Tempered Fractional Diffusion

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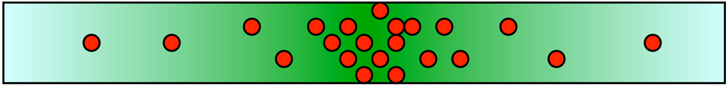
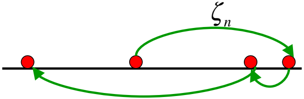
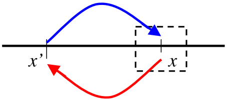
STATISTICAL FOUNDATION OF DIFFUSIVE TRANSPORT: THE BROWNIAN RANDOM WALK





ζ_n = jump $\qquad \lambda(\zeta)$ = jump size pdf

$\langle \zeta^2 \rangle$ =characteristic transport scale



Master equation

$$\partial_t P = \int_{-\infty}^{\infty} dx' \left[\underbrace{\lambda(x-x')\,P(x',t)}_{\text{gain}} - \underbrace{\lambda(x-x')\,P(x,t)}_{\text{loss}} \right]$$

$$\partial_t P = \int_{-\infty}^{\infty} dx' \; \lambda(x')\,P(x-x',t) - P(x,t)$$

Localization assumption

$$= \int_{-\infty}^{\infty} dx' \; \lambda(x') \left[P - x' \partial_x P + \frac{1}{2} x'^2 \, \partial^2{}_x P + \cdots \right] - P(x,t)$$

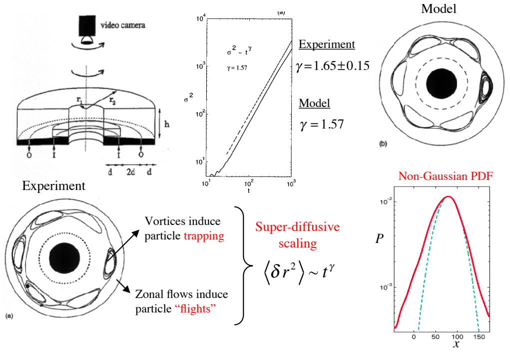
$$\boxed{\partial_t P = -V \partial_x P + D \, \partial^2{}_x P}$$

$$\boxed{\begin{aligned} \langle x \rangle &= V = \text{Advection} \\ \frac{1}{2}\langle x^2 \rangle &= D = \text{Diffusion} \end{aligned}}$$

NONDIFFUSIVE TRANSPORT AND COHERENT STRUCTURES

Chaotic transport by Rossby waves in zonal shear flows.

Problem *identical* to $\mathbf{E} \times \mathbf{B}$ transport by drift waves in zonal flows [1]



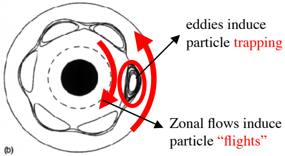
Solomon et al, Phys. Rev. Lett. **71,** 3975 (1993)   D. del-Castillo-Negrete, Phys. Fluids **10,** 576 (1998)

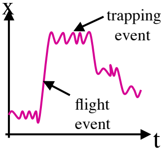
Signatures of anomalous transport: anomalous scaling of moments, $\langle \delta r^2 \rangle \sim t^\gamma$, $\gamma \neq 1$, and non-Gaussian (heavy tails) PDFs.

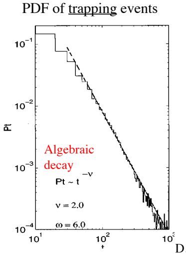
[1] DCN: Chaotic transport in zonal flows in analogous fluid and plasma systems. Phys. of Plasmas, 7, (5), 1702-1711, (2000).

UNDERLYING MECHANISM OF NONDIFFUSIVE TRANSPORT

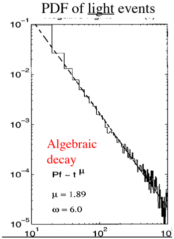
Levy flights induce by zonal flows and and long waiting times induced by trapping by Rossby (Drift) waves.







The algebraic decay of the trapping and flight pdf's gives rise to self-similar, scale-free transport

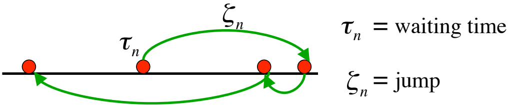


D. del-Castillo-Negrete, Phys. Fluids **10,** 576 (1998)

Levy flights: $P(\delta x) \sim \delta x^{-(1+\alpha)}$, $\langle \delta x^2 \rangle \to \infty$ for $\alpha < 2$.

STATISTICAL FOUNDATIONS OF NONDIFFUSIVE MODELS
The Continuous Time Random Walk (CTRW) model

Consider an ensemble of particles that at times $t_1, t_2, \ldots t_i \ldots$ experience a displacement $x_1, x_2, \ldots x_i \ldots$.
$\tau_i = t_i - t_{i-1}$ and x_i are assumed independent, identically distributed random variables



$\psi(\tau)$ =waiting time probability density function (pdf).
$\eta(x)$ =jump size pdf.

$$P[\tau_1 < \text{wait} < \tau_2] = \int_{\tau_1}^{\tau_2} \psi(\tau) d\tau \qquad P[x_1 < \text{jump} < x_2] = \int_{x_1}^{x_2} \eta(x) dx$$

Let $\phi(x, t)$ be the probability of finding a particle at x at time t if it was at $x = 0$ at time $t = 0$.

THE MONTROLL-WEISS CTRW MASTER EQUATION

- Probability that a particle has not moved during the time t

$$\Psi(t) = \int_t^\infty \psi(\tau) d\tau$$

- Probability of geting to x from any point x' during the time interval $(0, t)$

$$\int_0^t \psi(t - t') \int_{-\infty}^\infty \eta(x - x')\phi(x', t') dt' dx'$$

- The probability of finding a particle at x at time t if it was at $x = 0$ at time $t = 0$ is given by the master equation

$$\phi(x, t) = \delta(x)\Psi(t) + \int_0^t \psi(t - t') \int_{-\infty}^\infty \eta(x - x')\phi(x', t') dt' dx'$$

[Montroll-Weiss, 1969]

THE MONTROLL-WEISS CTRW MASTER EQUATION

Defining

$$\tilde{\Omega}(s) = \frac{s\tilde{\psi}}{1-\tilde{\psi}}, \qquad \tilde{H}(s) = \frac{1}{\tilde{\Omega}}$$

where $L[\psi](s) = \tilde{\psi}(s) = \int_0^\infty e^{-ts}\psi(t)dt$ denotes the Laplace transform, the Master equation can be rewritten as

$$\frac{\partial \phi}{\partial t} = \int_0^t dt'\Omega(t-t') \int_{-\infty}^{\infty} \left[\eta(x-x')\phi(x',t) - \eta(x'-x)\phi(x,t)\right]$$

where $\Omega(\tau)$ is the memory function. We can also write it as,

$$\int_0^t dt' H(t-t')\frac{\partial \phi}{\partial t'} = \int_{-\infty}^{\infty} dx' \left[\eta(x-x')\phi(x',t) - \eta(x'-x)\phi(x,t)\right]$$

The first term on the right hand side gives the accounts for transitions for x' to x and the second term accounts for contributions of transitions from x to x'.

SOLUTION IN FOURIER-LAPLACE SPACE AND FLUID LIMIT

- Let $\hat{f}(k)$ and $\tilde{f}(s)$ denote the Fourier and Laplace transforms.
- Application of the convolution theorem allow to transform the integral master equation into the algebraic equation

$$\hat{\tilde{\phi}}(k,s) = \frac{1-\tilde{\psi}}{s}\,\frac{1}{1-\tilde{\psi}(s)\,\hat{\eta}(k)}$$

  which explicitly determines $\phi(x,\tau)$ given $\psi(\tau)$ and $\eta(x)$.
- To simplify the highly nontrivial Fourier-Laplace inversion, and to focus on the time asymptotic, $t \gg 1$, long wavelength limit, we will consider

$$s \to 0\,, \qquad k \to 0\,,$$

RECOVERING THE STANDARD DIFFUSION MODEL

- In the absence of memory

$$\partial_t \hat{\phi} = \frac{\hat{\phi}}{\tau} \left[\hat{\eta} - 1\right] .$$

- In the long-wavelength limit, approximate

$$\partial_t \hat{\phi} \approx \frac{\hat{\phi}}{\tau} \left[\hat{\eta}(0) + \hat{\eta}'(0)k + \frac{\hat{\eta}''(0)}{2} k^2 + \ldots - 1\right] .$$

- Assuming the moments of η exist (key assumption!)

$$\langle x^n \rangle = (-i)^n \hat{\eta}^{(n)}(0) ,$$

- and using the identity

$$\mathcal{F}\left[\partial_x^n P\right] = (ik)^n \ \hat{P} ,$$

  the inversion of the Fourier transform gives the diffusion equation

$$\partial_t \phi = \chi \partial_x^2 \phi$$

FRACTIONAL IN SPACE MODEL OF SUPER-DIFFUSIVE TRANSPORT

- What happens if the moments of the jumps pdf does not exist? What is the macroscopic, effective transport equation in this case?
- Going back to the master equation without memory

$$\frac{\partial \phi}{\partial t} = \int_{-\infty}^{\infty} \left[\eta(x - x')\phi(x', t) - \eta(x' - x)\phi(x, t)\right] dx'$$

- In Fourier space

$$\frac{\partial \hat{\phi}}{\partial t} = [\hat{\eta}(k) - 1]\,\hat{\phi}$$

- To incorporate long jumps, we assume a Lévy process

$$\eta(x) \sim \frac{1}{|x|^{1+\alpha}}\,, \qquad \text{for} \qquad x \to \infty$$

  in Fourier space, $\hat{\eta}(k) \sim 1 - \chi|k|^{\alpha}\,, \qquad \text{for} \qquad |k| \to 0$
- For $1 < \alpha < 2$ this implies the divergence of moments

$$\langle x^n \rangle = \infty \qquad \text{for} \qquad n \geq 2$$

FRACTIONAL IN SPACE MODEL OF SUPER-DIFFUSIVE TRANSPORT

- Therefore, in the long wave-length limit

$$\frac{\partial \hat{\phi}}{\partial t} = -\chi |k|^{\alpha} \hat{\phi}$$

- Introducing the symmetric fractional spatial derivative:

$$D^{\alpha}_{|x|}\phi = F^{-1}\left[-|k|^{\alpha}\hat{\phi}\right] = \frac{\cos^{-1}(\pi\alpha/2)}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{\phi(y,t)}{|x-y|^{\alpha-1}}dy\,,$$

  for $1 < \alpha < 2$.

- We arrive to the following nonlocal in space model of superdiffusive-diffusive transport

$$\frac{\partial \phi}{\partial t} = \chi^{*} D^{\alpha}_{|x|}\phi$$

ASYMMETRIC SPACE-TIME FRACTIONL GENERAL MODEL

In flux conserving form

$$\partial_t \phi = -\partial_x \left[q_l + q_r \right] + S \,,$$

where q_l and q_r are the left and right nonlocal fluxes

$$q_l = -l \chi_l \, {}_0D_t^{\beta-1} \, {}_aD_x^{\alpha-1} \, \phi \,, \qquad q_r = r \chi_r \, {}_0D_t^{\beta-1} \, {}_xD_b^{\alpha-1} \, \phi \,,$$

where l and r determine the asymmetry of the nonlocal spatial operators

$${}_aD_x^{\alpha-1} \phi = \frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x} \int_a^x \frac{\phi(y,t)}{(x-y)^{\alpha-1}} \, dy \,,$$

$${}_xD_b^{\alpha-1} \phi = \frac{-1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x} \int_x^b \frac{\phi(y,t)}{(y-x)^{\alpha-1}} \, dy \,,$$

and the nonlocal temporal operator is

$${}_0^cD_t^{\beta} \phi = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial_\tau \phi(x,\tau)}{(t-\tau)^{\beta}} \, d\tau \,.$$

for $1 < \alpha < 2$, $0 < \beta < 1$.

GREEN'S FUNCTION

- Solution of the initial value problem

$$ {}_0^cD_t^\beta\,\phi = \chi\left[l\,{}_{-\infty}D_x^\alpha + r\,{}_xD_\infty^\alpha\right]\phi\,, \qquad \phi(x,t=0)=\phi_0(x) $$

$$ \phi(x,t)=\int_{-\infty}^{\infty}\phi_0(x')G(x-x',t)dx'\,, $$

- In Fourier-Laplace space, the Green's function is given by

$$ \hat{\tilde{G}} = \frac{s^{\beta-1}}{s^\beta-\Lambda(k)}\,, \qquad \Lambda=\chi\left[l(-ik)^\alpha + r(ik)^\alpha\right]\,, \qquad \alpha\neq 1 $$

- Introducing the Mittag-Leffler function

$$ E_\beta(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta n+1)}\,, \qquad \mathcal{L}\left[E_\beta(c\,t^\beta)\right]=\frac{s^{\beta-1}}{s^\beta-c}\,, $$

- The solution can be written in terms of the self-similar variable $\eta = x(\chi^{1/\beta}t)^{-\beta/\alpha}$ as

$$ G(x,t)=t^{-\beta/\alpha}K(\eta)\,, \qquad K(\eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\eta k}E_\beta\left[\Lambda(k)\right]dk\,. $$

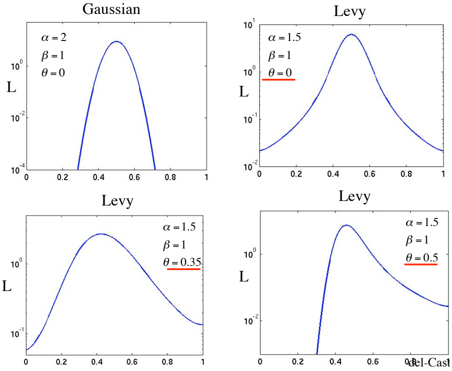
RELATION TO LEVY DISTRIBUTIONS

- Without memory ($\beta = 1$) i.e., only spatial fractional diffusion

$$G(x,t) = t^{-1/\alpha} L(\eta)$$

where $L(\eta)$ is the α-stable Lévy distribution

$$\hat{L}(k) = e^{\Lambda(k)} , \qquad \Lambda = \chi \left[l(-ik)^{\alpha} + r(ik)^{\alpha} \right] , \qquad \alpha \neq 1$$



SELF-SIMILARITY AND SCALING

- The scaling $\Lambda(\mu k) = \mu^{\alpha}\Lambda(k)$ implies the self-similar evolution

$$G(x, \mu t) = \mu^{-\beta/\alpha}\, G(\mu^{-\beta/\alpha} x, t)\,,$$

- From here it follows that the moments of G scale as

$$\langle x^q \rangle = C\, t^{q\beta/\alpha}\,, \qquad C = \int |\eta|^q\, K(\eta) d\eta\,.$$

$$2\beta/\alpha \begin{cases} > 1 & \text{super-diffusive scaling} \\ = 1 & \text{diffusive scaling} \\ < 1 & \text{sub-diffusive scaling} \end{cases}$$

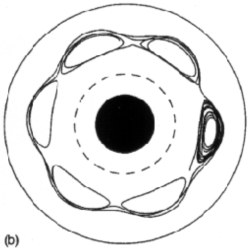
- Asymptotic scaling

$$G(x, t_0) \sim x^{-(1+\alpha)}\,, \qquad x \gg \left(\chi_f^{1/\beta} t_0\right)^{\beta/\alpha}$$

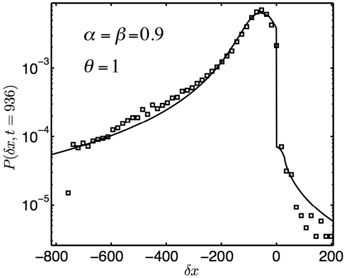
$$G(x_0, t) \sim \begin{cases} t^{\beta} & \text{for} \qquad t \ll \left(\chi_f^{-1} x_0^{\alpha}\right)^{1/\beta} \\ t^{-\beta} & \text{for} \qquad t \gg \left(\chi_f^{-1} x_0^{\alpha}\right)^{1/\beta} \end{cases}.$$

FRACTIONAL MODEL OF CHAOTIC TRANSPORT IN QUASIGEOSTROPHIC FLOWS

Model



Comparison with asymmetric neutral fractional diffusion equation



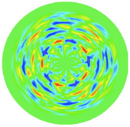
D. dCN, Phys. Fluids **10,** 576 (1998).
K. Gustafson, D. dCN, W. Dorland Phys. Of Plasmas **15,** 102309 (2008).

Fractional model (in space and time) reproduces quantitatively the PDF and scaling of moments in the strongly asymmetric regime

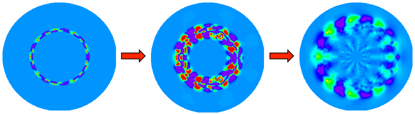
FRACTIONAL MODEL OF TURBULENT TRANSPORT IN MAGNETIZED PLASMAS

Test particle transport in electrostatic plasma turbulence

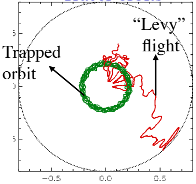
ExB flow velocity eddies induce particle trapping



"Avalanche like" phenomena induce flights that lead to spatial non-locality



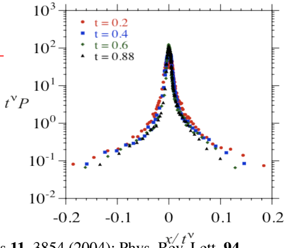
Tracer orbits



Particle trapping and flights leads to super-diffusive scaling

$$\boxed{\left\langle \delta r^{n} \right\rangle \sim t^{2n/3}}$$

Non-Gaussian (Levy) distribution



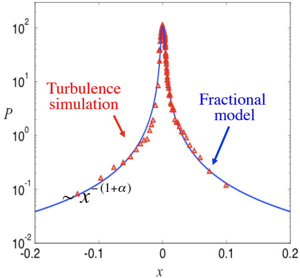
D. del-Castillo-Negrete, B. Carreras and V. Lynch, Phys. Plasmas **11**, 3854 (2004); Phys. Rev. Lett. **94**, 065003 (2005)

FRACTIONAL MODEL MODEL OF TURBULENT TRANSPORT IN MAGNETIZED PLASMAS

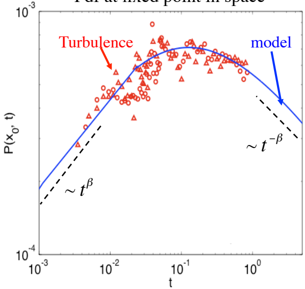
Test particle transport in the electrostatic plasma turbulence

$$\boxed{\partial_t^{\beta} P = -\partial_x q} \qquad \boxed{q = -D\partial_x^{\alpha-1} P} \qquad \alpha = 3/4 \quad \beta = 1/2 \qquad \langle x^2 \rangle \sim t^{2\beta/\alpha} \sim t^{4/3}$$

Levy distribution at fixed time



Pdf at fixed point in space



D. del-Castillo-Negrete, B. Carreras and V. Lynch, Phys. Plasmas **11**, 3854 (2004); Phys. Rev. Lett. **94**, 065003 (2005)

Fractional model reproduces quantitatively the PDF and scaling of moments

The need for tempered Levy processes

•As discussed before, there is experimental and numerical evidence of Levy flights in transport problems.

•The use of fractional diffusion to model these phenomena has proved to be very valuable.

•However, it is plausible that the finite-size domains and decorrelation effects (among other effects) might have an impact on the Levy flights.

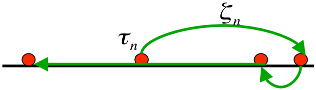
•Also, the divergence of the second moment of Levy pdfs can be physically questionable.

•These issues have motivated the introduction of tempered Levy processes [e.g. Mantegna&Stanley, 1994; Kopone, 1995; Cartea&dCN, 2007, Rosinski, 2007].

•Here we construct models that describe macroscopic transport driven by general Levy process and exponentially tempered processes in particular.

The importance of intermediate asymptotics

- Going back to the Continuous Time Random Walk (CTRW) model



$\lambda(\zeta)$ = jump size pdf $\qquad \psi(\tau)$ = waiting time pdf

$$\boxed{\begin{array}{c} \lambda \sim e^{-\zeta^2/2\sigma^2} \Rightarrow \langle \zeta^2 \rangle \text{ finite} \Rightarrow \text{Gaussian} \Rightarrow \partial_t P = \chi \partial_x^2 P \\ \lambda \sim \zeta^{-(1+\alpha)} \Rightarrow \langle \zeta^2 \rangle \text{ infinite} \Rightarrow \alpha-\text{stable Levy} \Rightarrow \partial_t P = \chi \partial_{|x|}^{\alpha} P \end{array}}$$

What happens when $\lambda \sim \zeta^{-(1+\alpha)} e^{-\lambda \zeta}$?

- Since $\langle \zeta^2 \rangle$ is finite, we expect the dynamics to converge asymptotically to Gaussian.

- However, the convergence rate is extremely slow, and in applications pure Gaussian behavior might never be observed but neither pure α-stable Levy!

- What is needed is a model that describes the interplay between long-jumps, truncation effects, and non-Markovian effects in the intermediate asymptotic regime.

CTRW FOR GENERAL LEVY PROCESSES

Going back to the Montroll-Weiss master equation for the CTRW

$$P = \delta(x) \int_t^{\infty} \psi(t')dt' + \int_0^t \psi\left(t - t'\right) \left[\int_{-\infty}^{\infty} \eta\left(x - x'\right) P(x', t')dx'\right] dt'$$

In the time-asymptotic limit, assuming $\psi \sim t^{-\beta-1}$, and in the long-wavelenght (fluid) limit

$$\hat{\eta}(k) = e^{\Lambda(k)} \approx 1 + \Lambda(k) + \dots$$

we get

$${}_0^c D_t^{\beta} \hat{P}(k, t) = \Lambda(k) \hat{P}(k, t) \, ,$$

where ${}_0^c D_t^{\beta}$ is the regularized (in the Caputo sense) fractional derivative in time.

GENERAL LEVY PROCESSES

- Λ is given by the Lévy-Khintchine representation

$$\Lambda = \ln \hat{\eta} = aik - \frac{1}{2}\sigma^2 k^2 + \int_{-\infty}^{\infty} \left[e^{ikx} - 1 - iku(x) \right] w(x) dx \, ,$$

where $w(x)$ is the Lévy density.

- Substituting into the dynamic equation and taking the inverse Fourier transform yields

$$ {}_0^c D_t^\beta P = -a \partial_x P + \frac{1}{2}\sigma^2 \partial_x^2 P +$$

$$+ \int_{-\infty}^{\infty} \left[P(x-y,t) - P(x,t) + u(y)\partial_x P \right] w(y) dy \, .$$

- This is the macroscopic transport equation describing the continuum, fluid limit of a CTRW with a general jump distribution function η characterized by a general Lévy density $w(y)$.

[Cartea and del-Castillo-Negrete, PRE, **76** 041105 (2007)]

α-STABLE LEVY PROCESSES

In the α-stable case the density is

$$w_{LS}(x) \quad = \quad \begin{cases} c\frac{(1+\theta)}{2}\,|x|^{-(1+\alpha)} & \text{for } x < 0, \\ \\ c\frac{(1-\theta)}{2}x^{-(1+\alpha)} & \text{for } x > 0, \end{cases} \tag{1}$$

Substituting and integrating

$$\Lambda_{LS} = iak - \frac{1}{2}\sigma^2 k^2 - \begin{cases} c|k|^\alpha \left\{1 + i\theta \mathrm{s}ign(k)\tan(\alpha\pi/2)\right\} & \alpha \neq 1, \\ \\ c|k|\left\{1 + \frac{2i\theta}{\pi}\mathrm{s}ign(k)\ln|k|\right\} & \alpha = 1\,, \end{cases} \tag{2}$$

where $\mathrm{sign}(k) = |k|/k$.

α-STABLE LEVY PROCESSES

From

$$ {}_0^cD_t^\beta \hat{P}(k,t) = \Lambda_{LS}\hat{P}\,, $$

inverting the Fourier transform we recover the fractional diffusion equation

$$ {}_0^cD_t^\beta P(x,t) = -a\partial_x P + \frac{1}{2}\sigma^2\partial_x^2 P + c\left[l\,{}_{-\infty}D_x^\alpha + r\,{}_xD_\infty^\alpha\right]\,P\,, $$

where the weighting factors are defined as

$$ l = -\frac{(1-\theta)}{2\cos(\alpha\pi/2)}\,, \qquad r = -\frac{(1+\theta)}{2\cos(\alpha\pi/2)}\,. $$

and

$$ \mathcal{F}\left[{}_{-\infty}D_x^\alpha f\right] = (-ik)^\alpha\,\hat{f}\,, \qquad \mathcal{F}\left[{}_xD_\infty^\alpha f\right] = (ik)^\alpha\,\hat{f}\,, $$

TEMPERED LEVY PROCESSES

- In the exponentially tempered case, the density is

$$w_{ET}(x) = \begin{cases} c\,\frac{(1+\theta)}{2}\,|x|^{-(1+\alpha)}\,e^{-\lambda|x|} & \text{for } x < 0, \\ c\,\frac{(1-\theta)}{2} x^{-(1+\alpha)} e^{-\lambda x} & \text{for } x > 0, \end{cases}$$

$0 < \alpha \leq 2$, $c > 0$, $-1 \leq \theta \leq 1$ and $\lambda \geq 0$.

- The corresponding characteristic exponent is

$$\Lambda_{ET} = -\frac{c}{2\cos(\alpha\pi/2)} \times$$

$$\times \begin{cases} (1+\theta)(\lambda+ik)^{\alpha} + (1-\theta)(\lambda-ik)^{\alpha} - 2\lambda^{\alpha}, \\ (1+\theta)(\lambda+ik)^{\alpha} + (1-\theta)(\lambda-ik)^{\alpha} - 2\lambda^{\alpha} - 2ik\alpha\theta\lambda^{\alpha-1} \end{cases}$$

for $0 < \alpha < 1$ and $1 < \alpha \leq 2$ respectively.

TEMPERED FRACTIONAL DIFFUSION

- From the fluid limit of the CTRW master equation,

$$ {}_0^cD_t^{\beta}\hat{P}(k,t) = \Lambda_{ET}\hat{P}\,, $$

inverting the Fourier transform we obtain the tempered fractional diffusion equation

$$ {}_0^cD_t^{\beta}P(x,t) = c\partial_x^{\alpha,\lambda}P\,. $$

- Where we have defined the tempered fractional diffusion operator

$$ \partial_x^{\alpha,\lambda}P = c\mathcal{D}_x^{\alpha,\lambda}P - V\partial_x P - \nu P\,. $$

- And we have defined the tempered fractional derivative

$$ \mathcal{D}_x^{\alpha,\lambda} = l e^{-\lambda x}\,{}_{-\infty}D_x^{\alpha}\,e^{\lambda x} + r e^{\lambda x}\,{}_xD_{\infty}^{\alpha}\,e^{-\lambda x}\,. $$

[Cartea and del-Castillo-Negrete, PRE, **76** 041105 (2007)]

TEMPERED DIFFUSION OPERATOR

$$\partial_x^{\alpha,\lambda} P = c\mathcal{D}_x^{\alpha,\lambda} P - V\partial_x P - \nu P$$

- Fourier transform of the tempered fractional derivative

$$\mathcal{F}\left[\mathcal{D}_x^{\alpha,\lambda} P\right] = \left[l\,(\lambda - ik)^{\alpha} + r\,(\lambda + ik)^{\alpha}\,\right]\hat{P}$$

- For $0 < \alpha < 1$, $V = 0$, but for for $1 < \alpha < 2$ there is a tempered induced drift in the asymmetric case

$$V = -\frac{c\alpha\theta\lambda^{\alpha-1}}{|\cos(\alpha\pi/2)|}$$

- The constant ν is defined as

$$\nu = -\frac{c\lambda^{\alpha}}{\cos(\alpha\pi/2)}\,,$$

although this term looks as a "damping" it actually guarantees the conservation of the probability, i.e., $\Lambda_{ET}(k=0,\lambda)=0$.

TEMPERED FRACTIONAL DIFFUSION

- Green's function

$$G = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} E_{\beta} \left[t^{\beta} \Lambda(k;\lambda) \right] dk$$

- Probability conservation

$$\int_{-\infty}^{\infty} G \, dx = \hat{G}(k=0,t) = E_{\beta} \left[t^{\beta} \Lambda(0;\lambda) \right] = E_{\beta}(0) = 1 \, .$$

- Truncation breaks the self-similarity

$$G\left(x, \mu t; \lambda\right) = \mu^{-\beta/\alpha} G\left(\mu^{-\beta/\alpha} x, t; \mu^{\beta/\alpha} \lambda\right) \, .$$

- Truncation guarantees finite moments.
  First moment $\langle x \rangle(t) = 0$ for $1 < \alpha < 2$ and

$$\langle x \rangle(t) = \frac{V}{\Gamma(\beta+1)} \, t^{\beta} \, , \qquad 0 < \alpha < 1 \, ,$$

where $V = \frac{-\chi \alpha \theta}{|\cos(\alpha\pi/2)| \lambda^{1-\alpha}}$ is the drift velocity defined before.

TEMPERED FRACTIONAL DIFFUSION

- Second moment:

$$\left\langle [x - \langle x \rangle]^2 \right\rangle (t) \begin{cases} C_\beta^2 V^2 \, t^{2\beta} + \frac{2\chi_*}{\Gamma(\beta+1)} \, t^\beta & 0 < \alpha < 1 \\ \frac{2\chi_*}{\Gamma(\beta+1)} \, t^\beta \, , & 1 < \alpha < 2 \, , \end{cases} \, , \quad (3)$$

where $C_\beta = 2/\Gamma(2\beta+1) - 1/\left[\Gamma\left(\beta+1\right)\right]^2$, and

$$\chi_* = \frac{\chi \alpha \left| \alpha - 1 \right|}{2 \left| \cos\left(\alpha \pi / 2 \right) \right| \lambda^{2-\alpha}} \, .$$

- Note that, as expected,

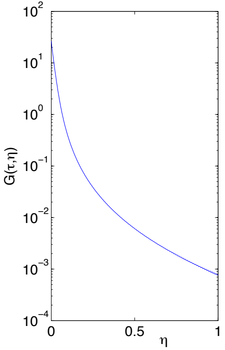
$$\lim_{\lambda \to 0} \chi_* = \infty$$

[Cartea and del-Castillo-Negrete, PRE, **76** 041105 (2007)]

GREEN'S FUNCTION OF SYMMETRIC TEMPERED FRACTIONAL DIFFUSION $\theta = \sigma = 0$ and $1 < \alpha < 2$

$$G(x,t) = \frac{1}{\pi}\int_0^\infty \cos(kx) E_\beta\left[t^\beta \Lambda_{ET}(k)\right] dk\,.$$

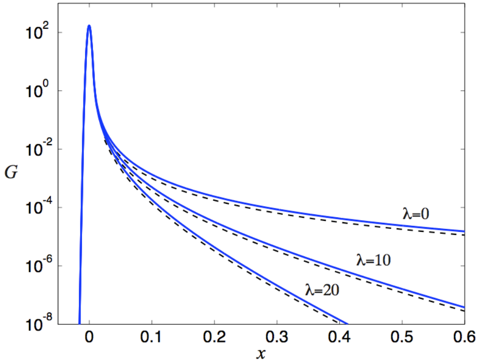
$$\eta = \lambda x\,, \qquad \tau = t/t_c\,, \qquad t_c = c^{-1/\beta}\lambda^{-\alpha/\beta}\,.$$



$\alpha = 1.25$, $\beta = 0.5$ and $\lambda = 1$

GREEN'S FUNCTION OF ASYMMETRIC TEMPERED FRACTIONAL DIFFUSION

$$G = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx+\chi t[(\lambda-ik)^{\alpha}-\lambda^{\alpha}]}\,dk\,, \qquad G \sim \chi t e^{-\chi\lambda^{\alpha}t}\,\frac{e^{-\lambda x}}{x^{1+\alpha}}\,.$$



$\alpha = 1.5,\ \beta = 1,\ \theta = -1.$

TEMPERED INDUCED ANOMALOUS SCALING TRANSITION

- Short time scaling

$$G\left(x,\mu t;\lambda\right)=\mu^{-\beta/\alpha}G\left(\mu^{-\beta/\alpha}x,t;\mu^{\beta/\alpha}\lambda\right)\ .$$

$$G(0,t;\lambda)=t^{-\beta/\alpha}G(0,1;t^{\beta/\alpha}\lambda)$$

$$G(0,t\ll 1;\lambda)\sim t^{-\beta/\alpha}$$

- Long time scaling

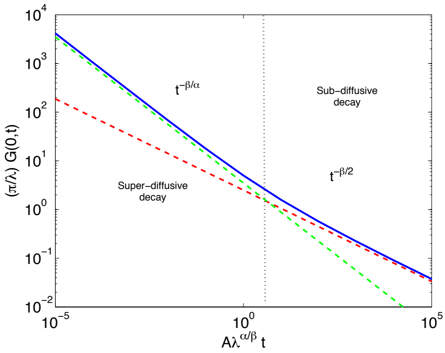
$$G(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}E_\beta\left[t^\beta\Lambda(k;\lambda)\right]dk$$

$$G(0,t)=\frac{1}{2\pi}\int_{0}^{\infty}E_\beta\left[t^\beta\Lambda_{ET}\right]dk$$

$$\sim\int_0^\infty E_\beta\left[-\chi t^\beta k^2\right]dk=\frac{1}{\sqrt{\chi t^\beta}}\int_0^\infty E_\beta\left(-u^2\right)du\,,$$

$$G(0,t\gg 1;\lambda)\sim t^{-\beta/2}\,.$$

ULTRA SLOW CONVERGENCE TO SUB-DIFFUSIVE SCALING



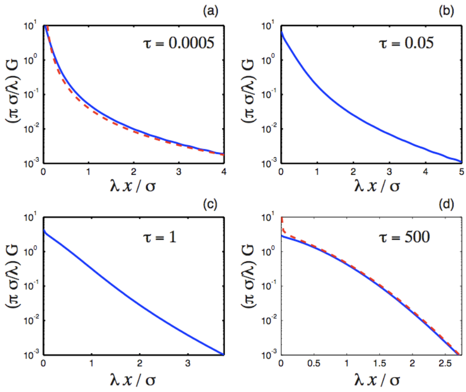
- Short times, $\lambda = 0$ scaling: $G(0, t; \lambda) \sim t^{-\beta/\alpha}$
- Large times, tempered scaling $\lim_{t\to\infty} G(0, t; \lambda) \sim t^{-\beta/2}$
- For $2\beta/\alpha > 1$ super-diffusion $\rightarrow$ sub-diffusion transition with cross-over time

$$\tau_c \sim c^{-1/\beta} \lambda^{-\alpha/\beta}$$

[Cartea and del-Castillo-Negrete, PRE, **76** 041105 (2007)]

TEMPERED INDUCED TRANSITION OF TAILS'S DECAY

- Short time (a) algebraic decay $G(\eta, t \ll t_c) \sim |\eta|^{-(1+\alpha)}$,
- Cross-over time (c) exponential decay $G(x, t \approx t_c) \sim e^{-a\eta}$,
- Long time (d) stretched Gaussian decay
  $G(x, t \gg t_c) \sim \eta^{a_1} \exp(-\eta^{a_2})]$



[Cartea and del-Castillo-Negrete, PRE, **76** 041105 (2007)]

LARGE TRUNCATION EXPANSION

- Fourier transform of tempered fractional derivative operator

$$\widehat{\mathcal{D}_x^{\alpha,\lambda} P} = \lambda^\alpha \left[l \left(1 - \frac{ik}{\lambda} \right)^\alpha + r \left(1 + \frac{ik}{\lambda} \right)^\alpha \right] \hat{P} \, ,$$

- Using the expansion $(1-z)^\alpha = \sum_{j=0}^{\infty} w_j^{(\alpha)} z^j \, , \qquad |z| < 1 \, ,$ we have

$$\widehat{\mathcal{D}_x^{\alpha,\lambda} P} = \lambda^\alpha \sum_{j=0}^{\infty} w_j^{(\alpha)} \left(\frac{ik}{\lambda} \right)^j \left[l + (-1)^j r \right] \hat{P} \, ,$$

- λ-Expansion of tempered fractional difusion

$$\widehat{\partial_x^{\alpha,\lambda} P} = -\frac{V_*}{\chi} H(1-\alpha) \widehat{\partial_x P} -$$

$$-\frac{\lambda^\alpha}{\cos(\alpha\pi/2)} \sum_{j=1}^{\infty} w_{2j}^{(\alpha)} \left(\frac{ik}{\lambda} \right)^{2j} \left[1 - \left(\frac{2j-\alpha}{2j+1} \right) \frac{ik\theta}{\lambda} \right] \hat{P}$$

[Kullberg and D. del-Castillo-Negrete, J. Phys. A: Math. Theor. 45 255101 (2012).]

LARGE TRUNCATION EXPANSION

- The expansion converges in general only for $|k| < \lambda$. To invert the Fourier transform we introduce the low-pass filter operator

$$\overline{f}(x) = \mathcal{F}^{-1}\left\{H\left(\lambda - |k|\right)\mathcal{F}\left[f\right]\right\} .$$

- Applying the low-pass filter to the operator, $\partial_x^{\alpha,\lambda} P$, we get

$$\overline{\partial_x^{\alpha,\lambda} P} = -\frac{V_*}{\chi} H(1-\alpha)\frac{\partial \overline{P}}{\partial x} - \frac{1}{\lambda^{2-\alpha}\cos\left(\alpha\pi/2\right)}$$

$$\sum_{j=1}^{\infty} \frac{w_{2j}^{(\alpha)}}{\lambda^{2(j-1)}} \frac{\partial^{2j}}{\partial x^{2j}} \left[1 + \left(\frac{2j-\alpha}{2j+1}\right)\frac{\theta}{\lambda}\frac{\partial}{\partial x}\right]\overline{P} ,$$

convergence is guaranteed because $\hat{\overline{P}}(k) = 0$ for $|k| > \lambda$.

- Equation for coarse grained PDF at scales larger than $1/\lambda$

$$\frac{\partial \overline{P}}{\partial t} + V_* H(1-\alpha)\frac{\partial \overline{P}}{\partial x} =$$

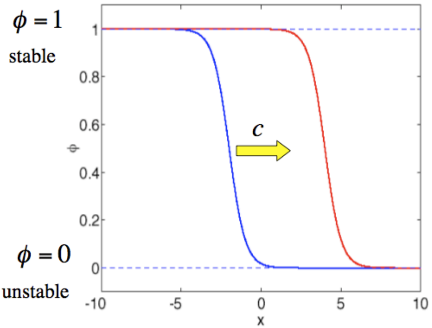
$$\chi_* \frac{\partial^2 \overline{P}}{\partial x^2} + \chi_* \frac{(2-\alpha)\,\theta}{3\lambda}\frac{\partial^3 \overline{P}}{\partial x^3} + \chi_* \frac{(3-\alpha)\,(2-\alpha)}{12\lambda^2}\frac{\partial^4 \overline{P}}{\partial x^4} + \ldots ,$$

FRONT PROPAGATION IN REACTION-DIFFUSION SYSTEMS

- One of the simplest reaction-diffusion systems is the extensively studied Fisher-Kolmogorov model

$$\partial_t \phi = \chi \partial_x^2 \phi + \gamma \phi (1 - \phi)$$

- The nontrivial dynamics of this type of systems arises from the competition between the reaction kinetics and diffusion.
- Front speed $c = 2\sqrt{\gamma\chi}$

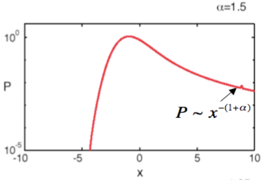


FRONT PROPAGATION IN THE PRESENCE OF LEVY FLIGHTS

As a simple model to explore the role of super-diffusive transport in reaction-diffusion systems we consider asymmetric fractional Fisher-Kolmogorov equation

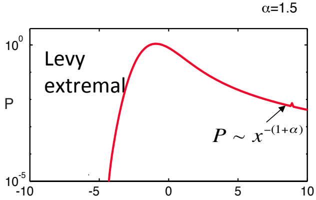
$$\partial_t \phi = \chi \, {}_{-\infty}D_x^{\alpha} \, \phi + \gamma \phi \, (1 - \phi)$$

$${}_{-\infty}D_x^{\alpha} \, \phi = \frac{1}{\Gamma(n-\alpha)} \partial_x^n \int_a^x \frac{\phi(u)}{(x-u)^{\alpha-n+1}} du \, .$$



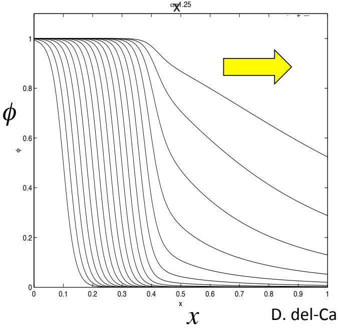
[del-Castillo-Negrete, Carreras, and Lynch, PRL, **91** 018302 (2003)]

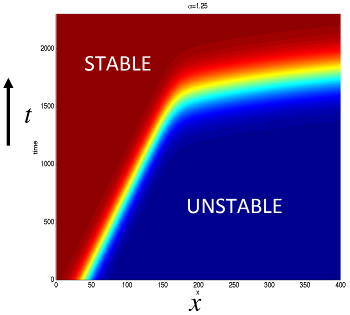
Algebraic decaying accelerated fronts



$$\phi(x,0) = 1 - \frac{1}{2}\left[1 + \tanh\left(\frac{x - x_0}{2W}\right)\right]$$

$$\boxed{\alpha = 1.25}$$





D. del-Castillo-Negrete et al, Phys.Rev. Lett. **91,** 018302-1 (2003)

Leading edge calculation for algebraic decaying, accelerated fronts

At large *x*, $\phi << 1$ implies

$$\boxed{\partial_t \phi = D\, \partial_x^\alpha \phi + \gamma\, \phi}$$

$$\phi(x,0) = \begin{cases} 1 & x < 0 \\ e^{-\lambda x} & x > 0 \end{cases}$$

$$\phi = e^{\gamma t}\, \psi(x,t) \qquad \boxed{\partial_t \psi = D\, \partial_x^\alpha \psi}$$

$$\psi(x,t) = \int_{-\infty}^{x} P_\alpha(\eta)\, \psi_0\left[x - (D t)^{1/\alpha}\eta\right] d\eta$$

$$\phi(x,t) = e^{\gamma t}\int_{z}^{\infty} P_\alpha(\eta)\, d\eta + e^{-\lambda x + \gamma t}\int_{-\infty}^{z} e^{\lambda (Dt)^{1/\alpha}\eta}\, P_\alpha(\eta)\, d\eta \qquad z = x\,(D t)^{-1/\alpha}$$

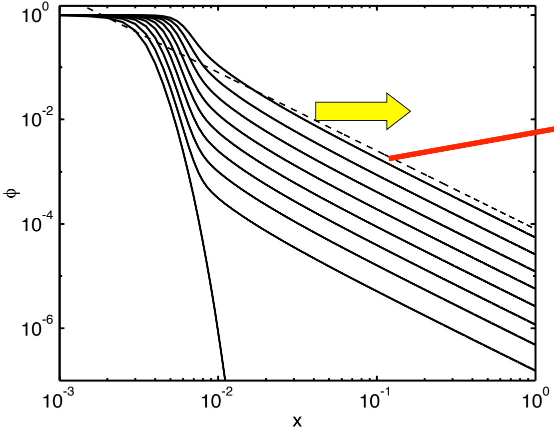
using $\boxed{P \sim \eta^{\alpha+1}}$ for large η

t fixed $z \to \infty$ $\boxed{\phi \sim x^{-\alpha}}$ algebraic tail

x fixed $t \to \infty$ $\phi \sim e^{\gamma t}$ $\boxed{V \sim e^{\gamma t/\alpha}}$ exponential acceleration

[del-Castillo-Negrete, Carreras, and Lynch, PRL, **91** 018302 (2003)]

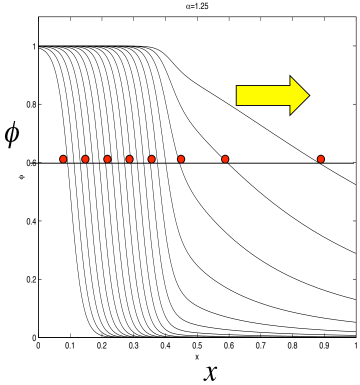
Algebraic decaying front



Analytical scaling

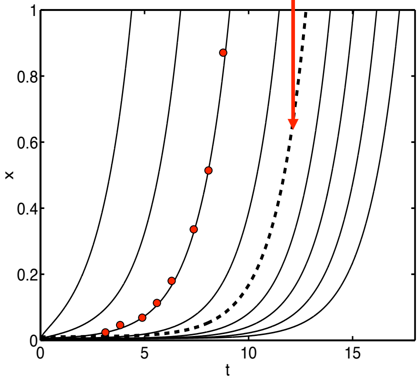
$$\boxed{\phi \sim x^{-\alpha}}$$

Exponential acceleration



Analytical scaling

$$\boxed{\phi \sim x^{-\alpha}\, e^{\gamma t}}$$



Space-time plot of front isocontours

TEMPERING EFFECTS IN SUPER-DIFFUSIVE FRONT ACCELERATION

- To study the role of truncation in fronts we propose the exponentially truncated fractional Fisher-Kolmogorov equation

$$\partial_t \phi = -V \partial_x \phi + c \mathcal{D}_x^{\alpha,\lambda} \phi - \mu \phi + \gamma \phi (1 - \phi)$$

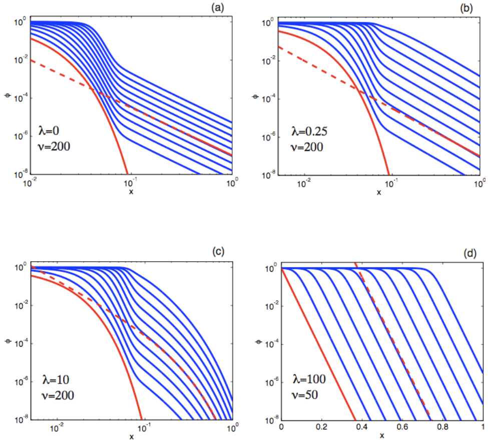
- Here we will focus attention in the left asymmetric truncated fractional case without drift

$$\partial_t \phi = \chi \left[e^{-\lambda x} \, {}_{-\infty}D_x^{\alpha} \left(e^{\lambda x} \phi \right) - \lambda^{\alpha} \phi \right] + \gamma \phi (1 - \phi)$$

- Without truncation Lévy flights lead to algebraic tails and exponential front acceleration.
- What is the role of role of tempering on these phenomena? [del-Castillo-Negrete, PRE **79**, 031120 (2009)]

FRONT REGIMENS: NUMERICAL RESULTS

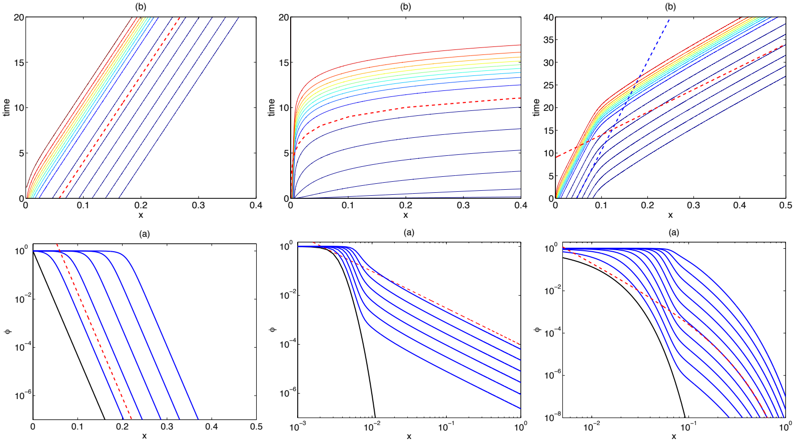
(a) Asymptotic algebraic regime for $\lambda = 0$; (b) Intermediate asymptotic algebraic regime for $\lambda \neq 0$; (c) Truncated regime for $0 < \lambda < \nu$; (d) Over-truncated regime for $\lambda > \nu$.



[del-Castillo-Negrete, PRE **79**, 031120 (2009)]

ANOMALOUS TRANSPORT AND FRONT PROPAGATION

Regular Diffusion | Fractional Diffusion | Tempered Fractional Diffusion



LEADING EDGE APPROXIMATION

- At the leading edge of the front $\phi \ll 1$, and therefore

$$\partial_t \phi = \chi e^{-\lambda x} \, {}_{-\infty}D_x^{\alpha} \left(e^{\lambda x} \phi \right) + (\gamma - \chi \lambda^{\alpha}) \, \phi$$

- Substituting $\phi = e^{-\lambda x + (\gamma - \chi \lambda^{\alpha})t} \psi(x,t)$ the equation reduces to the asymmetric fractional diffusion equation with general solution

$$\psi(x,t) = \int_{-\infty}^{\infty} \hat{G}_{\lambda=0}(\eta) \, \psi_0 \left[x - (\chi t)^{1/\alpha} \, \eta \right] d\eta$$

- For an initial condition of the form $\phi(x, t=0) = A$ for $x < 0$ and $\phi(x, t=0) = e^{-\nu x}$

$$\psi = e^{-(\nu-\lambda)x} \int_{-\infty}^{x/\tau} e^{(\nu-\lambda)\tau\eta} \hat{G}_{\lambda=0} d\eta + A e^{\lambda x} \int_{x/\tau}^{\infty} \hat{G}_{\lambda=0} e^{-\lambda \tau \eta} d\eta$$

LEADING EDGE APPROXIMATION

- In terms of ϕ the solution can be written as

$$\phi = e^{-\nu x+(\gamma-\chi\lambda^{\alpha})t}\,\mathcal{I}_1 + Ae^{(\gamma-\chi\lambda^{\alpha})t}\mathcal{I}_2\,,$$

- Where

$$\mathcal{I}_1 = \int_{-\infty}^{x/\tau} e^{(\nu-\lambda)\tau\eta}\,\hat{G}_{\lambda=0}(\eta)d\eta$$

$$\mathcal{I}_2 = \int_{x/\tau}^{\infty} \hat{G}_{\lambda=0}(\eta)e^{-\lambda\tau\eta}d\eta\,.$$

- The analysis is based on the asymptotic behavior of $\mathcal{I}_1$ and $\mathcal{I}_2$ for $x/\tau \to \infty$ where

$$\tau = (\chi t)^{1/\alpha}$$

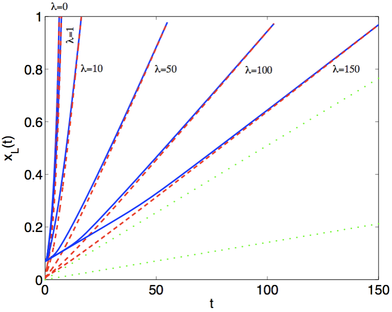
LAGRANGIAN FRONT SPACE-TIME PATH, $x_L(t)$,

- Intermediate asymptotic Levy tempered front path

$$\lambda x_L(t) + (\gamma - \chi\lambda^\alpha)\, t + \ln t - (\alpha+1) \ln x_L(t) = M$$

- Gaussian, diffusive front speed (green dotted lines)

$$c = \frac{\gamma}{\nu} + \nu\chi$$

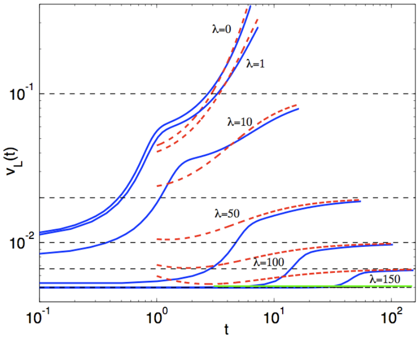


LAGRANGIAN FRONT SPEED $v_L(t) = dx_L/dt$

- Intermediate asymptotic Levy tempered front speed (red dashed lines)

$$v_L(t) = \frac{\gamma - \chi\lambda^\alpha + \frac{1}{t}}{\lambda + \frac{\alpha+1}{x_L(t)}}$$

- Terminal velocity (black lines) $v_* = \frac{\gamma - \lambda^\alpha \chi}{\lambda}$



[del-Castillo-Negrete, PRE **79**, 031120 (2009)]

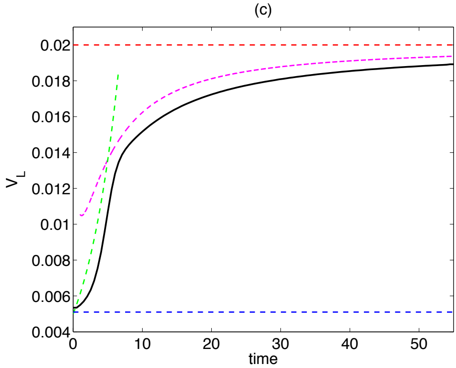
LAGRANGIAN FRONT SPEED $v_L(t) = dx_L/dt$

Blue: Diffusive front speed $c = \frac{\gamma}{\nu} + \nu\chi_d$

Red: Terminal speed $v_* = \frac{\gamma - \lambda^\alpha \chi}{\lambda}$

Green: Fractional speed $v_L(t) \approx v_{L0} e^{\gamma(t-t_0)/\alpha}$

Magenta: Tempered speed $v_L(t) = \frac{dx_L}{dt} = \frac{\gamma - \chi\lambda^\alpha + \frac{1}{t}}{\lambda + \frac{\alpha+1}{x_L(t)}}$



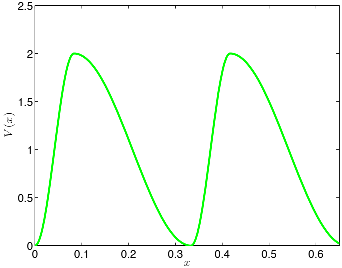
FOKKER-PLANCK EQUATION WITH RATCHET POTENTIAL

$$\partial_t P = \partial_x \left[P \partial_x V \right] + \chi \partial_x^2 P \,.$$

Periodic potential, $V(x) = V(x+L)$,

$$V = V_0 \begin{cases} 1 - \cos\left[\pi x / a_1\right] & \text{if } 0 \leq x < a_1 \\ 1 + \cos\left[\pi (x - a_1)/a_2\right] & \text{if } a_1 \leq x < L \,, \end{cases}$$

with broken symmetry parameter $A = (a_1 - a_2)/L$



As it is well-known, in this case, even when the potential is asymmetric, a net current cannot appear unless a non-equilibrium perturbation is added.

LEVY RATCHETS IN THE FRACTIONAL FOKKER-PLANCK EQUATION

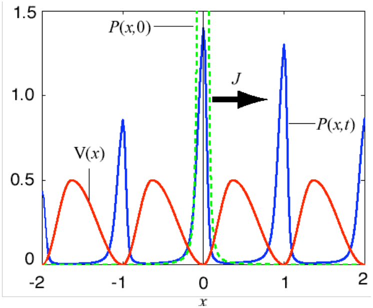
- In [del-Castillo-Negrete, Gonchar, Chechkin, arXiv:0710.0883 (2007), Physica A (2008)] a minimal model of ratchet transport driven by Levy noise was presented.
- Numerical integrations of the Fractional Fokker Planck equation

$$\partial_t P = \partial_x \left[P \partial_x V \right] + \chi \left[l_{-\infty} D_x^{\alpha} + r_x D_{\infty}^{\alpha} \right] P$$

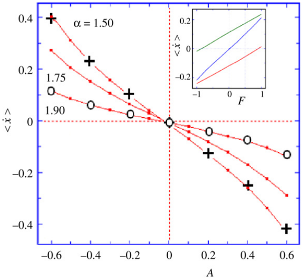
showed that even in the absence of an external tilting force or a bias in the noise, the Levy flights drive the system out of the thermodynamic equilibrium and generate a current in the presence of an asymmetric potential.

LEVY RATCHETS IN THE FRACTIONAL FOKKER-PLANCK EQUATION

Time evolution of of PDF



Current as function of asymmetry and α

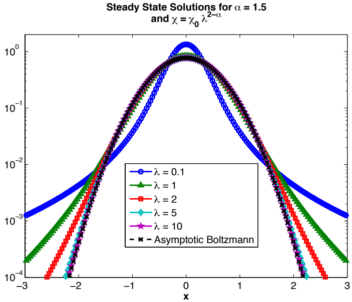


[del-Castillo-Negrete, Gonchar, Chechkin, arXiv:0710.0883 (2007), Physica A (2008)]

What is the role of truncation in this phenomena?

TEMPERED FOKKER-PLANCK EQUATION FOR QUADRATIC POTENTIAL $V(x) = Ax^2$

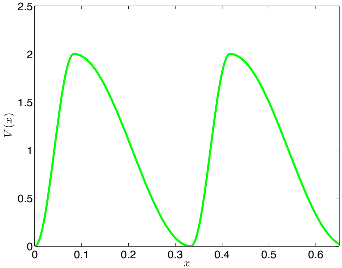
$$\partial_t P = \partial_x \left[P \partial_x V \right] + \chi \left[l e^{-\lambda x} \, {}_{-\infty}D_x^{\alpha} \, e^{\lambda x} + r e^{\lambda x} \, {}_{x}D_{\infty}^{\alpha} \, e^{-\lambda x} - \nu \right] P \, ,$$

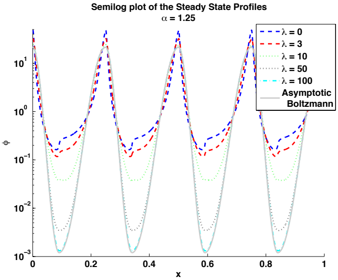


Non-Boltzmann steady state for finite λ, and approach to Boltzmann steady state in the limit $\lambda \to \infty$

[Kullberg and D. del-Castillo-Negrete, J. Phys. A: Math. Theor. 45 255101 (2012).]

TEMPERED FOKKER-PLANCK EQUATION FOR RATCHET POTENTIAL



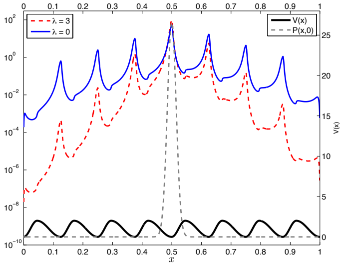


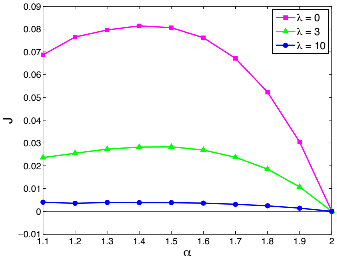
Non-Boltzmann steady state for finite λ, and approach to Boltzmann steady state in the limit $\lambda \to \infty$

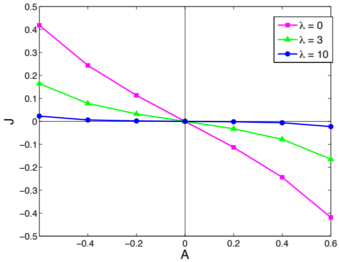
[Kullberg and D. del-Castillo-Negrete, J. Phys. A: Math. Theor. 45 255101 (2012).]

TIME DEPENDENT SOLUTION AND RATCHET CURRENT

$\alpha = 1.5$







[Kullberg and D. del-Castillo-Negrete, J. Phys. A: Math. Theor. 45

CONCLUSIONS

- We review fractional diffusion in the context of the CTRW model and discussed applications in fluids and plasmas. [dCN, Carreras, and Lynch, PRL, **94** 065003 (2005)]

- Following Ref.[Cartea and dCN, PRE, **76** 041105 (2007)] we discussed the CTRW for general stochastic processes and for Levy tempered processes in particular.

- The continuum limit of the CTRW for Levy tempered processes leads to the tempered fractional diffusion equation introduced in Ref.[Cartea and dCN, PRE, **76** 041105 (2007)].

- The non-Markovian, tempered fractional diffusion model exhibits ultra-slow convergence to sub-diffusive transport and the pdf exhibits a transition from algebraic decaying to stretched exponential

CONCLUSIONS

- Fronts in the fractional Fisher-Kolmogorov equation exhibit exponential acceleration [dCN, Carreras, and Lynch, PRL, **91** 018302 (2003)].
- With truncation, this phenomenology prevails in an intermediate asymptotic regime. Outside this regime, the front's velocity exhibits an algebraically slow convergence to a terminal velocity [dCN, PRE **79**, 031120 (2009)].
- Following Ref.[dCN, Gonchar, Chechkin, arXiv:0710.0883 (2007), Physica A (2008)] we discussed a minimal model for Levy ratchets.
- In the limit $\lambda \to \infty$ the steady state solution of the tempered Fractional Fokker-Planck equation approaches the Boltzmann distribution and the ratchet current vanishes. However, for finite λ, the steady state is non-Boltzmannian and a ratchet current persists. [Kullberg and D. del-Castillo-Negrete, J. Phys. A: Math. Theor. 45 255101 (2012).].