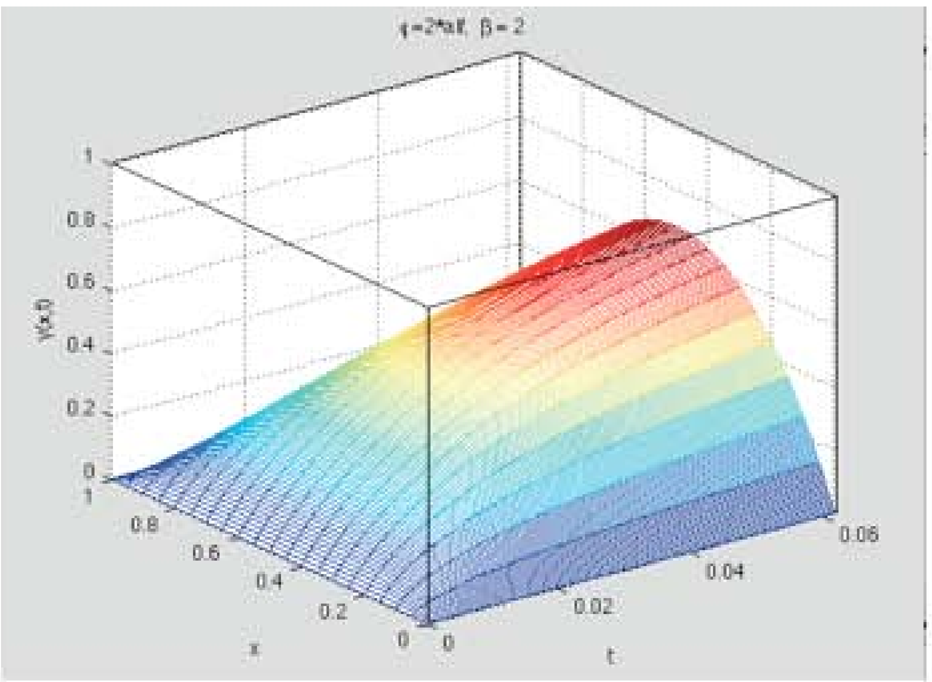
MATLAB: '0 + 100*(alf>0.99)'

Example T8(cont'd): DO-diffusion-wave equation

$$\varphi(\alpha) = 2\,(1-\alpha)$$



$$\varphi(\alpha) = 2\,\alpha$$



Variable step length?

As seen in MATLAB: **ode23.m** and **ode45.m** solvers

Method of "large steps"

ASME 2011 International Design Engineering Technical Conference
& Computers and Information in Engineering Conference
August 28-31, 2011 ~ Washington, DC, USA

DISCRETE FRACTIONAL CALCULUS: NON-EQUIDISTANT GRIDS AND VARIABLE STEP LENGTH

Podlubny I., Skovranek T, Verbitsky V., Vinagre B., Chen YQ., Petr...

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Matrix approach to discrete fractional calculus III: non-equidistant grids, variable step length and distributed orders

Igor Podlubny[1], Tomas Skovranek[1], Blas M. Vinagre Jara[2], Ivo Petras[1], Viktor Verbitsky[3] and YangQuan Chen[4]

Method of "large steps"

$$ {}_0D_t^{\alpha} y(t) = f(y(t), t), \quad (t > 0), $$
$$ y(0) = 0, $$

Suppose we obtained its solution in the interval (0,a) (and the final value y_a at $t = a$), then we can use this for transforming the above problem to

$$ {}_aD_t^{\alpha} y(t) = f(y(t), t) - {}_0R_a^{\alpha} y(t), \quad (t > a), $$
$$ y(a) = y_a, $$

where

$$ {}_0R_a^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^a (t-\tau)^{\alpha-1} y(\tau) d\tau, \quad (t > a). $$

$$ {}_0R_a^{\alpha} y(t) = {}_0D_t^{\alpha}\Big((1 - H(t-a))y(t)\Big) $$

First
"large step"
in [0, a]

$$
{}_0D_t^{\alpha}y(t) = f(y(t),t), \quad (t>0),
$$
$$
y(0) = 0,
$$

$$
{}_aD_t^{\alpha}y(t) = f(y(t),t) - {}_0R_a^{\alpha}y(t), \quad (t>a),
$$
$$
y(a) = y_a,
$$

Auxiliary function:

$$
y(t) \ = \ u(t) + y_a,
$$

Second
"large step"
in [a, b]

$$
{}_aD_t^{\alpha}u(t) = f(u(t)+y_a,t) - {}_0R_a^{\alpha}y(t) - y_a, \quad (t>a),
$$
$$
u(a) = 0.
$$

Method of "large steps": example (1)

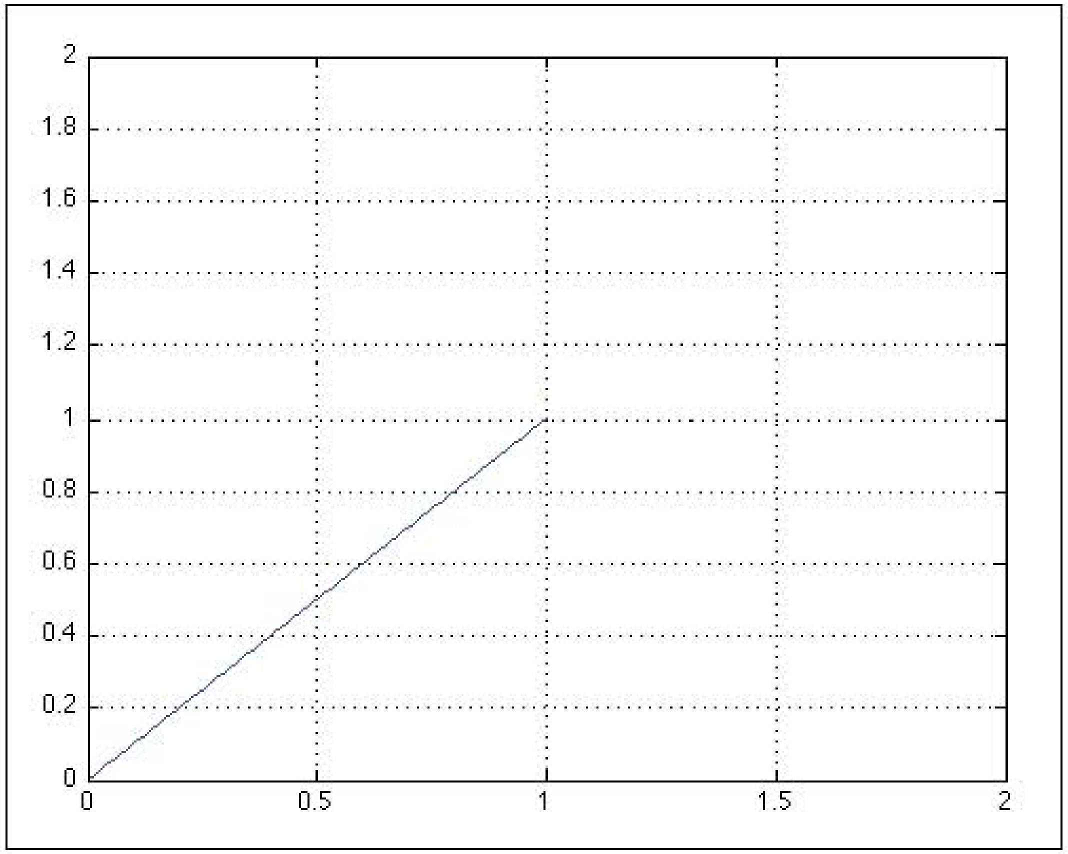
$$ {}_0D_t^{1/2}y(t) + y(t) = \frac{t^{1.5}}{\Gamma(1.5)} + t, \quad (t > 0), $$
$$ y(0) = 0. $$

Exact solution: $y(t) = t.$

First "large step": interval $[0, 1]$:

Using the matrix approach

```
clear all
h = 0.01;
t = 0:h:1;
N = 1/h + 1;
M = zeros(N,N);
M = ban(0.5, N, h) + eye(N,N);
F = (t.^(0.5)/gamma(1.5) + t)';
M = eliminator(N,[1])*M*eliminator(N, [1])';
F = eliminator(N,[1])*F;
Y = M\F;
Y0 = [0; Y];
plot (t,Y0,'b')
set(gca, 'xlim', [0 2], 'ylim', [0 2] )
grid on, hold on
```



Method of "large steps": example (2)

Second "large step": interval $[1, 2]$

$$ {}_0D_t^{1/2} y(t) = {}_1D_t^{1/2} y(t) + \frac{1}{\Gamma(0.5)} \int_0^1 \frac{y'(\tau)d\tau}{(t-\tau)^{1/2}}, \quad (t > 1) $$

$$ {}_1D_t^{1/2} y(t) + y(t) = \frac{t^{1.5}}{\Gamma(1.5)} + t - \frac{1}{\Gamma(0.5)} \int_0^1 \frac{d\tau}{(t-\tau)^{1/2}} \quad (t > 1). $$

$$ {}_1D_t^{\alpha} y(t) + y(t) = \frac{t^{1.5}}{\Gamma(1.5)} + t - \frac{2t^{0.5}}{\Gamma(0.5)} + \frac{2(t-1)^{0.5}}{\Gamma(0.5)}; \quad (t > 1) $$

$$ y(1) = 1. $$

$$ y(t) = u(t) + 1, $$

Method of "large steps": example (3)

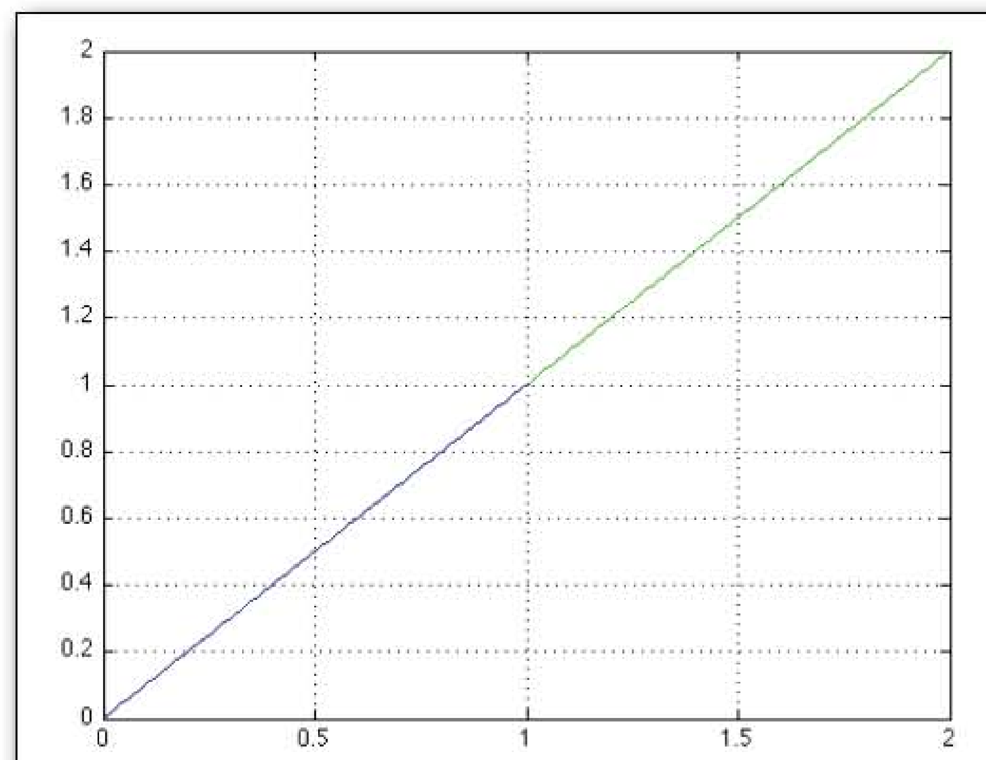
The problem to solve in $[1, 2]$:

$$ {}_1D_t^{\alpha} u(t) + u(t) = \frac{t^{1.5}}{\Gamma(1.5)} + t - \frac{2t^{0.5}}{\Gamma(0.5)} + \frac{2(t-1)^{0.5}}{\Gamma(0.5)} - 1; \qquad (t > 1) $$

$$ u(1) = 0. $$

Using the matrix approach

```
clear all
h = 0.01;
t = 1:h:2;
N = 1/h + 1;
M = zeros(N,N);
M = ban(0.5, N, h) + eye(N,N);
F = (t.^(0.5)/gamma(1.5) + t - 2*t.^(0.5)/gamma(0.5) ...
       + 2*(t-1).^(0.5)/gamma(0.5) - 1)';
M = eliminator(N,[1])*M*eliminator(N, [1])';
F = eliminator(N,[1])*F;
U = M \F;
U0 = [0; U];
Y0 = U0 + 1;
plot(t, Y0, 'g')
```



Method of "large steps" and the problem of initialization

C. Lorenzo and T. Hartley raised the question about initialization of fractional derivatives. Their motivation was to use or recover the information about the process $y(t)$ in the interval $(0, a)$, if we consider fractional derivatives of $y(t)$ in (a, ∞).

NOTE: in the second "large step" in the considered sample problem we used, in fact, the proper initialization of the fractional derivative in the interval $(1, 2)$ based on the known behavior of $y(t)$ in $(0, 1)$.

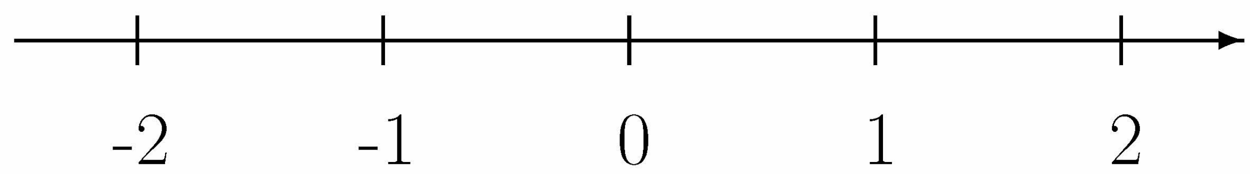
Finale

The Matrix Approach

- Uniform approach to CO, VO, and DO differentiation and integration.
- Easy, algorithmic, modular, ready to use.
- Allows solution of ODEs, including nonlinear problems.
- Allows solution of partial fractional differential equations.
- Allows consideration and solution of fractional differential equations with delays.
- Allows numerical solution of FDEs with a mixture of left-sided, right-sided, symmetric, CO,VO, DO fractional derivatives...
- Can be used on non-equidistant grids and in combination with the new method of "large steps".
- On the road:
  using sparse matrices;
  using parallel computations with the MATLAB Parallel Toolbox;
  applications to non-equidistantly sampled processes;
  and more...

Summary

1. The idea



$$\ldots, \frac{d^{-2}f}{dt^{-2}}, \frac{d^{-1}f}{dt^{-1}}, f, \frac{df}{dt}, \frac{d^2 f}{dt^2}, \ldots$$

2. Data to models

$$y = y_0 \, E_{\alpha,1}(a\, t^{\alpha}) \qquad {}_0^C D_t^{\alpha} y(t) - k\, y(t) = 0, \quad y(0) = y_0$$

3. Numerical solution

$$ {}_0D_t^{\alpha} U - a^2 \; {}_0D_x^{\beta} U = F$$

$$\left\{ B_n^{\alpha} \otimes E_m - a^2 \, E_n \otimes R_m^{\beta} \right\} u_{nm} = f_{nm}$$

Credits

Collaborators (in the order of appearance):

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Thank you!